Topological string theory
from Landau-Ginzburg models

based on: arXiv:0904.0862 [hep-th],

Nils Carqueville

LMU München
open topological string theory $\Longleftrightarrow$ Calabi-Yau $A_\infty$-algebra
open topological string theory $\iff$ Calabi-Yau $A_\infty$-algebra

$A_\infty$-algebras and relation to amplitudes
Outline

- open topological string theory $\iff$ Calabi-Yau $A_\infty$-algebra
- $A_\infty$-algebras and relation to amplitudes
- B-twisted Landau-Ginzburg models
open topological string theory $\iff$ Calabi-Yau $A_\infty$-algebra

$A_\infty$-algebras and relation to amplitudes

B-twisted Landau-Ginzburg models

bulk-deformed amplitudes
open topological string theory $\iff$ Calabi-Yau $A_\infty$-algebra

$A_\infty$-algebras and relation to amplitudes

B-twisted Landau-Ginzburg models

bulk-deformed amplitudes $\iff$ curved Calabi-Yau $A_\infty$-algebra
open topological string theory $\iff$ Calabi-Yau $A_\infty$-algebra

$A_\infty$-algebras and relation to amplitudes

B-twisted Landau-Ginzburg models

bulk-deformed amplitudes $\iff$ *curved* Calabi-Yau $A_\infty$-algebra

solution to deformation problem:

- “weak” deformation quantisation
- homological perturbation
open topological string theory $\iff$ Calabi-Yau $A_\infty$-algebra

$A_\infty$-algebras and relation to amplitudes

B-twisted Landau-Ginzburg models

bulk-deformed amplitudes $\iff$ curved Calabi-Yau $A_\infty$-algebra

solution to deformation problem:
  
  - “weak” deformation quantisation
  
  - homological perturbation

focus on general, conceptual results
Energy-momentum tensor $T$ is BRST exact:

$$T(z) = [Q, G(z)]$$
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Topological field theory correlators

$$\langle \psi_{i_1} \cdots \psi_{i_n} \rangle_{\text{disk}}, \quad \omega_{ij} = \langle \psi_i \psi_j \rangle_{\text{disk}}$$
Open topological string theory

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Topological string theory amplitudes

$$\langle \psi_{i_1} \psi_{i_2} \psi_{i_3} \int \psi_{i_4}^{(1)} \cdots \int \psi_{i_n}^{(1)} \rangle_{\text{disk}}, \quad \psi_i^{(1)} = [G_{-1}, \psi_i] d\tau$$
Open topological string theory

\[ W_{i_1...i_n} = \langle \psi_{i_1} \psi_{i_2} \psi_{i_3} \int \psi_{i_4}^{(1)} \ldots \int \psi_{i_n}^{(1)} \rangle \]
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Get **effective superpotential** from amplitudes:

\[ \mathcal{W}(u) = \sum_{n \geq 3} \frac{1}{n} W_{i_1...i_n} u_{i_1} \ldots u_{i_n} \]
Open topological string theory

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Ward identities and **BRST symmetry** imply cyclic symmetry and

\[ \sum_{r,s} \pm \omega^{ii'} W_{i_1 \ldots i_r j i_{r+s+1} \ldots i_n} \omega^{jj'} W_{j' i_{r+1} \ldots i_{r+s}} = 0 \]
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open topological string theory \( \iff \) Calabi-Yau \( A_\infty \)-algebra
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open topological string theory \(\iff\) Calabi-Yau \(A_\infty\)-algebra

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Hofman/Ma 2000, Herbst/Lazaroiu/Lerche 2004, Costello 2004
An $A_\infty$-algebra is a graded vector space $A$ together with a degree-one codifferential

$$\partial : T_A \longrightarrow T_A, \quad T_A = \bigoplus_{n \geq 1} A[1]^\otimes n, \quad \partial^2 = 0$$
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$$m_n = \pi_{A[1]} \circ \partial|_{A[1] \otimes^n} : A[1] \otimes^n \rightarrow A[1]$$
An $\infty$-algebra is a graded vector space $A$ together with a degree-one codifferential

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$$m_n = \pi_{A[1]} \circ \partial \big|_{A[1] \otimes^n} : A[1] \otimes^n \to A[1]$$

subject to the relations (from $\partial^2 = 0$)

$$\sum_{i \geq 0, j \geq 1, \atop i+j \leq n} m_{n-j+1} \circ (1 \otimes^i \otimes m_j \otimes 1 \otimes (n-i-j)) = 0$$
$A_\infty$-algebras

An $A_\infty$-algebra is a graded vector space $A$ together with linear maps $m_n : A[1] \otimes^n \rightarrow A[1]$ of degree +1 for all $n \geq 1$ such that

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$n = 1$ : $m_1 \circ m_1 = 0$

$n = 2$ : $m_1 \circ m_2 + m_2 \circ (m_1 \otimes 1) + m_2 \circ (1 \otimes m_1) = 0$

$n = 3$ : $m_2 \circ (m_2 \otimes 1) + m_2 \circ (1 \otimes m_2)$$
  + m_1 \circ m_3 + m_3 \circ (m_1 \otimes 1^{\otimes 2} + 1 \otimes m_1 \otimes 1 + 1^{\otimes 2} \otimes m_1) = 0$

$n = 4$ : $\ldots$
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$(A, m_n)$ is minimal iff $m_1 = 0$, and cyclic with respect to $\langle \cdot, \cdot \rangle$ iff

$$\langle \psi_{i_0}, m_n(\psi_{i_1} \otimes \ldots \otimes \psi_{i_n}) \rangle = \pm \langle \psi_{i_1}, m_n(\psi_{i_2} \otimes \ldots \otimes \psi_{i_n} \otimes \psi_{i_0}) \rangle$$
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An $A_\infty$-Algebra is Calabi-Yau if it is minimal and cyclic with respect to a non-degenerate pairing.
Relation to open topological string theory

Underlying TFT data (*Frobenius algebra*):

\[ H : \text{space of states} = \text{BRST cohomology with basis } \{\psi_i\} \]

\[ \langle \psi_{i_0} \ldots \psi_{i_n} \rangle : \text{correlators computed from OPE and topological metric} \]
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To get from TFT to topological string theory, need *Calabi-Yau A\(_\infty\)-algebra* \((H, m_n)\):

\[ W_{i_0 \ldots i_n} = \langle \psi_{i_0} \psi_{i_1} \psi_{i_2} \int \psi_{i_3}^{(1)} \ldots \int \psi_{i_n}^{(1)} \rangle \]
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\]

How to compute the products \( m_n \)?
\( A_\infty \)-algebras

**Minimal model theorem.** For any \( A_\infty \)-algebra \((A, \partial)\), its cohomology \( H = H_{m_1}(A) \) has a minimal \( A_\infty \)-structure \( \tilde{\partial} \)
Minimal model theorem. For any $A_\infty$-algebra $(A, \partial)$, its cohomology $H = H_{m_1}(A)$ has a minimal $A_\infty$-structure $\tilde{\partial}$ and an $A_\infty$-quasi-isomorphism

$$F : (H, \tilde{\partial}) \longrightarrow (A, \partial)$$

unique up to $A_\infty$-isomorphism.
**$A_\infty$-algebras**

**Minimal model theorem.** For any $A_\infty$-algebra $(A, \partial)$, its cohomology $H = H_{m_1}(A)$ has a minimal $A_\infty$-structure $\tilde{\partial}$ and an $A_\infty$-quasi-isomorphism $F : (H, \tilde{\partial}) \longrightarrow (A, \partial)$ unique up to $A_\infty$-isomorphism.

**Sketch of proof:** Compute Feynman diagrams.

\[ \tilde{m}_n(\psi_{i_1} \otimes \ldots \otimes \psi_{i_n}) = \sum_{\text{trees with } n \text{ leaves}} (-1)^{\#G} \]
General picture
General picture

topological string field theory
General picture

topological string field theory = off-shell space $A$
topological string field theory = off-shell space $A$ with DG structure $\partial$
General picture

topological string field theory = *off-shell* space $A$ with DG structure $\partial$

topological string theory
General picture

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General picture

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topological string theory = on-shell space $H$ with higher $A_\infty$-products $\tilde{\partial}$
General picture

topological string field theory = \textit{off-shell} space $A$ with DG structure $\partial$

minimal model

topological string theory = \textit{on-shell} space $H$ with higher $A_\infty$-products $\tilde{\partial}$
General picture

topological string field theory = \textit{off-shell} space $A$ with DG structure $\partial$

minimal model

topological string theory \quad = \textit{on-shell} space $H$ with higher $A_\infty$-products $\tilde{\partial}$

(Same for bulk sector if we replace “DG” by “DG Lie” and “$A_\infty$” by “$L_\infty$”.)
B-twisted affine Landau-Ginzburg models

\[ Z = \int D\Phi \ e^{-\int K(\Phi, \bar{\Phi}) - (\frac{i}{2} \int W(\Phi) + \text{c.c.})} \ \text{str} \left( \mathcal{P} e^{-\oint \left( \frac{1}{2} \rho^i \nabla_i D + \frac{i}{2} \eta^i \nabla_i D^\dagger + \frac{i}{2} \{D, D^\dagger\} \right)} \right) \]
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**Bulk sector. Jacobi algebra** \( \text{Jac}(W) = \mathbb{C}[x_1, \ldots, x_N]/(\partial_i W) \)
B-twisted affine Landau-Ginzburg models

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\]

**Bulk sector.** Jacobi algebra \( \text{Jac}(W) = \mathbb{C}[x_1, \ldots, x_N] / (\partial_i W) \)

\[= \text{BRST cohomology of } (\Gamma(\mathbb{C}^N, \bigwedge T^{(1,0)}(\mathbb{C}^N), [-W, \cdot]_{\text{SN}}) \]
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**Bulk sector.** *Jacobi algebra* \( \text{Jac}(W) = \mathbb{C}[x_1, \ldots, x_N]/(\partial_i W) \)

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topological metric:

\[ \langle \phi_1 \phi_2 \rangle = \text{Res} \left[ \phi_1 \phi_2 \frac{dx_1 \wedge \ldots \wedge dx_N}{\partial_1 W \ldots \partial_N W} \right] \]

Vafa 1990
B-twisted affine Landau-Ginzburg models

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Boundary sector.
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**Boundary sector.** **Matrix factorizations** \( D \in \text{Mat}_{2r \times 2r}(\mathbb{C}[x]) \) with \( D^2 = W \cdot \mathbb{1}_{2r \times 2r} \)

B-twisted affine Landau-Ginzburg models

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off-shell open string space: \( A = \text{Mat}_{2r \times 2r}(\mathbb{C}[x]) \) with BRST differential \([D, \cdot]\)
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topological metric (**Kapustin-Li pairing**):

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topological metric (**Kapustin-Li pairing**):

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**Defect sector.** \( \ldots \Rightarrow \text{extended 2d TFT} \)
Open top. string theory for Landau-Ginzburg models

Apply minimal model theorem to off-shell algebra

\[
\left( A = \text{Mat}_{2r \times 2r}(\mathbb{C}\{x\}), [D, \cdot], \text{matrix multiplication} \right)
\]
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Apply minimal model theorem to off-shell algebra

\[ A = \text{Mat}_{2r \times 2r} (\mathbb{C}[x]), \ [D, \cdot], \text{matrix multiplication} \]

to obtain higher \( A_\infty \)-products on on-shell space \( H = H_{[D,\cdot]}(A) \).
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**Complication.** Generically the \(A_\infty\)-products will not be cyclic with respect to the Kapustin-Li pairing.
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**Complication.** Generically the \( A_\infty \)-products will *not* be cyclic with respect to the Kapustin-Li pairing.

**Solution.** Reformulate theory in terms of formal *non-commutative geometry.*
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Complication. Generically the \( A_\infty \)-products will \textit{not} be cyclic with respect to the Kapustin-Li pairing.

Solution. Reformulate theory in terms of formal \textit{non-commutative geometry}.

Result. General algorithm to construct all open tree-level amplitudes.
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**Complication.** Generically the \(A_\infty\)-products will not be cyclic with respect to the Kapustin-Li pairing.

**Solution.** Reformulate theory in terms of formal non-commutative geometry.

**Result.** General algorithm to construct all open tree-level amplitudes, and (another) first-principle derivation of Kapustin-Li pairing.
Bulk deformations

\[ W_{i_1 \ldots i_n} \]
Bulk deformations

\[ W_{i_1 \ldots i_n} \mapsto W_{i_1 \ldots i_n}(t) = \left\langle \psi_{i_1} \psi_{i_2} \psi_{i_3} \int \psi_{i_4}^{(1)} \ldots \int \psi_{i_n}^{(1)} e^{\sum_i t_i \int \phi_i^{(2)}} \right\rangle \]
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**Fact.** Bulk-deformed amplitudes are described by *curved* $A_\infty$-products $m_0(t), m_1(t), m_2(t), \ldots$
Bulk deformations

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**Fact.** Bulk-deformed amplitudes are described by *curved* $A_\infty$-products $m_0(t)$, $m_1(t)$, $m_2(t)$, \ldots

**Fact.** Curvature screws everything up:

- $m_1$ no longer a differential
- cannot apply minimal model theorem
- need a new approach
Deformations and Maurer-Cartan equations

Given an $A_\infty$-algebra $(A, \partial)$, a deformation is $\delta \in \text{End}^1(T_A)$ such that $(A, \partial + \delta)$ is a curved $A_\infty$-algebra.
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\[ \iff \delta \in \text{Coder}^1(T_A), \quad [\partial, \delta] + \frac{1}{2} [\delta, \delta] = 0 \]
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This is the **Maurer-Cartan equation** for the DG Lie algebra

\[ \left( \text{Coder}(T_A), [\partial, \cdot], [\cdot, \cdot] \right) \]
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**Fact.** Let $L$ be an $L_\infty$-quasi-isomorphism between DG Lie algebras. Then

$$\delta \mapsto \sum_{n \geq 1} \frac{1}{n!} L_n(\delta^{\wedge n})$$

is an *isomorphism* between the spaces of Maurer-Cartan solutions modulo gauge transformations.
Back to Landau-Ginzburg models

Want to find bulk-induced deformations of open string algebra \((H, \tilde{\partial})\)
Back to Landau-Ginzburg models

Want to find bulk-induced deformations of open string algebra \((H, \tilde{\partial})\),
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Off-shell bulk sector is also a DG Lie algebra:

\[
\left( T_{\text{poly}} = \Gamma(\mathbb{C}^N, \bigwedge T^{(1,0)}\mathbb{C}^N), [-W, \cdot]_{\text{SN}}, [\cdot, \cdot]_{\text{SN}} \right)
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\]

The solutions to its Maurer-Cartan equation are the \textbf{on-shell bulk fields}:

\[
\delta|_C : 1 \mapsto \sum_i t_i \phi_i , \quad \phi_i \in \text{Jac}(W) = H[-W, \cdot]_{\text{SN}}(T_{\text{poly}})
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Back to Landau-Ginzburg models

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Transport them to deformations of \((H, \tilde{\partial})\) via an \(L_\infty\)-map

\[
\left( T_{\text{poly}}, [-W, \cdot]_{\text{SN}}, [\cdot, \cdot]_{\text{SN}} \right) \rightarrow \left( \text{Coder}(T_H), [\tilde{\partial}, \cdot], [\cdot, \cdot] \right)
\]
Back to Landau-Ginzburg models

\[
\left( T_{\text{poly}}, [-W, \cdot]_{\text{SN}}, [\cdot, \cdot]_{\text{SN}} \right) \longrightarrow \left( \text{Coder}(T_A), [\partial, \cdot], [\cdot, \cdot] \right)
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First step: deformations of the off-shell open string algebra
Back to Landau-Ginzburg models

\[
\left( T_\text{poly}, [-W, \cdot]_{\text{SN}}, [\cdot, \cdot]_{\text{SN}} \right) \longrightarrow \left( \text{Coder}(T_A), [\partial, \cdot], [\cdot, \cdot] \right) \\
\longrightarrow \left( \text{Coder}(T_H), [\tilde{\partial}, \cdot], [\cdot, \cdot] \right)
\]

First step: deformations of the off-shell open string algebra

**Theorem.** There is a sequence of explicit $L_\infty$-quasi-isomorphisms

\[
\left( T_\text{poly}, [-W, \cdot]_{\text{SN}}, [\cdot, \cdot]_{\text{SN}} \right) \xrightarrow{\text{"weak" deform. quant.}} \left( \text{Coder}(T_{C[x]}), \left[\hat{\partial}_0 + \hat{\partial}_2, \cdot\right], [\cdot, \cdot] \right) \\
\xrightarrow{\text{Morita equivalence}} \left( \text{Coder}(T_A), \left[\partial_0 + \partial_2, \cdot\right], [\cdot, \cdot] \right) \\
\xrightarrow{\text{tadpole cancellation}} \left( \text{Coder}(T_A), \left[\partial_1 + \partial_2, \cdot\right], [\cdot, \cdot] \right)
\]
Digression: deformation quantisation à la Kontsevich

Consider classical theory with phase space $M = \mathbb{R}^d$ and associative, commutative algebra of observables $(C^\infty(M, \mathbb{R}), \cdot) \equiv (C^\infty(M, \mathbb{R}), \hat{\partial}_2)$. 
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**Deformation quantisation** constructs algebra of quantum observables $(C^\infty(M, \mathbb{R})[\hbar], \star)$ by deforming the product to

$$f \star g = f \cdot g + B_1(f, g)\hbar + B_2(f, g)\hbar^2 + \ldots$$
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This is the same as solving the Maurer-Cartan equation of

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Kontsevich constructs an explicit $L_\infty$-quasi-isomorphism

$$K : \left( \Gamma(M, \bigwedge TM), 0, [\cdot, \cdot]_{\text{SN}} \right) \longrightarrow \left( \text{Coder}(T_{C^\infty(M, \mathbb{R})}), [\hat{\partial}_2, \cdot], [\cdot, \cdot] \right)$$
Digression: deformation quantisation à la Kontsevich

\[
\left( K_n(\gamma_1 \wedge \ldots \wedge \gamma_n) \right)_m (f_1 \otimes \ldots \otimes f_m) \\
= \sum_{\Gamma \in G(n,m)} \frac{1}{(2\pi)^{2n+m-2}} \int_{\mathcal{C}^{n,m}} \left( d\varphi_{e_k^1} \wedge \ldots \wedge d\varphi_{e_k^{\tilde{\gamma}_k}} \right) \\
\cdot \sum_{I} \left[ \prod_{i=1}^{n} \left( \prod_{e \in \Gamma \rightarrow i} \partial_{I(e)} \right) \gamma_{I(e_i^1)} \ldots \gamma_{I(e_i^{\tilde{\gamma}_i})} \right] \left[ \prod_{\bar{j}=1}^{m} \left( \prod_{e \in \Gamma \rightarrow \bar{j}} \partial_{I(e)} \right) f_{\bar{j}} \right]
\]
Digression: deformation quantisation à la Kontsevich

Kontsevich 1997
Weak deformation quantisation

**Theorem.** Kontsevich’s map

\[ K : \left( T_{\text{poly}}, 0, [\cdot, \cdot]_{\text{SN}} \right) \rightarrow \left( \text{Coder}(T_{\mathbb{C}[x]}), [\hat{\partial}_2, \cdot], [\cdot, \cdot] \right) \]
Theorem. Kontsevich’s map

\[ K : \left( T_{\text{poly}}, 0, [\cdot , \cdot ]_{\text{SN}} \right) \rightarrow \left( \text{Coder}(T_{\mathbb{C}[x]}), [\hat{\partial}_2, \cdot ], [\cdot , \cdot ] \right) \]
is also an \( L_\infty \)-quasi-isomorphism

\[ \left( T_{\text{poly}}, [\cdot , W \cdot ]_{\text{SN}}, [\cdot , \cdot ]_{\text{SN}} \right) \rightarrow \left( \text{Coder}(T_{\mathbb{C}[x]}), [\hat{\partial}_0 + \hat{\partial}_2, \cdot ], [\cdot , \cdot ] \right) \]
Weak deformation quantisation

**Theorem.** Kontsevich’s map

\[ K : (T_{\text{poly}}, 0, [\cdot, \cdot]_\text{SN}) \longrightarrow (\text{Coder}(T_{\mathbb{C}[[x]]}), [\hat{\partial}_2, \cdot], [\cdot, \cdot]) \]

is also an \( L_\infty \)-quasi-isomorphism

\[ (T_{\text{poly}}, [-W, \cdot]_\text{SN}, [\cdot, \cdot]_\text{SN}) \longrightarrow (\text{Coder}(T_{\mathbb{C}[[x]]}), [\hat{\partial}_0 + \hat{\partial}_2, \cdot], [\cdot, \cdot]) \]

Thus we have constructed the first part of our bulk deformation map

\[ (T_{\text{poly}}, [-W, \cdot]_\text{SN}, [\cdot, \cdot]_\text{SN}) \longrightarrow (\text{Coder}(T_A), [\partial, \cdot], [\cdot, \cdot]) \]

\[ \longrightarrow (\text{Coder}(T_H), [\tilde{\partial}, \cdot], [\cdot, \cdot]) \]
Theorem. Let \((A, \partial)\) be an \(A_\infty\)-algebra and \((H, \tilde{\partial})\) its minimal model.
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\[
\begin{array}{c}
(T_H, \tilde{\partial}) \xleftarrow{F} (T_A, \partial) \xrightarrow{U} (T_H, \tilde{\partial})
\end{array}
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Theorem. Let \((A, \partial)\) be an \(A_\infty\)-algebra and \((H, \tilde{\partial})\) its minimal model. Then we have an explicit \textit{deformation retraction}

\[
\begin{array}{c}
(T_H, \tilde{\partial}) \quad \xrightarrow{F} \quad (T_A, \partial)
\end{array}
\]

in standard form, i.e.

\[
\begin{align*}
\bar{F}F &= 1_{T_H}, \\
1_{T_A} - F\bar{F} &= \partial U + U\partial, \\
U^2 &= UF = \bar{FU} = 0
\end{align*}
\]
Homological perturbation

**Theorem.** Let \((A, \partial)\) be an \(A_\infty\)-algebra and \((H, \tilde{\partial})\) its minimal model. Then we have an explicit deformation retraction

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(T_H, \tilde{\partial}) \xleftrightarrow{F \quad \bar{F}} (T_A, \partial)
\]

in standard form, i.e.

\[
\bar{F}F = 1_{T_H}, \quad 1_{T_A} - F\bar{F} = \partial U + U\partial, \quad U^2 = UF = \bar{F}U = 0
\]

This gives rise to an \(L_\infty\)-morphism

\[
\left( \text{Coder}(T_A), [\partial, \cdot], [\cdot, \cdot] \right) \longrightarrow \left( \text{Coder}(T_H), [\tilde{\partial}, \cdot], [\cdot, \cdot] \right)
\]

\[
\delta \mapsto \sum_{n \geq 1} \bar{F}(\delta U)^n \delta F
\]
Recursive formulas for $\bar{F}, U$

$$U_n^1 = -\frac{1}{2} G \partial_2^1 \left( \sum_{l=1}^{n-1} (U_l^1 \otimes (\mathbb{1} + F\bar{F})_{n-l}^1 + (\mathbb{1} + F\bar{F})_{n-l}^1 \otimes U_l^1) \right)$$

$$\bar{F}_n^1 = -\frac{1}{2} \pi_H \partial_2^1 \left( \sum_{l=1}^{n-1} (U_l^1 \otimes (\mathbb{1} + F\bar{F})_{n-l}^1 + (\mathbb{1} + F\bar{F})_{n-l}^1 \otimes U_l^1) \right)$$
Bulk-deformed amplitudes for Landau-Ginzburg models

\[
\left( \text{Coder}(T_A), [\partial, \cdot], [\cdot, \cdot] \right) \longrightarrow \left( \text{Coder}(T_H), \tilde{\partial}, \cdot], [\cdot, \cdot] \right)
\]

\[
\delta \longmapsto \sum_{n \geq 1} \tilde{F}(\delta U)^n \delta F
\]
Bulk-deformed amplitudes for Landau-Ginzburg models

\[
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\[
\delta \longmapsto \sum_{n \geq 1} F(\delta U)^n \delta F
\]

This is true in particular for Landau-Ginzburg models.
Bulk-deformed amplitudes for Landau-Ginzburg models

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All off-shell deformations \( \delta \) are bulk-induced, i.e. uniquely defined by

\[
\delta \big|_C : 1 \longrightarrow \sum_i t_i \phi_i \cdot 1, \quad \phi_i \in \text{Jac}(W)
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All off-shell deformations \( \delta \) are bulk-induced, i.e. uniquely defined by

\[
\delta|_C : 1 \longrightarrow \sum_i t_i \phi_i \cdot 1, \quad \phi_i \in \text{Jac}(W)
\]

Thus the curved \( A_\infty \)-products describing bulk-deformed open topological string amplitudes are explicitly encoded in

\[
\tilde{\partial} + \sum_{n \geq 1} \bar{F}(\delta U)^n \delta F
\]
open topological string theory $\iff$ Calabi-Yau $A_\infty$-algebra
Conclusion

- open topological string theory $\iff$ Calabi-Yau $A_\infty$-algebra
- gives explicit algorithm to compute open amplitudes
open topological string theory $\iff$ Calabi-Yau $A_\infty$-algebra

gives explicit algorithm to compute open amplitudes

bulk-deformed amplitudes
open topological string theory $\iff$ Calabi-Yau $A_\infty$-algebra

gives explicit algorithm to compute open amplitudes

bulk-deformed amplitudes computable via weak deformation quantisation and homological perturbation:

$$\langle \psi_{i_0}, \tilde{m}_n^t(\psi_{i_1} \otimes \cdots \otimes \psi_{i_n}) \rangle$$

$$= \langle \psi_{i_0}, \psi_{i_1} \psi_{i_2} \int \psi_{i_3}^{(1)} \cdots \int \psi_{i_n}^{(1)} e^{\sum t_i \int \phi_i^{(2)}} \rangle$$