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MMAs and the MMP (with Iyama)

$/\mathbb{C}$

Today: R will always denote a commutative Gorenstein 3-fold (e.g. $R = \mathbb{C}[u,v,x,y]/(f)$ or $R = \mathbb{C}[x,y,z]^{F}$)

Will later assume: R is normal, and R has only rational singularities

2 aspects of MMP Input $\operatorname{Spec} R$

① (Reid, Kawamata) Existence of minimal models (= $\mathbb{Q}$-factorial terminalizations)

$Y \xrightarrow{f} \operatorname{Spec} R$ where f is projective birational, $f^*\omega_R = \omega_Y$ and Y has only "mild" sings ($\mathbb{Q}$-factorial terminal)

Special case: crepant resolutions (but in general, Y need not be smooth)

② (many people: Mori, Kollár) running of MMP (contraction of curves, flops, ..)

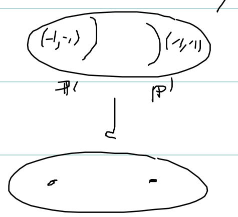
Main new idea

The existence of NC minimal models (MMAs) $\overset{\text{should}}{\Longrightarrow}$ existence of commutative ones (i.e. $\mathbb{Q}$-factorial terminalizations)

MMAs = maximal modification algebra

Example 1

$R = \mathbb{C}[u,v,x,y]/uv = xy(y-2)$



Spec R

There are 4 crepant resolutions.

$\operatorname{End}_R(R \oplus (u,x))$ is a noncommutative crepant resolution (NCCR)

(Can check at complete local rings ✓)

Quiver: $y \circlearrowright R \underset{y-2}{\overset{u}{\rightleftarrows}} (u,x) \circlearrowleft y$ and relations

Fix $\alpha=(1,1)$, There are only 2 stability conditions ($\Rightarrow$) (Craw-Ishii finite)

<u>Can't</u> get all NCCR's from this one.

<u>Why?</u> R is not Krull-Schmidt, so we shouldn't have taken a direct sum ---.

Let $M = R \oplus (u,x)$, $N = R \oplus (u,xy) \oplus (u, x(y-2))$.

Then $\operatorname{add} M = \operatorname{add} M$ (where $\operatorname{add} X$ = all direct summands of (finite) direct summands of X.

($\Rightarrow$ $\operatorname{End}_R(M) \underset{\text{Morita}}{\sim} \operatorname{End}_R(N)$: Morita-equiv, <u>not</u> derived Morita-equiv)

$\operatorname{End}_R(N)$ is also NCCR

$$\begin{tikzcd} & (u,xy) \arrow[dl, shift left] \\ R \arrow[ur, shift left] \arrow[r, shift left] & (u, x(y-2)) \arrow[l, shift left] \end{tikzcd}$$

/ relations. <u>Not</u> a symmetric algebra!

$\alpha=(1,1,1)$: check by changing θ to get all commutative crepant resolutions

$\Rightarrow$ forces us to use $\operatorname{add} M$ language (doesn't make a choice)

<u>Example 2</u>

$R = \mathbb{C}[u,v,x,y]/uv = x(x^2+y^3)$ <u>no</u> crepant resolution, b/c singularity.

sing pt $\to$ $\mathbb{P}^1$

$\downarrow$

Spec R

The MMA here is
$$\begin{tikzcd} \bullet \arrow[loop left, "y_1"] \arrow[r, shift left=3, "a"] \arrow[r, shift left=1, "b"'] & \bullet \arrow[loop right, "y_2"] \arrow[l, shift left=1, "s"'] \arrow[l, shift left=3, "t"] \end{tikzcd}$$

(infinite global dimension)

relations:

$asb = bsa$ $\quad$ $y_1 a = a y_2$ $\quad$ $at = (bs)^2 + y_1^3$

not needed { ~~$atb = bta$~~ $\quad$ $y_1 b = b y_2$ $\quad$ $ta = (sb)^2 + y_2^3$

~~$sat = tas$~~ $\quad$ $y_2 s = s y_1$

$sbt = tbs$ $\quad$ $y_2 t = t y_1$

(above is t/p)

It turns out that $\operatorname{End}_R(R \oplus (u,x)) \cong kQ/R$

Forces us to use $\operatorname{End}_R(M)$ language

(assume R normal) <u>Definition 1</u> Let $M \in \operatorname{ref} R$ ($\dim R = 3$, ref = reflexive modules)

M is called modifying if $\operatorname{End}_R(M) \in \operatorname{CM} R$

(i.e. is a Cohen-Macaulay module /R)

Definition 2 (Van den Bergh)

$\mathrm{End}_R(M)$ is called NCCR if (i) M is modifying, & (ii) $\mathrm{gl\,dim}\,\mathrm{End}_R(M) < \infty$.

Definition 3 $M \in \mathrm{refl}\,R$ is called maximal modifying if (i) M is modifying, and (ii) maximal with respect to this property, i.e.,

$$X \in \mathrm{refl}(R),\ \mathrm{End}_R(M \oplus X) \in \mathrm{CM}\,R \Rightarrow X \in \mathrm{add}\,M.$$

Equivalently, $\mathrm{add}\,M = \{X \in \mathrm{refl}\,R \mid \mathrm{End}_R(M \oplus X) \in \mathrm{CM}\,R\}$.

Note that this condition is almost impossible to check.

Facts

1) If $\exists$ NCCR, MMAs = NCCRs (geometrically: if one minimal model is smooth, they all are)

2) Fix R. Then all MMAs are derived equivalent (analogous to geometric resolutions)

3) If T is a tilting bundle on a $\mathbb{Q}$-factorial terminalization $Y \xrightarrow{f} \mathrm{Spec}\,R$, then

$\mathrm{End}_Y(T) \cong \mathrm{End}_R((f_*T)^{**})$ is a MMA.

Theorem

Suppose have $X \xrightarrow{f} \mathrm{Spec}\,R$ with X Gorenstein, isolated sings $\{x_1, \dots, x_n\}$.

and suppose $D^b(X) \simeq D^b(\mathrm{End}_R(M))$. Assume f is projective birational.

(1) If $\mathrm{End}_R(M)$ is MMA then X is locally factorial.

$\Updownarrow$ $\nearrow$

see Iyer's talk yesterday $\rightarrow$ $D_{sg}(\mathrm{End}_R M)$ has no rigid objects

(2) $\overline{D_{sg}(\mathrm{End}_R(M))}$ has no rigid objects $\Rightarrow$ X is complete locally factorial.

— means "idempotent completion"

If in addition each $\mathcal{O}_{X,x}$ is a hypersurface, then

(3) $\mathrm{End}_R(M)$ is MMA $\Leftrightarrow$ $D_{sg}(\mathrm{End}_R(M))$ no rigids $\Leftrightarrow$ X is $\mathbb{Q}$-factorial

(4) $\overline{D_{sg}(\mathrm{End}_R(M))}$ no rigids $\Leftrightarrow$ X is complete locally $\mathbb{Q}$-factorial

<u>Key point</u>

$$D_{sg}(X) \hookrightarrow \bigoplus_{i=1}^{n} D_{sg}(\mathcal{O}_{X,x_i}) \hookrightarrow \bigoplus_{i=1}^{n} D_{sg}(\widehat{\mathcal{O}_{X,x_i}})$$

all equivalences up to summands (but maybe not essentially surjective)