Majorana Representations, the Monster and the Straight Flush Conjecture

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The notion of Majorana representation of groups was introduced by the speaker in 2008 at an Oberwoflach meeting by axiomatizing certain properties of the action of the Monster group on its 196 884-dimensional non-associative Conway–Griess–Norton algebra. This axiomatization was motivated by results by M. Miyamoto and S. Sakuma proved in the context of Vertex Operator Algebras and the study of the Monster algebra by J. Conway and S. Norton. The first description of the Monster algebra was given by R. Griess as a foundation of his first construction of the Monster. Basically the Majora axiomatic amounts to the statements that the algebra is generated by a special class of vectors called Majorana axes. If a is a Majorana axis then its adjoint actions have four eigenvalues 1, 0, $\frac{1}{4}$, $\frac{1}{32}$ and the corresponding eigenspaces satisfy the fusion rules of the Majorana fermion. The fusion rules imply that the involution $\tau(a)$ which negates every $\frac{1}{32}$-eigenvector and centralizes the remaining eigenvectors, is an automorphism of the algbera. In the Vertex Operator Algebra context $\tau(a)$ corresponds to the Miyamoto involution and in the Monster algebra it is a $2A$-involution.

By a theorem proved by S. Sakuma there are nine possibilities for the isomorphism type of a subalgebra generated by two Majorana axes a_0 and a_1 which match the nine subalgebras in the Monster algebra naturally indexed and named by the conjugacy classes

$$1A,\ 2A,\ 2B,\ 3A,\ 3C,\ 4A,\ 4B,\ 5A,\ 6A$$

of the Monster. The numerical part of the algebra name is the order of the product $\tau(a_0)\tau(a_1)$. These nine algebras are now called the Norton–Sakuma algebras. Every subgroup of the Monster (including the Monster itself) generated by $2A$ involutions contained in that group possesses a Majorana representation which is said to be based on the embedding into the Monster. Currently the study of Majorana representation is a new dramatically growing research area which goes under the name Majorana Theory. There are Majorana representations which are not based on embeddings into the Monster, a highly non-trivial one being the 70-dimensional representation of A_6 of the shape $(2A, 3C, 4B, 5A)$. In all the non-embaddable examples constructed so far at least one of the six numerical values in the Norton–Sakuma subalgebras is missing (the $6A$-algebra is missing in the A_6 example). Two Majorana axes generating a $2B$-algebra are perpendicular and annihilate each other. A Majorana representation is *connected* if such is the graph on the generating Majorana axes where non-edges correspond to the $2B$-subalgebras. All the representations constructed so far fit in the following *Straight Flush Conjecture*.

Conjecture 1 *A connected Majorana representation which involves Norton–Sakuma algebra with the numerical parts* 2, 3, 4, 5 *and* 6 *is based on an embedding into the Monster.*

The Straight Flush Conjecture can be restated as a claim that the Majorana representation $\mathcal{R}_M$ of the Monster is universal in the sense that every other representation which involves enough Norton–Sakuma algebras is the restriction of $\mathcal{R}_M$ to a subgroup. The universality of the Monster in a proper setting was believed to hold and informaly discussed by S. Norton and M. Miyamoto.