Topological string theory from Landau-Ginzburg models

based on: arXiv:0904.0862 [hep-th],
arXiv:1104.5438 & 1111.1749 [hep-th] with **Michael Kay**

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Outline

□ open topological string theory $\Longleftrightarrow$ Calabi-Yau A_∞-algebra

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- □ A_∞-algebras and relation to amplitudes

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- □ bulk-deformed amplitudes $\Longleftrightarrow$ *curved* Calabi-Yau A_∞-algebra

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- □ solution to deformation problem:
  - “weak” deformation quantisation
  - homological perturbation

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- □ B-twisted Landau-Ginzburg models
- □ bulk-deformed amplitudes $\Longleftrightarrow$ *curved* Calabi-Yau A_∞-algebra
- □ solution to deformation problem:
  - – "weak" deformation quantisation
  - – homological perturbation
- □ focus on general, conceptual results

Open topological string theory

Energy-momentum tensor T is BRST exact:

$$T(z) = \left[Q, G(z)\right]$$

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Topological *field* theory **correlators**

$$\big\langle \psi_{i_1} \dots \psi_{i_n} \big\rangle_{\mathsf{disk}}\,, \qquad \omega_{ij} = \big\langle \psi_i \psi_j \big\rangle_{\mathsf{disk}}$$

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$$\Big\langle \psi_{i_1} \psi_{i_2} \psi_{i_3} \int \psi_{i_4}^{(1)} \ldots \int \psi_{i_n}^{(1)} \Big\rangle_{\mathsf{disk}} \,, \qquad \psi_i^{(1)} = \big[G_{-1}, \psi_i\big] \mathrm{d}\tau$$

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$$W_{i_1 \dots i_n} = \left\langle \psi_{i_1} \psi_{i_2} \psi_{i_3} \int \psi^{(1)}_{i_4} \dots \int \psi^{(1)}_{i_n} \right\rangle$$

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Get **effective superpotential** from amplitudes:

$$\mathcal{W}(u) = \sum_{n \geqslant 3} \frac{1}{n} W_{i_1 \dots i_n} u_{i_1} \dots u_{i_n}$$

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Ward identities and **BRST symmetry** imply cyclic symmetry and

$$\sum_{r,s} \pm \omega^{ii'} W_{i' i_1 \ldots i_r j i_{r+s+1} \ldots i_n} \omega^{jj'} W_{j' i_{r+1} \ldots i_{r+s}} = 0$$

Hofman/Ma 2000, Herbst/Lazaroiu/Lerche 2004

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open topological string theory $\Longrightarrow$ Calabi-Yau A_∞-algebra

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A_∞-algebras

An ***A_∞*-algebra** is a graded vector space A together with a degree-one codifferential

$$\partial : T_A \longrightarrow T_A\,, \qquad T_A = \bigoplus_{n\geqslant 1} A[1]^{\otimes n}\,, \qquad \partial^2 = 0$$

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subject to the relations (from $\partial^2 = 0$)

$$\sum_{\substack{i\geqslant 0, j\geqslant 1,\\ i+j\leqslant n}} m_{n-j+1} \circ \left(\mathbb{1}^{\otimes i} \otimes m_j \otimes \mathbb{1}^{\otimes(n-i-j)}\right) = 0$$

Stasheff 1963

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$$\begin{aligned}
n = 1: &\quad m_1 \circ m_1 = 0\\
n = 2: &\quad m_1 \circ m_2 + m_2 \circ (m_1 \otimes \mathbb{1}) + m_2 \circ (\mathbb{1} \otimes m_1) = 0\\
n = 3: &\quad m_2 \circ (m_2 \otimes \mathbb{1}) + m_2 \circ (\mathbb{1} \otimes m_2)\\
&\qquad + m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathbb{1}^{\otimes 2} + \mathbb{1} \otimes m_1 \otimes \mathbb{1} + \mathbb{1}^{\otimes 2} \otimes m_1) = 0\\
n = 4: &\quad \dots
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An A_∞-Algebra is **Calabi-Yau** if it is minimal and cyclic with respect to a non-degenerate pairing.

Relation to open topological string theory

Underlying TFT data (*Frobenius algebra*):

$$H : \text{space of states} \;=\; \text{BRST cohomology with basis } \{\psi_i\}$$

$$\big\langle \psi_{i_0} \ldots \psi_{i_n} \big\rangle : \text{correlators computed from OPE and topological metric}$$

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To get from TFT to topological string theory, need *Calabi-Yau* A_∞*-algebra* (H, m_n):

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How to compute the products m_n?

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Minimal model theorem. For any A_∞-algebra (A, ∂), its cohomology $H = H_{m_1}(A)$ has a minimal A_∞-structure $\widetilde{\partial}$

Kadeishvili 1980, Merkulov 1999, Kontsevich/Soibelman 2000

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$$F : (H, \widetilde{\partial}) \longrightarrow (A, \partial)$$

unique up to A_∞-isomorphism.

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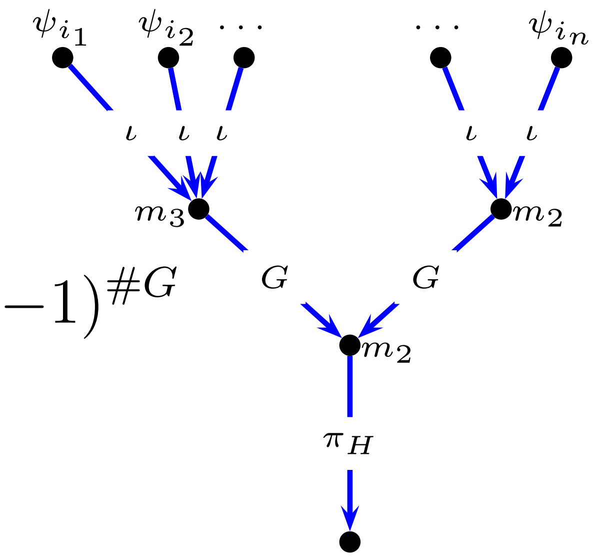
Sketch of proof: Compute Feynman diagrams.

$$\widetilde{m}_n(\psi_{i_1} \otimes \ldots \otimes \psi_{i_n}) = \sum_{\text{trees with } n \text{ leaves}} (-1)^{\#G}$$



Kadeishvili 1980, Merkulov 1999, Kontsevich/Soibelman 2000

General picture

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topological string field theory

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topological string field theory $=$ *off-shell* space A

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topological string field theory $=$ *off-shell* space A with DG structure ∂

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topological string theory $=$ *on-shell* space H with higher A_∞-products $\widetilde{\partial}$

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(Same for bulk sector if we replace "DG" by "DG Lie" and "A_∞" by "L_∞".)

B-twisted affine Landau-Ginzburg models

$$Z = \int \mathcal{D}\Phi \; \mathrm{e}^{-\int K(\Phi,\bar{\Phi}) - (\frac{\mathrm{i}}{2}\int W(\Phi) + \mathsf{c.\,c.})} \, \mathrm{str}\left(\mathcal{P}\, \mathrm{e}^{-\oint (\frac{1}{2}\rho^i \nabla_i D + \frac{\mathrm{i}}{2}\eta^{\bar{\imath}} \nabla_{\bar{\imath}} D^\dagger + \frac{\mathrm{i}}{2}\{D, D^\dagger\})} \right)$$

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Bulk sector. **Jacobi algebra** $\mathrm{Jac}(W) = \mathbb{C}[x_1, \ldots, x_N]/(\partial_i W)$

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$= \text{BRST cohomology of } \left(\Gamma(\mathbb{C}^N, \bigwedge T^{(1,0)}\mathbb{C}^N), [-W, \cdot\,]_{\mathrm{SN}}\right)$

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topological metric:

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Boundary sector. **Matrix factorisations** $D \in \mathrm{Mat}_{2r\times 2r}(\mathbb{C}[\boldsymbol{x}])$ with $D^2 = W \cdot \mathbb{1}_{2r\times 2r}$

Kapustin/Li 2002, Brunner/Herbst/Lerche/Scheuner 2003, Lazaroiu 2003

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topological metric (**Kapustin-Li pairing**):

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Defect sector. $\ldots \Rightarrow$ extended 2d TFT

Kapustin/Li 2002, Brunner/Herbst/Lerche/Scheuner 2003, Lazaroiu 2003, Kapustin/Li 2003, Herbst/Lazaroiu 2004

Open top. string theory for Landau-Ginzburg models

Apply minimal model theorem to off-shell algebra

$$\Big(A = \mathrm{Mat}_{2r\times 2r}(\mathbb{C}[\boldsymbol{x}]),\ [D,\ \cdot\,],\ \text{matrix multiplication} \Big)$$

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$$\Big(A = \mathrm{Mat}_{2r\times 2r}(\mathbb{C}[\boldsymbol{x}]),\ [D,\,\cdot\,],\ \text{matrix multiplication}\Big)$$

to obtain higher A_∞-products on on-shell space $H = H_{[D,\,\cdot\,]}(A)$.

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Result. General algorithm to construct all open tree-level amplitudes, and (another) first-principle derivation of Kapustin-Li pairing.

Kontsevich/Soibelman 2006, Carqueville 2009

Bulk deformations

$$W_{i_1 \ldots i_n}$$

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$$W_{i_1 \dots i_n} \longmapsto W_{i_1 \dots i_n}(t) = \left\langle \psi_{i_1} \psi_{i_2} \psi_{i_3} \int \psi_{i_4}^{(1)} \dots \int \psi_{i_n}^{(1)} \, \mathrm{e}^{\sum_i t_i \int \phi_i^{(2)}} \right\rangle$$

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Fact. Curvature screws everything up:

- m_1 no longer a differential
- cannot apply minimal model theorem
- need a new approach

Deformations and Maurer-Cartan equations

Given an A_∞-algebra (A, ∂), a *deformation* is $\delta \in \mathrm{End}^1(T_A)$ such that $(A, \partial + \delta)$ is a curved A_∞-algebra.

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This is the **Maurer-Cartan equation** for the DG Lie algebra

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Fact. Let L be an L_∞-quasi-isomorphism between DG Lie algebras. Then

$$\delta \longmapsto \sum_{n \geqslant 1} \frac{1}{n!}\, L_n(\delta^{\wedge n})$$

is an *isomorphism* between the spaces of Maurer-Cartan solutions modulo gauge transformations.

Back to Landau-Ginzburg models

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Off-shell bulk sector is also a DG Lie algebra:

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Transport them to deformations of $(H, \widetilde{\partial})$ via an L_∞-map

$$\Big(T_{\mathrm{poly}},\, [-W,\, \cdot\,]_{\mathrm{SN}},\, [\,\cdot\,,\,\cdot\,]_{\mathrm{SN}}\Big) \longrightarrow \Big(\mathrm{Coder}(T_H),\, [\widetilde{\partial},\, \cdot\,],\, [\,\cdot\,,\,\cdot\,]\Big)$$

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First step: deformations of the off-shell open string algebra

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$$\Big(T_{\text{poly}},\ [-W,\ \cdot\,]_{\text{SN}},\ [\,\cdot\,,\ \cdot\,]_{\text{SN}}\Big) \longrightarrow \Big(\text{Coder}(T_A),\ [\partial,\ \cdot\,],\ [\,\cdot\,,\ \cdot\,]\Big)$$
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First step: deformations of the off-shell open string algebra

Theorem. There is a sequence of explicit L_∞-quasi-isomorphisms

$$\Big(T_{\text{poly}},\ [-W,\ \cdot\,]_{\text{SN}},\ [\,\cdot\,,\ \cdot\,]_{\text{SN}}\Big) \xrightarrow[\text{deform. quant.}]{\text{“weak”}} \Big(\text{Coder}(T_{\mathbb{C}[\boldsymbol{x}]}),\ [\widehat{\partial}_0 + \widehat{\partial}_2,\ \cdot\,],\ [\,\cdot\,,\ \cdot\,]\Big)$$
$$\xrightarrow[\text{equivalence}]{\text{Morita}} \Big(\text{Coder}(T_A),\ [\partial_0 + \partial_2,\ \cdot\,],\ [\,\cdot\,,\ \cdot\,]\Big)$$
$$\xrightarrow[\text{cancellation}]{\text{tadpole}} \Big(\text{Coder}(T_A),\ [\partial_1 + \partial_2,\ \cdot\,],\ [\,\cdot\,,\ \cdot\,]\Big)$$

Digression: deformation quantisation à la Kontsevich

Consider classical theory with phase space $M = \mathbb{R}^d$ and associative, commutative algebra of observables $(C^\infty(M,\mathbb{R}),\ \cdot\ ) \equiv (C^\infty(M,\mathbb{R}), \widehat{\partial}_2)$.

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Deformation quantisation constructs algebra of quantum observables $(C^\infty(M,\mathbb{R})[\![\hbar]\!], \star)$ by deforming the product to

$$f \star g = f \cdot g + B_1(f,g)\hbar + B_2(f,g)\hbar^2 + \dots$$

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Kontsevich constructs an explicit L_∞-quasi-isomorphism

$$K:\ \Big(\Gamma(M, \textstyle\bigwedge TM),\ 0,\ [\,\cdot\,,\ \cdot\,]_{\mathrm{SN}}\Big) \longrightarrow \Big(\mathrm{Coder}(T_{C^\infty(M,\mathbb{R})}),\ [\widehat{\partial}_2,\ \cdot\,],\ [\,\cdot\,,\ \cdot\,]\Big)$$

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$$
\begin{aligned}
&\Big(K_n(\gamma_1\wedge\ldots\wedge\gamma_n)\Big)_m(f_1\otimes\ldots\otimes f_m)\\
&=\sum_{\Gamma\in\mathcal{G}(n,m)}\frac{1}{(2\pi)^{2n+m-2}}\int_{\iota(\overline{C}^{n,m})}\bigwedge_{k=1}^{n}\big(\,\mathrm{d}\varphi_{e_k^1}\wedge\ldots\wedge\mathrm{d}\varphi_{e_k^{\widetilde{\gamma}_k}}\big)\\
&\qquad\cdot\sum_I\Big[\prod_{i=1}^{n}\Big(\prod_{e\in\Gamma_{\bullet\to i}}\partial_{I(e)}\Big)\gamma_i^{I(e_i^1)\ldots I(e_i^{\widetilde{\gamma}_i})}\Big]\Big[\prod_{\bar{\jmath}=\bar{1}}^{\bar{m}}\Big(\prod_{e\in\Gamma_{\bullet\to\bar{\jmath}}}\partial_{I(e)}\Big)f_{\bar{\jmath}}\Big]
\end{aligned}
$$

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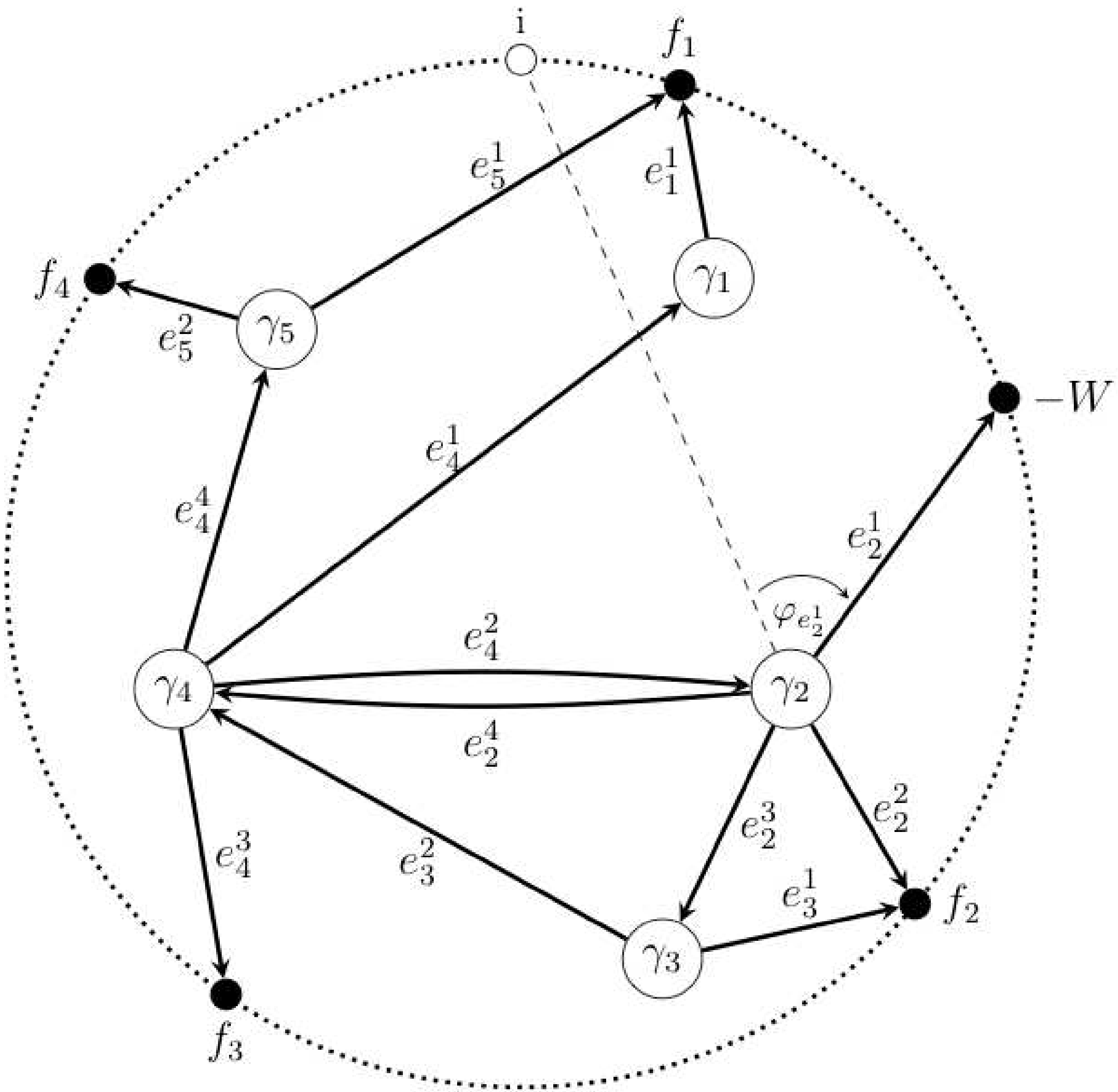

Weak deformation quantisation

Theorem. Kontsevich's map

$$K : \Big(T_{\mathrm{poly}},\, 0,\, [\,\cdot\,,\,\cdot\,]_{\mathrm{SN}} \Big) \longrightarrow \Big(\mathrm{Coder}(T_{\mathbb{C}[\boldsymbol{x}]}),\, [\widehat{\partial}_2,\,\cdot\,],\, [\,\cdot\,,\,\cdot\,] \Big)$$

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Thus we have constructed the first part of our bulk deformation map

$$\begin{aligned}\Big(T_{\mathrm{poly}},\, [-W,\,\cdot\,]_{\mathrm{SN}},\, [\,\cdot\,,\,\cdot\,]_{\mathrm{SN}}\Big) &\longrightarrow \Big(\mathrm{Coder}(T_A),\, [\partial,\,\cdot\,],\, [\,\cdot\,,\,\cdot\,]\Big)\\ &\longrightarrow \Big(\mathrm{Coder}(T_H),\, [\widetilde{\partial},\,\cdot\,],\, [\,\cdot\,,\,\cdot\,]\Big)\end{aligned}$$

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Theorem. Let (A, ∂) be an A_∞-algebra and $(H, \widetilde{\partial})$ its minimal model.

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in standard form, i. e.

$$\bar{F}F = \mathbb{1}_{T_H} \, , \qquad \mathbb{1}_{T_A} - F\bar{F} = \partial U + U\partial \, , \qquad U^2 = UF = \bar{F}U = 0$$

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This gives rise to an L_∞-morphism

$$
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\delta &\longmapsto \sum_{n\geqslant 1} \bar{F}(\delta U)^n \delta F
\end{aligned}
$$

Recursive formulas for $\bar{F}, U$

$$U_n^1 = -\frac{1}{2} G \partial_2^1 \Big(\sum_{l=1}^{n-1} (U_l^1 \otimes (\mathbb{1} + F\bar{F})_{n-l}^1 + (\mathbb{1} + F\bar{F})_{n-l}^1 \otimes U_l^1 \Big)$$

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All off-shell deformations δ are bulk-induced, i. e. uniquely defined by

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Thus the curved A_∞-products describing bulk-deformed open topological string amplitudes are explicitly encoded in

$$\widetilde{\partial} + \sum_{n\geqslant 1} \bar{F}(\delta U)^n \delta F$$

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- open topological string theory $\Longleftrightarrow$ Calabi-Yau A_∞-algebra
- gives explicit algorithm to compute open amplitudes
- **bulk-deformed amplitudes** computable via *weak deformation quantisation* and *homological perturbation*:

$$\left\langle \psi_{i_0}, \widetilde{m}_n^t(\psi_{i_1} \otimes \ldots \otimes \psi_{i_n}) \right\rangle$$
$$= \left\langle \psi_{i_0}, \psi_{i_1} \psi_{i_2} \int \psi_{i_3}^{(1)} \ldots \int \psi_{i_n}^{(1)} \, \mathrm{e}^{\sum_i t_i \int \phi_i^{(2)}} \right\rangle$$