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\[ \text{MMA}_\text{a} \text{c the } \text{MMP} \quad \text{(Ian Iyama)} \]

\[ \text{today, } R \text{ will always denote a commutative } \mathbb{C}-\text{algebra} \]
\[ \text{e.g. } R = \mathbb{C}[u,v,w]/(w^2 + v^2) \]
\[ \text{or } \quad R = \mathbb{C}[x,y]/(x^2 - y^2) \]

\[ \text{will later assume } R \text{ is normal, and } R \text{ has only rational singularities} \]

2 aspects of MMP

\[ \text{Input: } \text{Spec } R \]

(1) (Heid, Katzarkov) Existence of minimal models (= d.f. terminal singularities)

\[ Y \leftarrow \text{Spec } R \quad \text{where } f \text{ in proj. b.}, \quad f^*: \mathbb{C}^* \rightarrow \mathbb{C} \text{ has only mild sing.} \]

Special case: crepant resolutions (i.e. in good, Y need not be smooth)

(2) (many people: Mori, Kollár) Running of MMP (includes y cases, fppf, etc.)

Main new idea

The existence of NC minimal models (\( \text{MMA}_\text{a} \)) \[ \Rightarrow \] existence of commutative \( \text{MMP} \) (i.e. d.f. terminal singularities)

Example

\[ R = \mathbb{C}[u,v,x,y]/(uv = xy(y-2)) \]

[Diagram]

\[ \xrightarrow{d} \quad \text{Spec } R \]

There are 4 crepant resolutions

End \( R \otimes (u,x) \) is a noncommutative crepant resolution (NCCR)

(Can check det, coker, and map)

\[ \text{Qwiz: } (u) \subseteq (x) \quad \text{and relation} \]
Fix $a = (1, 1)$, then we get $2$ such notions $(x)$ (Yamada-Hashimoto).

Can't get all NCCR's from this one.

Why? $R$ is not Krull-Schmidt, so we shouldn't bother c what $\text{End}_R$ does.

Let $N = R \oplus (uy)$, $N = R \oplus (uxg) \oplus (u, x, y)$.

Thus $\text{add } A = \text{add } M$, where $\text{add } A$ is all direct sums of $A$ that summand $A$ $x$.

(=) $\text{End}_R(M)$, $\text{End}_R(N)$: Morita equiv. not derived Morita equiv.

$\text{End}_R(N)$ is also NCCR.

Relate $\text{End}_R(N)$ to $\text{End}_K$.

\[ (u, y, z) \quad \text{relation} \quad \text{Not a symmetric algebra!} \]

$A = (uy, z)$: clue by changing $B$ to get all commuting project relations.

$\Rightarrow$ from $m$-times add $M$ language (doesn't need $A$ chain). Example 2.

\[ R = C[U, V, x, y, Z]/UV = x(x^2y^2) \] is not a regular relation.

The MMA here is $y, \quad \circ \quad \circ \quad \circ \quad \circ R_2$

(\text{infinite global dimension})

Relation: $s = s' a$, $y a = a y$, $a t = (s)^2 + y$, $t = \frac{1}{2} (s)^2 + y$.

\[ \text{not needed} \]

\[ s = t = s \]

\[ y = t = y \]

\[ c = c' y \]

It turns out that $\text{End}_R(R \oplus (ux)) \cong K R / R$

Thus in this case, $\text{End}_R(M)$ language.

Definition: Let $M \in \text{mod } R$ (di $R$-$\Omega$, ret $\Omega$ - reflective module).

$M$ is called modifying if $\text{End}_R(M) \in \text{Csr}_{R}$.

(i.e., is a Cohen-Macaulay module $/R$.)
Definition 2 (Van der Put)

$\text{End}_R(M)$ is called NCCM if (i) $M$ is minimal, & (ii) global $\text{End}_R(M) < \infty$.

Definition 3

$\text{MCM}_R$ is called maximal modifying at (i) $M$ is modifying, & (ii) maximal with respect to this property, i.e.,

$$\exists \text{End}_R(M)\text{C} \subseteq \text{CM}_R \Rightarrow X \in \text{add} M.$$

Equality, $\text{add} M = \{X \in \text{End}_R(M)\text{C} \subseteq \text{CM}_R \}$

Note that this condition is almost impossible to check.

Fact

1) If $T = \text{NCCM}$, then $T = \text{NCCM}$ (sufficient: $M$ is minimal, & global $\text{End}_R(M) < \infty$).

2) Fix $R$. Then all $\text{MCM}_R$ are defined equivalent (analogy to geometry reduced).

3) If $T$ is a thick subcategory of an $\mathcal{O}$-functorial derived category $\mathcal{D} = \mathcal{D}\mathcal{P} R$, then $\text{End}_R(T) = \text{End}_R(\mathcal{D}\mathcal{P} R)$ is a MMA.

Theorem

Suppose $X \xrightarrow{f} \text{Spec} R$ with $X$ connected, isolated sing $\{x_1, \ldots, x_n\}$.

If $X$ satisfies $D^+(X) \cong D^+(\text{End}_R(M)\text{C})$. Assume $f$ is projective bounded.

1) If $\text{End}_R(M)\text{C}$ is MMA then $X$ is locally factorial.

2) $D^+(\text{End}_R(M)\text{C})$ has no rigid objects $\Rightarrow X$ is complete locally factorial.

3) $\text{End}_R(M)\text{C}$ is MMA $\Rightarrow D^+(\text{End}_R(M)\text{C})$ has no rigid objects $\Rightarrow X$ is complete locally factorial.

4) $D^+(\text{End}_R(M)\text{C})$ has no rigid objects $\Rightarrow X$ is complete locally factorial.
Key point

\[ D_{sg}(X) \supset \bigoplus_{i=1}^{n} D_{sg}(A_{x_i}) \supset \bigoplus_{i=1}^{n} D_{sg}(G_{x_i}) \]

all equations with summands (but maybe not essentially surjective)