The d-orbifold programme, with applications to moduli spaces of J-holomorphic curves
Lecture 1 of 5: Overview

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Mainly work in progress. Some partial references:
For d-manifolds and d-orbifolds, see arXiv:1206.4207 (survey), arXiv:1208.4948, and preliminary version of book available at http://people.maths.ox.ac.uk/∼joyce/dmanifolds.html.
For C^∞ geometry, see arXiv:1104.4951 (survey), and arXiv:1001.0023.

Plan of talk:

1. Introduction

These lectures concern new classes of geometric objects I call *d-manifolds* and *d-orbifolds* — 'derived' smooth manifolds, in the sense of Derived Algebraic Geometry. Some properties:

- D-manifolds form a *strict 2-category* **dMan**. That is, we have objects $\mathbf{X}$, the d-manifolds, 1-morphisms $\mathbf{f}, \mathbf{g} : \mathbf{X} \to \mathbf{Y}$, the smooth maps, and also 2-morphisms $\eta : \mathbf{f} \Rightarrow \mathbf{g}$.
- Smooth manifolds embed into d-manifolds as a full (2)-subcategory. So, d-manifolds generalize manifolds.
- There are also 2-categories $\mathbf{dMan^b}$, $\mathbf{dMan^c}$ of d-manifolds *with boundary* and *with corners*, and orbifold versions $\mathbf{dOrb}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$ of these, *d-orbifolds*.
- Much of differential geometry extends nicely to d-manifolds: submersions, immersions, orientations, submanifolds, transverse fibre products, cotangent bundles, ...

- Almost any moduli space used in any enumerative invariant problem over $\mathbb{R}$ or $\mathbb{C}$ has a d-manifold or d-orbifold structure, natural up to equivalence. There are truncation functors to d-manifolds and d-orbifolds from structures currently used — Kuranishi spaces, polyfolds, $\mathbb{C}$-schemes or Deligne–Mumford $\mathbb{C}$-stacks with obstruction theories.
  Combining these truncation functors with known results gives d-manifold/d-orbifold structures on many moduli spaces.
- I will also outline an approach to prove existence of d-manifold/d-orbifold structures on moduli spaces directly, using 'representable 2-functors'.
- Virtual classes/cycles/chains can be constructed for compact oriented d-manifolds and d-orbifolds.

So, d-manifolds and d-orbifolds provide a unified framework for studying enumerative invariants and moduli spaces. They also have other applications, and are interesting for their own sake.

In the rest of this talk I want to discuss four different 'big ideas', which are the subjects of Lectures 2–5:

2. D-manifolds and d-orbifolds: what are they, why are they a 2-category, how are they related to other classes of spaces (e.g. Kuranishi spaces and polyfolds)?
3. A new approach to moduli problems using 'representable 2-functors' to define a d-orbifold structure on a moduli space.
4. 'Stratified manifolds', a new analytic tool for dealing with 'bubbling', 'neck stretching' and 'nodes' for J-holomorphic curves, and compactification of moduli spaces in general.
5. 'D-orbifold (co)homology', new (co)homology theories $dH_*(Y,R)$, $dH^*(Y,R)$ of a manifold or orbifold Y, isomorphic to ordinary (co)homology, but in which the (co)chains are 1-morphisms $\mathbf{f}:\mathbf{X}\to Y$ for $\mathbf{X}$ a compact, oriented d-orbifold with corners, plus extra data. Forming virtual classes for moduli spaces in d-orbifold (co)homology is almost trivial.

2. D-manifolds and d-orbifolds
2.1 Why do we need 'derived geometry'?

In very general terms, we want to solve the following:

Problem

Find a (hopefully canonical) geometric structure $\mathcal{G}$ on moduli spaces of J-holomorphic curves $\overline{\mathcal{M}}$, such that (compact, oriented) spaces $\overline{\mathcal{M}}$ with structure $\mathcal{G}$ can be 'counted' in $\mathbb{Z},\mathbb{Q}$ or some (co)homology theory, so that we can do Gromov–Witten invariants, Lagrangian Floer cohomology, Symplectic Field Theory, ...

I claim that the structure $\mathcal{G}$ must be 'derived' in the sense of Derived Algebraic Geometry of Lurie, Toën–Vezzosi, ..., and that a basic understanding of derived geometry really helps here.

Derived schemes, deformations, and obstructions

Consider a moduli space $\mathcal{M}$ of some objects Σ (e.g. J-holomorphic curves). Linearizing the deformation theory at $\Sigma \in \mathcal{M}$, we have vector spaces of deformations $\mathscr{D}_\Sigma$ (typically the kernel of an operator L_Σ) and obstructions $\mathcal{O}_\Sigma$ (typically the cokernel of L_Σ).

If we model $\mathcal{M}$ as a *classical* space (e.g. a scheme or stack), then $\mathcal{M}$ remembers $\mathscr{D}_\Sigma$ as the tangent space $T_\Sigma\mathcal{M}$, but *forgets* $\mathcal{O}_\Sigma$.

If we model $\mathcal{M}$ as a *derived* space (e.g. a derived scheme or stack), then $\mathcal{M}$ remembers both $\mathscr{D}_\Sigma, \mathcal{O}_\Sigma$, as $\mathscr{D}_\Sigma = H^0(\mathbb{T}_\mathcal{M}|_\Sigma)$, $\mathcal{O}_\Sigma = H^1(\mathbb{T}_\mathcal{M}|_\Sigma)$ for $\mathbb{T}_\mathcal{M}$ the tangent complex of $\mathcal{M}$.

Derived geometry was introduced exactly to give geometric structures which remember all the deformation theory.

To 'count' moduli spaces $\mathcal{M}$ of J-holomorphic curves correctly, the geometric structure $\mathcal{G}$ on $\mathcal{M}$ must encode the obstructions $\mathcal{O}_\Sigma$ as well as the deformations $\mathscr{D}_\Sigma$. So $\mathcal{G}$ must be 'derived'.

Introduction
D-manifolds and d-orbifolds
D-orbifolds as representable 2-functors, and moduli spaces
'Stratified manifolds', including singular curves in moduli spaces
D-orbifold (co)homology, and virtual classes/cycles/chains

Why do we need higher categories?

Suppose we have a category $\mathcal{C}$ (e.g. a category of complexes), and we want to invert (localize) some class of morphisms $\mathcal{W}$ in $\mathcal{C}$ called 'weak equivalences' (e.g. quasi-isomorphisms) which are not isomorphisms, to form a new 'derived' category $\bar{\mathcal{C}}$. A fundamental insight in derived geometry is that $\bar{\mathcal{C}}$ should be a *higher category* (usually an ∞-category), not an ordinary category.

Derived objects (e.g. derived schemes) always form higher categories. You can truncate to an ordinary category, but you lose too much information (e.g. the universal property for fibre products of derived objects only makes sense in the higher category).

D-manifolds and d-orbifolds form 2-*categories*, the simplest kind of higher category.

2.2. Other classes of spaces. Kuranishi spaces versus d-orbifolds

A *Kuranishi structure* on a space $\mathcal{M}$ (Fukaya–Oh–Ohta–Ono) involves *Kuranishi neighbourhoods* (V_p, E_p, s_p) and (non-invertible) *coordinate changes* $(f_{pq}, \hat{f}_{pq}) : (V_p, E_p, s_p) \to (V_q, E_q, s_q)$.
Think of the Kuranishi neighbourhoods (V_p, E_p, s_p) as the objects in a category $\mathcal{C}$, and the coordinate changes $(f_{pq}, \hat{f}_{pq})$ as the weak equivalences $\mathcal{W}$ we want to invert. Derived geometry says that to do the job properly, we *must* use higher categories, but FOOO do not. This is the source of some of the problems in the theory.
I regard d-orbifolds as *the 'correct' definition of Kuranishi spaces, what Kuranishi spaces should have been*. It should be possible to give a FOOO-style definition of Kuranishi spaces involving 2-categories, and get a 2-category equivalent to d-orbifolds **dOrb**.

Polyfolds, and schemes with obstruction theory

Polyfolds (Hofer–Wysocki–Zehnder) do not use higher categories, so far as I know. They get away with this because they never localize. Therefore a polyfold comprises a huge amount of information, the whole of the functional-analytic moduli problem, as nothing is forgotten by localizing. This works, but I feel polyfolds are unwieldy, and unsatisfying as geometric spaces.

Schemes $\mathcal{M}$ with obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ in algebraic geometry (Behrend–Fantechi) work very nicely — much more so than Kuranishi spaces or polyfolds in the (rather more difficult) differential-geometric context. Note that they are 'derived' objects (as I said $\mathcal{M}$ must be), since $\mathcal{E}^\bullet$ is an object in the derived category $D^b \mathrm{coh}(\mathcal{M})$, and should be understood as the cotangent complex $\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ of an underlying derived scheme $\boldsymbol{\mathcal{M}}$.

2.3. D-manifolds – sketch of the definition

Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes. In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces.

There is a little-known theory of schemes in differential geometry, C^∞-*schemes*, going back to Lawvere, Dubuc, Moerdijk and Reyes, ... in synthetic differential geometry in the 1960s-1980s.

C^∞-schemes are essentially *algebraic* objects, on which smooth real functions and calculus make sense.

Our d-manifolds are a special kind of *derived* C^∞*-scheme*, combining C^∞-algebraic geometry and derived geometry.

C^∞-rings

Let X be a manifold, and write $C^\infty(X)$ for the smooth functions $c : X \to \mathbb{R}$. Then $C^\infty(X)$ is an $\mathbb{R}$-*algebra*: we can add smooth functions $(c, d) \mapsto c + d$, and multiply them $(c, d) \mapsto cd$, and multiply by $\lambda \in \mathbb{R}$.

But there are many more operations on $C^\infty(X)$ than this, e.g. if $c : X \to \mathbb{R}$ is smooth then $\exp(c) : X \to \mathbb{R}$ is smooth, giving $\exp : C^\infty(X) \to C^\infty(X)$, which is algebraically independent of addition and multiplication.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by $\Phi_f(c_1, \ldots, c_n)(x) = f\bigl(c_1(x), \ldots, c_n(x)\bigr)$ for all $x \in X$. Then addition comes from $f : \mathbb{R}^2 \to \mathbb{R}$, $f : (x, y) \mapsto x + y$, multiplication from $(x, y) \mapsto xy$, etc. This huge collection of algebraic operations Φ_f make $C^\infty(X)$ into an algebraic object called a C^∞-*ring*.

Definition

A C^∞-*ring* is a set $\mathfrak{C}$ together with n-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying:
Let $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathfrak{C}$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have $\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j$.
A *morphism* of C^∞-rings is $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

C^∞-algebraic geometry

We can now develop the whole machinery of scheme theory and stack theory in algebraic geometry, replacing rings or algebras by C^∞-rings throughout — see my arXiv:1104.4951, arXiv:1001.0023.
This gives a category of C^∞-*schemes* $\mathbf{C^\infty Sch}$ and a 2-category of C^∞-*stacks* $\mathbf{C^\infty Sta}$, which contain manifolds $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ and orbifolds $\mathbf{Orb} \subset \mathbf{C^\infty Sta}$ as full (2-)subcategories.
We also define (*quasi*)*coherent sheaves* on C^∞-schemes and C^∞-stacks, generalizing vector bundles on manifolds and orbifolds.

Derived C^∞-algebraic geometry

Derived smooth manifolds (or orbifolds) should be defined as special examples of suitable higher categories of *derived C^∞-schemes* or *derived C^∞-stacks*. This was the approach taken by David Spivak (arXiv:0810.5175, Duke Math. J.), a student of Jacob Lurie, who defined an ∞-category of simplicial-C^∞-ringed spaces, with an ∞-subcategory of 'derived manifolds'.
My approach is roughly a simplified 2-category truncation of Spivak's (for the precise relation, see Borisov arXiv:1212.1153).
I define 2-categories of *d-spaces* **dSpa** and *d-stacks* **dSta** containing d-manifolds **dMan** $\subset$ **dSpa** and d-orbifolds **dOrb** $\subset$ **dSta** as full 2-subcategories.

What is a d-space?

A derived scheme (dg-scheme) $\mathbf{X}$ is roughly a topological space X equipped with a sheaf (or homotopy sheaf) $\mathcal{O}_{\mathbf{X}}$ of dg-rings, with points of X corresponding to prime ideals of the dg-rings.
Similarly, a derived C^∞-scheme $\mathbf{X}$ should roughly be a topological space X equipped with a (homotopy?) sheaf $\mathcal{O}_{\mathbf{X}}$ of dg C^∞-rings (or possibly simplicial C^∞-rings), with points of X corresponding to ideals of the dg-rings with residue field $\mathbb{R}$.
A d-space $\mathbf{X}$ is a topological space X with a sheaf (not a homotopy sheaf) $\mathcal{O}_{\mathbf{X}}$ of dg-C^∞-rings $\mathfrak{C}^\bullet$ of a special kind: they are two-step dg-C^∞-rings $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$ such that $\mathfrak{C}^{-1} \cdot \mathrm{d}[\mathfrak{C}^{-1}] = 0$, which implies that $\mathrm{d}[\mathfrak{C}^{-1}]$ is a square zero ideal in the (ordinary) C^∞-ring $\mathfrak{C}^0$, and $\mathfrak{C}^{-1}$ is a module over the 'classical' C^∞-ring $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.

What is a d-manifold?

We have inclusions of (2-)categories **Man** $\subset$ **C$^\infty$Sch** $\subset$ **dSpa**, so manifolds are examples of d-spaces. A *d-manifold* **X** of *virtual dimension* $n \in \mathbb{Z}$ is a d-space **X** whose topological space X is Hausdorff and second countable, and such that **X** is covered by open d-subspaces **Y** $\subset$ **X** with equivalences $\mathbf{Y} \simeq U \times_{g,W,h} V$, where U, V, W are manifolds with $\dim U + \dim V - \dim W = n$, and $g : U \to W$, $h : V \to W$ are smooth maps, and $U \times_{g,W,h} V$ is the fibre product in the 2-category **dSpa**. (The 2-category structure is *essential* to define the fibre product here.)
Alternatively, we can write the local models as $\mathbf{Y} \simeq V \times_{0,E,s} V$, where V is a manifold, $E \to V$ a vector bundle, $s : V \to E$ a smooth section, and $n = \dim V - \operatorname{rank} E$. Then (V, E, s) is a *Kuranishi neighbourhood* on **X** (compare with Kuranishi spaces).
We call such $V \times_{0,E,s} V$ *affine d-manifolds*.

3. D-orbifolds as representable 2-functors, moduli spaces

Recall the Grothendieck approach to moduli spaces in algebraic geometry, using *moduli functors*. Write $\mathbf{Sch}_{\mathbb{C}}$ for the category of $\mathbb{C}$-schemes, and $\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}}$ for the subcategory of affine $\mathbb{C}$-schemes. Any $\mathbb{C}$-scheme X defines a functor $\mathrm{Hom}(-, X) : \mathbf{Sch}^{\mathbf{op}}_{\mathbb{C}} \to \mathbf{Sets}$ mapping each $\mathbb{C}$-scheme S to the set $\mathrm{Hom}(S, X)$. By the Yoneda Lemma, the $\mathbb{C}$-scheme X is determined up to isomorphism by the functor $\mathrm{Hom}(-, X)$ up to natural isomorphism. This still holds if we restrict to $\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}}$. Thus, given a functor $F : (\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}})^{\mathbf{op}} \to \mathbf{Sets}$, we can ask if there exists a $\mathbb{C}$-scheme X (necessarily unique up to canonical isomorphism) with $F \cong \mathrm{Hom}(-, X)$. If so, we call F a *representable functor*. More generally, *Artin $\mathbb{C}$-stacks* are defined as a class of functors $F : (\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}})^{\mathbf{op}} \to \mathbf{Groupoids}$.

Grothendieck's moduli schemes

Suppose we have an algebro-geometric moduli problem (e.g. vector bundles on a smooth projective $\mathbb{C}$-scheme Y) for which we want to form a moduli scheme. Grothendieck tells us that we should define a *moduli functor* $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Sets}$, such that for each affine $\mathbb{C}$-scheme S, $F(S)$ is the set of isomorphism classes of families of the relevant objects over S (e.g. vector bundles over $Y \times S$). Then we should try to prove F is a representable functor, using some criteria for representability. If it is, $F \cong \mathrm{Hom}(-, \mathcal{M})$, where $\mathcal{M}$ is the (*fine*) *moduli scheme*.
To form a moduli stack, we define $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$, so that for each affine $\mathbb{C}$-scheme S, $F(S)$ is the category of families of objects over S, with morphisms isomorphisms of families, and try to show F satisfies the criteria to be an Artin stack.

D-orbifolds as representable 2-functors

D-orbifolds $\mathbf{dOrb}$ are a 2-category with all 2-morphisms invertible. Thus, if $\mathbf{S}, \mathbf{X} \in \mathbf{dOrb}$ then $\mathbf{Hom}(\mathbf{S}, \mathbf{X})$ is a groupoid, and $\mathbf{Hom}(-, \mathbf{X}) : \mathbf{dOrb}^{\mathbf{op}} \to \mathbf{Groupoids}$ is a 2-functor, which determines $\mathbf{X}$ up to equivalence in $\mathbf{dOrb}$. This is still true if we restrict to affine d-manifolds $\mathbf{dMan}^{\mathbf{aff}} \subset \mathbf{dOrb}$. Thus, we can consider 2-functors $F : (\mathbf{dMan}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$, and ask whether there exists a d-orbifold $\mathbf{X}$ (unique up to equivalence) with $F \simeq \mathbf{Hom}(-, \mathbf{X})$. If so, we call F a *representable 2-functor*.
I expect there are nice criteria for when F is representable:

(A) F satisfies a sheaf-type condition;
(B) the 'coarse topological space' $\mathcal{M} = F(\text{point})/\text{isos}$ is Hausdorff and second countable;
(C) F admits a 'Kuranishi neighbourhood' of dimension $n \in \mathbb{Z}$ near each $x \in \mathcal{M}$, a local model with a universal property.

Moduli 2-functors in differential geometry

Suppose we are given a moduli problem in differential geometry (e.g. J-holomorphic curves in a symplectic manifold) and we want to form a moduli space $\boldsymbol{\mathcal{M}}$ as a d-orbifold. I propose that we should define a *moduli 2-functor* $F : (\mathbf{dMan}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$, such that for each affine d-manifold $\mathbf{S}$, $F(\mathbf{S})$ is the category of families of the relevant objects over $\mathbf{S}$. Then we should try to prove F satisfies (A)–(C), and so is represented by a d-orbifold $\boldsymbol{\mathcal{M}}$; here (A),(B) will usually be easy, and (C) the difficult part.

To define F, we need a good notion of 'families of J-holomorphic curves over a base $\mathbf{S}$, an affine d-manifold'. I will explain a short, natural definition for F for J-holomorphic curves in Lecture 3.

Some remarks:

- Current definitions of differential-geometric moduli spaces (e.g. Kuranishi spaces, polyfolds) are generally long, complicated ad hoc constructions, with no obvious naturality. In contrast, if we allow differential geometry over d-manifolds, my approach gives you a short, natural definition of the moduli functor F, followed by a long proof that F is representable. The effort moves from a construction to a theorem.
- The definition of F involves only finite-dimensional families of smooth objects, with no analysis, Banach spaces, etc. (But the proof of (C) will involve analysis and Banach spaces.) This enables us to sidestep some analytic problems, e.g. non-smoothness of action of diffeomorphisms on Banach spaces of maps (cf. sc-smoothness in polyfold theory).

4. 'Stratified manifolds', and curves with nodes

In symplectic geometry, one considers moduli spaces $\mathcal{M}$ of J-holomorphic curves $u : \Sigma \to S$, for (S, ω) a symplectic manifold with almost complex structure J. For counting problems (Gromov–Witten, etc.) it is essential that $\mathcal{M}$ be compact. But if we consider only nonsingular Riemann surfaces Σ, we generally get noncompact $\mathcal{M}$. To get compactified moduli spaces $\bar{\mathcal{M}}$, we must include Riemann surfaces Σ with singularities (nodes).

Two closely related problems are moduli spaces of J-holomorphic curves with ends in Symplectic Field Theory including curves which 'stretch' along an infinite cylinder, and (simpler) moduli spaces of gradient flow lines in Morse homology, including 'broken flow lines'.

Modelling such moduli spaces $\bar{\mathcal{M}}$ analytically near $\Sigma \in \bar{\mathcal{M}}$ with nodes is rather messy, and apparently *not smooth*: the obvious constructions yield Kuranishi neighbourhoods (V, E, s) on $\bar{\mathcal{M}}$ in which the section $s : V \to E$ is not smooth normal to the nodal stratum $V_{\rm node} \subset V$, but only continuous. (Curiously, the algebraic geometry version does not suffer from this problem.) Non-smooth sections $s : V \to E$ would be *bad* in our C^∞-geometry approach.

In the polyfold picture, one deals with this using a *gluing profile* φ: basically, one changes the smooth structure on V along $V_{\rm node}$ in the normal directions, to get a new manifold $\tilde{V}$ with the same topological space as V, such that s is smooth w.r.t. the smooth structure on $\tilde{V}$. Roughly, $\tilde{V} = V \times_{r^2,[0,\infty),\varphi} [0, \infty)$, where $r : V \to [0, \infty)$ is the distance from $V_{\rm node}$, and $\varphi : [0, \infty) \to [0, \infty)$ is $\varphi(0) = 0$, $\varphi(x) = e^{-1/x}$, $x > 0$.

In the 'representable 2-functor' approach of §3, I expect this problem comes up as follows: one can define a moduli functor $F : \mathbf{dMan^{aff}} \to \mathbf{Groupoids}$ for $\bar{\mathcal{M}}$, including curves with nodes, in a straightforward way. But F may not be representable near singular curves Σ in $\bar{\mathcal{M}}$. To deal with this, one should modify F using a gluing profile φ to get a new, representable functor $\tilde{F}$.

However, I wish to propose an alternative approach I believe is analytically more natural; my ideas here are still at an early stage. I want to define a new category $\mathbf{Man^{st}}$ of 'stratified manifolds', which are roughly manifolds V with designated 'strata' $V_{\rm node} \subset V$ (e.g. the boundary of V) such that the smooth structure on V along $V_{\rm node}$ in the normal directions is exotic, nonstandard; so V is not actually a conventional smooth manifold near $V_{\rm node}$.

These stratified manifolds should be included in d-manifolds and d-orbifolds to give 2-categories $\mathbf{dMan^{st}}, \mathbf{dOrb^{st}}$. A choice of gluing profile φ should induce 'smoothing functors' from $\mathbf{Man^{st}}, \mathbf{dMan^{st}}$, $\mathbf{dOrb^{st}}$ to $\mathbf{Man^{c}}, \mathbf{dMan^{c}}, \mathbf{dOrb^{c}}$.

I hope that this class of analytic moduli problems can be more naturally modelled using stratified manifolds, yielding Kuranishi neighbourhoods (V, E, s) for $\bar{\mathcal{M}}$ in which V is a stratified manifold, with strata at the points representing nodal curves, and canonical moduli functors which are representable in $\mathbf{dOrb^{st}}$, though not necessarily in $\mathbf{dOrb^{c}}$.

There appear to be connections with the work of Richard Melrose on analysis on manifolds with boundary and corners.

For more details, see Lecture 4.

5. D-orbifold (co)homology, and virtual cycles

Note: you can find a previous version of this project using Kuranishi spaces instead of d-orbifolds at arXiv:0710.5634 (survey), and arXiv:0707.3572. *These papers should not be trusted*, as the Kuranishi spaces material is dodgy, but they do show the basic ideas.

Once we have given moduli spaces of J-holomorphic curves $\bar{\mathcal{M}}$ the structure of d-orbifolds (or whatever), to do Gromov–Witten invariants, Lagrangian Floer cohomology, Symplectic Field Theory, ..., we need to associate a *virtual class* (or *virtual cycle*, or *virtual chain*) to $\bar{\mathcal{M}}$, in some (co)homology theory. That is, we need a bridge between moduli spaces and homological algebra.

In [FOOO], this is done by perturbing the Kuranishi spaces using multisections to get a ($\mathbb{Q}$-weighted, non-Hausdorff) manifold, triangulating this by simplices to get a chain in singular homology. This process is acutely painful, because singular homology does not play at all nicely with Kuranishi spaces, and much of the algebraic complexity of [FOOO] is due to the problems this causes – especially, perturbing Kuranishi spaces to transverse.

I propose to define new (co)homology theories $dH_*(Y,R)$, $dH^*(Y,R)$ of a manifold or orbifold Y, isomorphic to ordinary (co)homology, but in which the (co)chains are 1-morphisms $\mathbf{f}:\mathbf{X}\rightarrow Y$ for $\mathbf{X}$ a compact, oriented d-orbifold with corners, plus extra data. Forming virtual classes for moduli spaces in d-orbifold (co)homology is almost trivial, there is no need to perturb; the homological algebra in [FOOO] can be drastically simplified.

D-orbifold homology

Let Y be a manifold or orbifold, and R a $\mathbb{Q}$-algebra. We define a complex of R-modules $\bigl(dC_*(Y,R);\partial\bigr)$, whose homology groups $dH_*(Y,R)$ are the *d-orbifold homology* of Y. Chains in $dC_k(Y;R)$ for $k\in\mathbb{Z}$ are R-linear combinations of equivalence classes $[\mathbf{X},\mathbf{f},\mathbf{G}]$, where $\mathbf{X}$ is a compact, oriented d-orbifold with corners of dimension k, $\mathbf{f}:\mathbf{X}\to Y$ is a 1-morphism in $\mathbf{dMan^c}$, and $\mathbf{G}$ is some extra 'gauge-fixing data' associated to $\mathbf{X}$, for which there will be many possible choices. If we did not include $\mathbf{G}$ then chains $(\mathbf{X},\mathbf{f})$ might have infinite automorphism groups, leading to bad behaviour.
The boundary operator $\partial: dC_k(Y;R)\to dC_{k-1}(Y;R)$ maps

$$\partial:\bigl[\mathbf{X},\mathbf{f},\mathbf{G}\bigr]\longmapsto\bigl[\partial\mathbf{X},\mathbf{f}\circ\mathbf{i}_{\mathbf{X}},\mathbf{G}|_{\partial\mathbf{X}}\bigr].$$

Note that $\partial^2\mathbf{X}$ has an orientation-reversing involution $\sigma:\partial^2\mathbf{X}\to\partial^2\mathbf{X}$. Using this we show that $\partial^2=0$.

Singular homology $H_*^{\rm sing}(Y;R)$ may be defined using $\bigl(C_*^{\rm sing}(Y;R);\partial\bigr)$, where $C_k^{\rm sing}(Y;R)$ is spanned by *smooth* maps $f:\Delta_k\to Y$, for Δ_k the standard k-simplex, thought of as a manifold with corners.
We define an R-linear map $F_{\rm sing}^{\rm dH}: C_k^{\rm sing}(Y;R)\to dC_k(Y;R)$ by

$$F_{\rm sing}^{\rm dH}: f\longmapsto\bigl[\Delta_k,f,\mathbf{G}_{\Delta_k}\bigr],$$

with $\mathbf{G}_{\Delta_k}$ some standard choice of gauge-fixing data for Δ_k.
Then $F_{\rm sing}^{\rm dH}\circ\partial=\partial\circ F_{\rm sing}^{\rm dH}$, so that $F_{\rm sing}^{\rm dH}$ induces morphisms $F_{\rm sing}^{\rm dH}: H_k^{\rm sing}(Y;R)\to dH_k(Y;R)$. I hope to show these are isomorphisms. Proving this will involve perturbing d-orbifolds to orbifolds or manifolds and triangulating by simplices — the messy bits of [FOOO]. But we only have to do this once, not every time we use the theory.