Monopole operators, moduli spaces and dualities in 3d CS matter theories

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2011

D. Berenstein and M. R.(to appear in ATMP), arXiv:0909.2856 [hep-th]
M. R. JHEP **1109**, 122, arXiv:1011.4733 [hep-th]
D. Berenstein and M. R., arXiv:1108.4013 [hep-th]

$\mathbf{AdS_4/CFT_3}$

The Setup

- M2-brane worldvolume theory

$$AdS_4 \times X_7 \leftrightarrow 3d \;\; \text{SCFT}.$$

- The cone CX_7 over X_7 is contained in $\mathcal{M}_{vac}$, the moduli space of vacua of the corresponding SCFT.

- Our purpose is to get X_7 computing $\mathcal{M}_{vac}$.

- If $X_7 = S^7/\mathbb{Z}_k$. The theory corresponds to M2 branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity (ABJM theory).

- Type IIA string theory on $AdS_4 \times X_6$, plus flux.

- $\mathcal{M}_{vac} \sim \mathbb{C}^*$ fiber over a CY_3.

Outline

- BPS states on CS matter theories
- Examples
- Seiberg-like duality?
- Monopole operators ($\mathcal{N} = 3$ case)
- Conclusions

$\mathcal{M}_{\mathbf{vac}}$

- $\mathcal{M}_{vac}$ is characterized by the VEVs $\langle \mathcal{O}^I \rangle$ of the different scalar operators (order parameters). The set of numbers $\{\langle \mathcal{O}^I \rangle\}$ labels a vacuum.

- There can exist relations between them

$$\sum_{\{I_i\}} a_{I_1 \cdots I_n} \langle \mathcal{O}^{I_1} \rangle \cdots \langle \mathcal{O}^{I_n} \rangle = 0$$

- $\mathcal{M}_{vac}$ correspond to the variety parameterized by these VEVs modulo relations.

$\mathcal{M}_{\mathbf{vac}}$ and the chiral ring in 4d SCFTs

The operators that compose the coordinate ring of $\mathcal{M}_{4d}$ are elements of the chiral ring. These are holomorphic operators.

Their VEVs are classified by the solution of the F-term and D-term equations. For a theory with bifundamental matter fields ϕ^a

$$\frac{\partial W}{\partial \phi^I} = 0 \qquad W = Tr(\sum_l a_{[l]} \phi^{[l]}).$$

$$\sum_{t(\phi)=i} \phi \phi^\dagger - \sum_{h(\phi)=i} \phi^\dagger \phi = \zeta_{FI}$$

We will set $\zeta_{FI} = 0$.

$\mathcal{M}_{\mathbf{4d}}$ and quiver representations

F-terms can be re-written as a path algebra by associating a nilpotent operator $P^{(a)}$ to each vertex.

$$\begin{aligned}\mathcal{A} &= \langle \phi^I, P^{(a)} \rangle / \{\partial W = 0\} \\ &= \mathbb{C}Q / \{\partial W = 0\}\end{aligned}$$

Representations are labeled by their dimension vector $\vec{d} \in \mathbb{N}^{Q_0}$ and the values of the linear maps ϕ^I.
In the case of D3-branes (4d SCFTs) D-terms (with $\zeta_{FI} = 0$) will give us a moment map, which will be equivalent to consider $GL(N, \mathbb{C})$-classes of $\mathcal{A}$-modules and we have the correspondences

$$\mathcal{Z}\mathcal{A} \leftrightarrow \text{singularity} \qquad R_{\vec{d}} \leftrightarrow \text{branes}$$

For M2-branes this is more complicated for many reasons. But we still want to do the identification

$$R_{\vec{d}} \leftrightarrow \text{branes}$$

Semiclassical methods will help us to identify the CY_4 singularity.

$\mathcal{M}_{\mathbf{3d}}$ with CS gauge fields

We will work on theories with at least $\mathcal{N} = 2$ SUSY in 3d $\Leftrightarrow$ $\mathcal{N} = 1$ in 4d. So, we can use the usual $\mathcal{N} = 1$ superspace formalism and holomorphy. The vector multiplet will look like

$$V = 2i\bar{\theta}\theta\sigma + 2\theta\sigma^{\mu}\bar{\theta}A_{\mu} + i\sqrt{2}\theta\theta\bar{\theta}\chi^{\dagger} - i\sqrt{2}\bar{\theta}\bar{\theta}\theta\chi + \theta\theta\bar{\theta}\bar{\theta}D.$$

and the supersymmetric CS action

$$S_{CS}(A) = \frac{k}{4\pi}\int \mathrm{Tr}\left(AdA + \frac{2}{3}A^3 - \overline{\chi}\chi + 2D\sigma\right)$$

canonical kinetic terms will have couplings of the form

$$\int \phi^{\dagger}D\phi$$

integration of the auxiliary field D will give us the following vacuum equations...

$\mathcal{M}_{\mathbf{3d}}$ with CS gauge fields

$$\sigma_\alpha \phi_{\alpha\beta} - \phi_{\alpha\beta}\sigma_\beta = 0$$

$$\frac{\partial W}{\partial \phi_{\alpha\beta}} = 0$$

$$\sum_{t(\phi)=\alpha} \phi\phi^\dagger - \sum_{h(\phi)=\alpha} \phi^\dagger\phi = k_\alpha \sigma_\alpha$$

In addition

$$A_D \equiv \sum_{i=1}^{N_G} A_i \qquad S_{CS} = \int A' \wedge dA_D + \ldots$$

decouples from matter.

The chiral ring

$$P_\mu|0\rangle = Q|0\rangle = \overline{Q}|0\rangle = M_{\mu\nu}|0\rangle = 0,$$

the expectation value w.r.t. $|0\rangle$ of a general superfield $\mathcal{O}(\theta,\bar{\theta},x)$ satisfies

$$\partial_\mu\langle\mathcal{O}\rangle = \partial_\theta\langle\mathcal{O}\rangle = \partial_{\bar{\theta}}\langle\mathcal{O}\rangle = 0,$$

Moreover

$$\mathcal{O}(\theta,\bar{\theta},x) = \{\overline{D},\mathcal{G}(\theta,\bar{\theta},x)\} \Rightarrow \langle\mathcal{O}\rangle = 0$$

Therefore we only have to worry about equivalence classes of chiral operators

$$\overline{D}\mathcal{O} = 0.$$

For $\mathcal{O}$ chiral we have the nice properties

$$\partial_{x_1}\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \partial_{x_2}\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = 0,$$

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \langle\mathcal{O}(x_1)\rangle\langle\mathcal{O}(x_2)\rangle,$$

Definition. The **chiral ring** is the subset of chiral operators ($\overline{D}\mathcal{O} = 0$)

$$\mathfrak{R} = \left\{\mathcal{O}|\overline{D}_\alpha\mathcal{O}(\theta,\bar{\theta},x) = 0\right\} \Big/ \left\{\mathcal{O} = \left\{\overline{D}, G(\theta,\bar{\theta},x)\right\}\right\},$$

The chiral ring

For SCFTs additional constrains can be imposed over the operators on $\mathfrak{R}$ due to the large amount of (super-)symmetry.

This boils down to consider operators whose lowest component ϕ is a superprimary in the chiral ring (i.e. its equivalence class can be represented by a superprimary). More importantly this casts ϕ as a BPS state satisfying $\Delta_\phi \sim R_\phi$, with Δ_ϕ the scaling dimension of ϕ and R_ϕ its R-charge. In particular, for $d = 3$

$$\Delta_\phi = R_\phi.$$

So, the moduli space of these theories can be written as

$$\mathcal{M} \cong \left\{ \langle\phi\rangle | \mathcal{O} = \phi + \bar{\theta}\psi + \ldots, \mathcal{O} \in \mathfrak{R} \right\},$$

$\mathcal{M}_{\mathbf{vac}}$ in SCFTs

When working in the cylinder $\mathbb{R} \times S^d$, then we can identify Δ with the Hamiltonian of the system. BPS equations can be solved classically.

The classical Hamiltonian and R-charge in terms of the momenta are given by

$$
\begin{aligned}
H &= \int_{S^2} \left((K_{,a\bar{a}})^{-1} \Pi_{\phi_a} \Pi_{\bar{\phi}_{\bar{a}}} + K_{,a\bar{a}} \nabla \phi^a \nabla \bar{\phi}^{\bar{a}} + \frac{1}{4} K + V_D + V_F \right) \\
Q_R &= i \int_{S^2} \left(\Pi_{\phi_a} \gamma_a \phi^a - \Pi_{\bar{\phi}^{\bar{a}}} \gamma_{\bar{a}} \bar{\phi}^{\bar{a}} \right)
\end{aligned}
$$

The classical BPS eqs. $H - Q_R = 0$ reduce to a sum of squares that have to vanish separatedly

$$
\begin{aligned}
\dot{\phi}^a &= i \gamma_a \phi^a, \\
\nabla \phi^a &= 0 \\
V_D &= V_F = 0
\end{aligned}
$$

$\mathcal{M}_{\mathbf{vac}}$ in SCFTs

Additionally we have the constraint coming from the A_0 e.o.m

$$-\frac{k_i F^{(i)}}{\pi} = \int_{S^2} -i \sum_{t(a)=i} \Pi_{\phi_a} \phi^a + i \sum_{h(a)=i} \Pi_{\phi_a} \phi^a + i \sum_{t(\bar{a})=i} \Pi_{\bar{\phi}_{\bar{a}}} \bar{\phi}^{\bar{a}} - i \sum_{h(\bar{a})=i} \Pi_{\bar{\phi}_{\bar{a}}} \bar{\phi}^{\bar{a}}$$

The pullback of ω, the symplectic form of the ϕ^a phase space, to the manifold of BPS solutions can be written as

$$\omega = i K_{,a\bar{a}} d\phi_a \wedge d\bar{\phi}_{\bar{a}} = -2 d\phi_a \wedge d\Pi_{\phi_a}$$

this shows that we can holomorphically quantize the ϕ^a's. Wave functions will take the form

$$\prod_a \phi_a^{m_a}$$

the A_0 equations can be written as a constraint on the exponents (for the $U(1)^l$ case)

$$-\frac{k_i F^{(i)}}{\pi} = -i \sum_{t(a)=i} m_a + i \sum_{h(a)=i} m_a$$

$\mathcal{M}_{\mathbf{vac}}$ in SCFTs

Summarizing

$$\frac{\partial W}{\partial \phi_a} = 0$$

$$k_i F^{(i)} \psi = \left(\sum_{t(a)=i} \phi^a \partial_{\phi^a} - \sum_{h(a)=i} \phi^a \partial_{\phi^a} \right) \psi$$

$F^{(i)} \in \mathbb{Z}$

$$\psi = \prod_a \phi_a^{m_a}$$

Example 1: ABJM

Stack of N M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity ($\mathcal{N}=6$). $G=U(N)\times U(\overline{N})$

O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, JHEP **0810**, (2008)

$$
\begin{aligned}
S_{ABJM} \quad = \quad & \int d^3x\Big[2K\varepsilon^{\mu\nu\lambda}Tr\Big(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \widehat{A}_\mu\partial_\nu\widehat{A}_\lambda - \frac{2i}{3}\widehat{A}_\mu\widehat{A}_\nu\widehat{A}_\lambda\Big) \\
- \quad & 4KD\sigma + 4K\widehat{D}\widehat{\sigma}\Big] \\
+ \quad & \int d^3xd^4\theta Tr\Big[-\bar{\mathcal{Z}}e^{-V}\mathcal{Z}e^{\widehat{V}} - \bar{\mathcal{W}}e^{-\widehat{V}}\mathcal{W}e^{V}\Big] \\
+ \quad & \frac{1}{4K}\int d^3xd^2\theta Tr\Big[\varepsilon_{AC}\varepsilon^{BD}\mathcal{Z}^A\mathcal{W}_B\mathcal{Z}^C\mathcal{W}_D\Big] \\
+ \quad & \frac{1}{4K}\int d^3xd^2\theta Tr\Big[\varepsilon^{AC}\varepsilon_{BD}\bar{\mathcal{Z}}_A\bar{\bar{\mathcal{W}}}^B\bar{\mathcal{Z}}_C\bar{\mathcal{W}}^D\Big]
\end{aligned}
$$

Field	$U(N)$	$U(\overline{N})$
$\mathcal{Z}$, $\bar{\mathcal{W}}$	$\square$	$\overline{\square}$
$\mathcal{W}$, $\bar{\mathcal{Z}}$	$\overline{\square}$	$\square$

Example 1: ABJM

Point-like brane

$$G = U(1)_k \times U(1)_{-k}$$

$$W_c = Tr\left(Z_1 W^1 Z_2 W^2 - Z_1 W^2 Z_2 W^1\right),$$

Classical Moduli equations

$$\mathcal{A}_c = \langle Z_i, W^j, P_a \rangle / \{dW_c = 0\}$$

$$Z_A = \begin{bmatrix} 0 & z_A \\ 0 & 0 \end{bmatrix} \qquad W^A = \begin{bmatrix} 0 & 0 \\ w^A & 0 \end{bmatrix}$$

Wave functions

$$(z_1)^{i_1}(z_2)^{i_2}(w^1)^{j_1}(w^2)^{j_2} \qquad i_1 + i_2 - j_1 - j_2 \in k\mathbb{Z}$$

The variables (z, w) describe the coordinate ring of $\mathbb{C}^4/\mathbb{Z}_k$. This is the moduli space of one M2-brane in the *bulk*.

Example 2: Non-toric quiver

$$W \sim Tr(ABC)$$

$$F \begin{pmatrix} k_A & 0 & 0 & 0 \\ 0 & k_B & 0 & 0 \\ 0 & 0 & k_C & 0 \\ 0 & 0 & 0 & k_C \end{pmatrix} \psi = \begin{pmatrix} B\partial_B - C\partial_C & 0 & 0 & 0 \\ 0 & C\partial_C - A\partial_A & 0 & 0 \\ 0 & 0 & A^1\partial_{A_1} - B_1\partial_{B_1} & A^1\partial_{A_2} - B_2\partial_{B_1} \\ 0 & 0 & A^2\partial_{A_1} - B_1\partial_{B_2} & A^2\partial_{A_2} - B_2\partial_{B_2} \end{pmatrix} \psi$$

$$F \in \mathbb{Z}$$

$$\tilde{M} = B^{2k_C} C^{2k_C + k_A} \qquad M = B^{2k_C} C^{2k_C + k_A}$$

$$M\tilde{M} \sim (ACB)^{2k_C}$$

Seiberg-like duality?

Studied for the case of a single gauge group

$$U(N_c)_k \to U(N_f - N_c + |k|)_{-k}$$

What about quivers?

$$\begin{aligned} [B] &\to [\overline{B}] \\ [B_i] &\to [\widehat{B_i}] \equiv [B_i] + n_i[B] \end{aligned}$$

If $([B], [B_i])$ have ranks (N, N_i), then

$$N \to \sum_i N_i n_i - N.$$

so

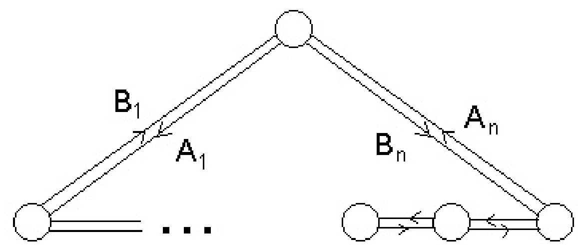
$$\begin{aligned} k &\to -k \\ k_i &\to k_i + n_i k. \end{aligned}$$

equivalently we can have

$$\begin{aligned} k &\to -k \\ k_i &\to k_i + \tilde{n}_i k. \end{aligned}$$

Seiberg-like duality?

Consider a $\mathcal{N} = 3$ theory with superpotential and field content given by



$$W = Tr\left[\sum_{i=1}^{n} \varphi_i (B_i A_i - A_{i-1} B_{i-1}) - \frac{1}{2}\sum_{i=1}^{n} k_i \varphi_i^2\right]$$

$$uv = \prod_i (w - \varphi r_i)$$

$$r_i = \sum_{j=1}^{n-i} k_j + \sum_{j=1}^{n} \frac{j}{n} k_j$$

Seiberg-like duality?

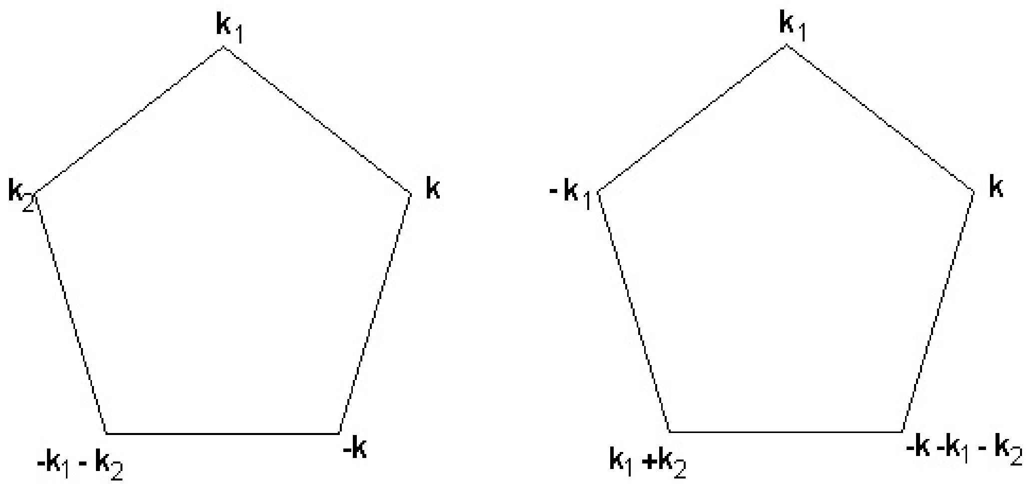
Fractional brane branches are allowed iff $r_i = r_l$ for $i \neq l$, this is equivalent to

$$\sum_{j=i}^{l} k_j = 0$$

i.e. some consecutive subset of the k_i's add up to zero.
This also is equivalent to have a singularity at a point distinct than $u = v = w = \varphi = 0$.
Therefore fractional branes have dimension vector

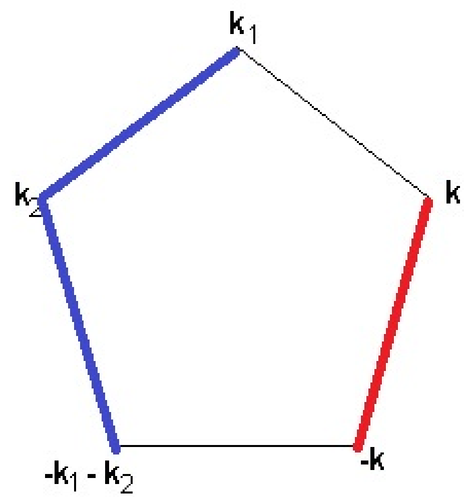
$$d_j = 1 \text{ if } \ j \in [i, l] \qquad d_j = 0 \text{ otherwise}$$

Seiberg-like duality?



In both cases the bulk moduli space is the same but fractional brane branches describe different singularities

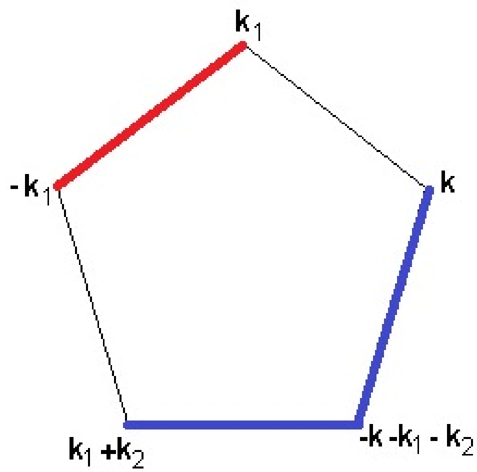
Seiberg-like duality?



$$M = a_1^{k_1} a_2^{k_2+k_1} \qquad \widetilde{M} = b_1^{k_1} b_2^{k_2+k_1} \Rightarrow M\widetilde{M} \sim z^{2k_1+k_2}$$

$$M' = a^k \qquad \widetilde{M}' = b^k \Rightarrow M'\widetilde{M}' \sim z^k$$

Seiberg-like duality?



$$M = a_1^k b_2^{k_2+k_1} \qquad \widetilde{M} = b_1^k a_2^{k_2+k_1} \Rightarrow M\widetilde{M} \sim z^{k+k_1+k_2}$$

$$M' = a^{k_1} \qquad \widetilde{M}' = b^{k_1} \Rightarrow M'\widetilde{M}' \sim z^{k_1}$$

Monopole operators

In CS theories with matter, monopole operators (operators with vortex charge) can be BPS. Bare monopoles are not gauge invariant, but they can be paired with matter fields to form an operator in $\mathfrak{R}$

$$\mathcal{O}_H(X) = T_H \prod_i X_i^{m_i}$$

Bare monopoles T_H are characterized by an element of $\mathfrak{h} \subseteq \mathfrak{g}$

$$H = \sum_a n_a h_a \qquad\qquad n_a \in \mathbb{Z}$$

classically

$$F \sim *d\left(\frac{1}{|x|}\right) H$$

Monopole operators

In the particular case of quiver gauge theories $G = \prod_i U(N_i)$ and so

$$H_i = (n_{i,1}, \ldots, n_{i,N_i}) \qquad i = 1, \ldots N_G$$

The spectrum of fermions changes in the presence of a monopole. Therefore, bare monopoles can have non-trivial global charges. In particular the R-charge (without fundamental matter)

$$R[T_H] = -\frac{1}{2}\sum_{X_{ij}}(R[X_{ij}]-1)\sum_{k=1}^{N_i}\sum_{l=1}^{N_j}|n_{i,k}-n_{j,l}|-\frac{1}{2}\sum_{i=1}^{N_G}\sum_{k=1}^{N_i}\sum_{l=1}^{N_i}|n_{i,k}-n_{i,l}|$$

We will focus on a particular class of monopole operators.

Monopole operators

Diagonal monopoles: (denote $N_i = N + M_i$ for all i)

$$n_{i,k} = n \text{ for all } i \quad \text{and for} \quad k = 1, \ldots, N$$

and

$$n_{i,k} = 0 \text{ for all } i \quad \text{and for} \quad k = N+1, \ldots, N_i$$

Then, since $R[\mathcal{O}_H(X)] = R[T_H] + R[X]$, we can show that for diagonal monopoles

$$R[T_H] \sim N_i \qquad R[X] \sim k$$

therefore semiclassical techniques computes the leading order relations of the chiral ring ($k \gg N$).

Monopole operators

Back to the $\mathcal{N}=3$ case (A_{n-1} quiver), suppose $\sum_{a=1}^{s} k_a = 0$ and $s < n-1$ (so, we don't have zero CS levels). Then (for simplicity $N=1$)

$$R[\mathcal{O}_H(X)] = 1 + \frac{1}{2}(N_{s+1} + N_n - N_s - N_1) + \frac{1}{2}(|k_1| + |k_1 + k_2| + \cdots + |-k_s|)$$

after SD duality at node 1, $\sum_{a=2}^{s} k'_a = 0$ (taking on account Hanany-Witten effect, so $N_1 \to N_2 + N_n - N_1 + |k_1|$)

$$R[\mathcal{O}'_H(X)] = 1 + \frac{1}{2}(N_{s+1} + N_n - N_s - N_1) + \frac{1}{2}|k_1| + \frac{1}{2}(|k_1 + k_2| + \cdots + |-k_s|)$$

Therefore the fractional brane branches coincide after the duality.

Future directions

- We presented a semiclassical method to compute moduli spaces of CS gauge theories with matter. We can apply it to theories arising from M2-branes probing non-toric singularities.

- The procedure differs from the D3-brane case. Non-perturbative BPS operators (monopoles) play a crucial role.

- Incorporate 1-loop corrections to global charges of monopole operators.

- How can we describe $\mathcal{M}_{3d}$ in terms of CY algebras?

- Is there a general rule for Seiberg-like dualities for 3d CS-matter theories?