Consider the case where \( F \) is simply connected, we see that there is a map \( \alpha \). From the long exact sequence of a fibration (here we use simply connected so \( S^g \)), suppose \( g \) is the inclusion. So we obtain \( o \). Moreover, \( \alpha \) is a cocycle and \( \theta(g) \in H^{n+1}(X; \pi_n(Y)) \) vanishes if and only if \( f|_{X^{(n-1)}} \) can be extended to \( X^{(n+1)} \); that is, \( f \) may need to be redefined on the \( n \)-cells.

**Obstructions to lifting a map**

Given a fibration \( F \to E \xrightarrow{p} B \) and a map \( f : X \to B \), when can \( f \) be lifted to a map \( g : X \to E \)? If \( X = B \) and \( f = \text{id}_B \), then we are asking when \( p \) has a section. For convenience, we will only consider the case where \( F \) and \( B \) are simply connected, from which it follows that \( E \) is simply connected. For a more general statement, see Theorem 7.37 of [2].

Suppose \( g \) has been defined on \( X^{(n)} \). Let \( e^{n+1} \) be an \( n \)-cell and \( \alpha : S^n \to X^{(n)} \) its attaching map, then \( p \circ g \circ \alpha : S^n \to B \) is equal to \( f \circ \alpha \) and is nullhomotopic (as \( f \) extends over the \( (n+1) \)-cell).

From the long exact sequence of a fibration (here we use simply connected so \([S^n, F] = \pi_n(F)\) etc.), we see that there is a map \( \beta : S^n \to F \) such that \( g \circ \alpha \) is homotopic to \( i \circ \beta \) where \( i : F \to E \) is the inclusion. So we obtain \( \theta(g) \in C^{n+1}(X; \pi_n(F)) \) which vanishes if and only if \( g \) extends to \( X^{(n+1)} \). As before, \( \theta(g) \) is a cocycle and \( \theta(g) \in H^{n+1}(X; \pi_n(F)) \) vanishes if and only if \( g|_{X^{(n-1)}} \) extends to \( X^{(n+1)} \).

Lots of interesting problems can be analysed using obstructions to lifting a map. For example:

- When does a vector bundle have a nowhere-zero section?
- When is a smooth manifold orientable?
- When is a smooth manifold spin?
- When does a smooth manifold admit an almost complex structure?
- When does a topological manifold admit a PL structure or smooth structure?

We’re going to focus on the fourth one.

**Almost Complex Structures**

A linear complex structure on a real vector space \( V \) is an endomorphism \( J : V \to V \) such that \( J \circ J = -\text{id}_V \). If \( V \) is endowed with a linear complex structure \( J \), then we can view \( V \) as a complex vector space by defining \((a + bi) \cdot v = av + bJ(v)\). In particular, if \( V \) is finite-dimensional, then \( \dim V = 2\dim C V \) is even. Moreover, if \( \{e_1, \ldots, e_n\} \) is a basis for \( V \) as a complex vector space, then \( \{e_1, J(e_1), \ldots, e_n, J(e_n)\} \) is a basis for \( V \) as a real vector space and \( e_1 \wedge J(e_1) \wedge \cdots \wedge e_n \wedge J(e_n) \) defines an orientation; this orientation is independent of the choice of basis \( \{e_1, \ldots, e_n\} \).

Let \( E \to B \) be a real vector bundle. An almost complex structure on \( E \) is a bundle endomorphism \( J : E \to E \) such that \( J \circ J = -\text{id}_E \). It follows that in order for an almost complex structure to
exist, \( E \) must have even rank and be orientable. Note, given an almost complex structure, one can view \( E \) as a complex vector bundle.

**Remark:** The reason I use the terminology ‘linear almost complex structure’ on \( V \) rather than ‘almost complex structure’ is that the latter could be interpreted as an almost complex structure on the manifold \( V \), i.e. an almost complex structure on the vector bundle \( TV \).

An almost complex structure on a smooth manifold \( M \) is defined to be an almost complex structure on \( TM \). Again, if \( M \) admits an almost complex structure then \( M \) has even dimension and is orientable. Moreover, \( TM \) can be viewed as a complex vector bundle.

**Question:** Does every even-dimensional orientable smooth manifold admit an almost complex structure?

**Answer:** No, there are obstructions.

### Classifying Spaces

A **topological group** is a group \((G, *)\) such that \( G \) is a topological space, and the maps \(* : G \times G \to G\) and \( i : G \to G, \ g \mapsto g^{-1}\) are continuous. If \( G \) is a smooth manifold and the maps \(*\) and \( i\) are smooth, then \((G, *)\) is called a **Lie group**.

A **fiber bundle** with fiber \( F \) is a continuous map \( \pi : E \to B \) such that for every \( b \in B \), there is an open neighbourhood \( U \subseteq B \) of \( b \) and a homeomorphism \( \varphi : \pi^{-1}(U) \to U \times F \) such that \( \pi = \text{pr}_1 \circ \varphi \).

Let \( G \) be topological group. A **principal \( G\)-bundle** is a fiber bundle \( \pi : E \to B \) together with a continuous right action \( E \times G \to E \) which preserves fibers (i.e. \( \pi(e \cdot g) = \pi(e) \)), and acts freely and transitively on them. As the action is free and transitive, we can (non-canonically) identify the fibers of \( \pi \) with \( G \).

An isomorphism between principal \( G\)-bundles \( P \to B \) and \( Q \to B \) is a \( G\)-equivariant map \( \phi : P \to Q \) covering the identity. Denote the isomorphism classes of principal \( G\)-bundles on a topological space \( B \) by \( \text{Prin}_G(B) \).

Fiber bundles, and hence principal bundles, are Serre fibrations; see Proposition 4.48 of [4]. Note however, they are not necessarily Hurewicz fibrations, see [1].

**Examples**

1. If \( G \) a discrete group, a principal \( G \)-bundle is a normal covering with group of deck transformations isomorphic to \( G \).

2. If \( H \) is a closed subgroup of a Lie group \( G \), then \( G \to G/H \) is a principal \( H \)-bundle.

3. Main example, frame bundles.

Let \( E \to B \) be a real rank \( n \) vector bundle. The frame bundle of \( E \) is a space \( F(E) \) together with a map \( \pi : F(E) \to B \) such that \( \pi^{-1}(p) \) is the collection of ordered bases, or frames, for \( E_p \).

Any two frames are related by a unique element of \( GL(n, \mathbb{R}) \). This is a principal \( GL(n, \mathbb{R})\)-bundle. Conversely, given a principal \( GL(n, \mathbb{R})\)-bundle, one can build a real vector bundle via a process known as the associated bundle construction. This defines a bijection between \( \text{Prin}_{GL(n, \mathbb{R})}(B) \) and \( \text{Vect}_n(B) \), the collection of isomorphism classes of real rank \( n \) vector bundles.

Equipping \( E \) with a Riemannian metric, we can take the orthogonal frame bundle which is a principal \( O(n)\)-bundle. Different Riemannian metrics give isomorphic principal \( O(n)\)-bundles. Again by the associated bundle construction, there is a bijection between \( \text{Prin}_{O(n)}(B) \) and \( \text{Vect}_n^{\text{orth}}(B) \).

If \( E \) also admits an orientation, we can take the oriented orthonormal frame bundle which is a principal \( SO(n)\)-bundle. Now we obtain a bijection between \( \text{Prin}_{SO(n)}(B) \) and \( \text{Vect}_n^{\text{orth}}(B) \), the collection of isomorphism classes of oriented real rank \( n \) vector bundles.

If \( E \) has rank \( 2n \) and is the underlying real vector bundle of a complex vector bundle, then one can take the bundle of complex frames which is a principal \( GL(n, \mathbb{C})\)-bundle. If \( E \) is equipped
with a hermitian metric, we can take the bundle of unitary frames which is a principal $U(n)$-bundle. As in the real case, there is a bijection $\text{Prin}_{GL(n,\mathbb{C})}(B)$ and $\text{Prin}_{U(n)}(B)$, and a bijection $\text{Prin}_{GL(n,\mathbb{C})}(B)$-bundles and $\text{Vect}_n^C(B)$, the collection of isomorphism classes of rank $n$ complex vector bundles.

**Theorem.** Let $G$ be a topological group. There is a space $BG$ and a principal $G$-bundle $G \to EG \to BG$ such that for every paracompact topological space $B$, isomorphism classes of principal $G$-bundles on $B$ are in bijection with $[B, BG]$.

The space $BG$ is unique up to homotopy and is called the classifying space. Milnor gave an explicit model for $BG$ using the join construction, see [5]. We call $G \to EG \to BG$ the universal principal $G$-bundle; it is characterised by the fact that $EG$ is weakly contractible; it follows from the long exact sequence in homotopy that $\pi_n(BG) \cong \pi_{n-1}(G)$. Given a map $f : B \to BG$, we can associate to it the principal $G$-bundle $f^*EG \to B$. If $P \to B$ is a principal $G$-bundle, a map $f : B \to BG$ such that $f^*EG \cong P$ is called a classifying map for $P$.

The association $G \to BG$ is functorial. In particular, given a continuous group homomorphism $\rho : H \to G$, there is an associated continuous map $B\rho : BH \to BG$. If $i : H \to G$ is inclusion, then the homotopy fiber of $Bi : BH \to BG$ is $G/H$.

**Characteristic Classes**

From the theorem, we see that there is a bijection between $\text{Vect}_n^+(B)$ and $[B, BSO(n)]$, as well as a bijection between $\text{Vect}_n^C(B)$ and $[B, BU(n)]$. The grassmannians $Gr^+_n(\mathbb{R}^\infty)$ and $Gr^+_n(\mathbb{C}^\infty)$ are explicit models for $BSO(n)$ and $BU(n)$, and the tautological bundles over them are the universal bundles.

One can show that $H^*(BSO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_2, \ldots, w_n]$ where $\deg w_i = i$. Given a principal $SO(n)$-bundle $P \to B$, we define $w_i(P) = f^*w_i$ where $f : B \to BSO(n)$ is any classifying map for $P$ — these are the Stiefel-Whitney classes for $P$. Note, the class $w_i(P)$ doesn’t depend on the choice of classifying map as homotopic maps induce the same map on cohomology.

Similarly, we have $H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n]$ where $\deg c_i = 2i$. Given a principal $U(n)$-bundle $P \to B$, we define $c_j(P) = f^*c_j$ where $f : B \to BU(n)$ is any classifying map for $P$ — these are the Chern classes of $P$.

The integral cohomology of $BSO(2n)$ is more complicated than that of $BU(n)$. There are elements $p_i \in H^{4i}(BSO(2n); \mathbb{Z})$ for $i = 1, \ldots, n$ and $e \in H^{2n}(BSO(2n); \mathbb{Z})$. Modulo torsion, these classes generate the cohomology, but not freely. More precisely, $H^*(BSO(2n); \mathbb{Q}) \cong \mathbb{Q}[p_1, \ldots, p_n, e]/(p_n - e^2)$. Given a principal $SO(2n)$-bundle $P \to B$, we define $p_i(P) = f^*p_i$ where $f : B \to BSO(2n)$ is any classifying map for $P$ — these are the Pontryagin classes of $P$. We define $e(P) = f^*e$ — this is the Euler class of $P$.

**Obstructions to the Existence of an Almost Complex Structure**

Let $p : BU(n) \to BSO(2n)$ be the map induced by the inclusion $i : U(n) \to SO(2n)$; i.e. $p = Bi$. Postcomposition with $p$ gives a map $[B, BU(n)] \to [B, BSO(2n)]$ and hence a map from complex rank $n$ vector bundles to orientable rank $2n$ real vector bundles; this just forgets the almost complex structure. We want to know when a principal $SO(2n)$-bundle comes from a principal $U(n)$-bundle, that is when $f : B \to BSO(2n)$ admits a lift $g : B \to BU(n)$. Suppose $g$ is a lift of $f$, i.e. then $f = g \circ p$. It follows that if $E$ is a complex rank $n$ vector bundle, $c_i(E) \equiv w_{2i}(E) \mod 2$ and $w_{2i+1}(E) = 0$.

The obstructions to a such a lift lie in $H^{k+1}(X; \pi_k(F))$ where $F$ is the homotopy fiber of $BU(n) \to BSO(2n)$. As the map $BU(n) \to BSO(2n)$ is induced by inclusion, the homotopy fiber is $SO(2n)/U(n)$ which can be identified with the space of linear complex structures on $\mathbb{R}^{2n}$ which are compatible with a given inner product and orientation. It is a closed manifold of dimension
In order to do obstruction theory, we need to determine the first non-zero homotopy group of $SO(2n)/U(n)$. From the long exact sequence in homotopy associated to the fibration $U(n) \to SO(2n) \to SO(2n)/U(n)$ together with the fact that $\pi_2(G) = 0$ for Lie groups we see that

$$0 \to \pi_2(SO(2n)/U(n)) \to \mathbb{Z} \to \pi_1(SO(2n)/U(n)) \to 0.$$  

As $\text{ker}(\mathbb{Z} \to \mathbb{Z}_2) \cong \mathbb{Z}$, regardless of the map, we see that $\pi_2(SO(2n)/U(n)) \cong \mathbb{Z}$. So either $\mathbb{Z} \to \mathbb{Z}_2$ is given by $1 \mapsto 1$, in which case $\pi_1(SO(2n)/U(n)) = 0$, or $1 \mapsto 0$, in which case $\pi_1(SO(2n)/U(n)) = 0$. Using the five lemma, we can show the following.

**Lemma.** For $n > 1$, $\pi_1(SO(2n)/U(n)) = 0$ and $\pi_2(SO(2n)/U(n)) \cong \mathbb{Z}$.

In fact, we see that $\pi_1(SO(2n)/U(n)) \cong \pi_1(SO(4)/U(2))$ and $\pi_2(SO(2n)/U(n)) \cong \pi_2(SO(4)/U(2))$ for all $n > 1$ (then use the fact that $SO(4)/U(2) = S^2$). More generally, $\pi_1(SO(2n+2)/U(n+1)) \cong \pi_i(SO(2n)/U(n))$ for $i \leq 2n - 2$. This is called the stable range (pass to the direct limit $SO/U$ which is $(\Omega^\infty)_0$ by Bott periodicity).

Therefore, the first obstruction to a lift $g$ lies in $H^3(B; \mathbb{Z})$. What is it? This is the hardest part of obstruction theory, actually identifying the obstructions. The following result gets us started, see Theorem 5.7 of [3].

**Theorem.** The first non-trivial obstruction is natural.

This means that the first obstruction to lifting $f : B \to BSO(2n)$ to $BU(n)$ is the pullback by $f$ of the first obstruction to lifting $id : BSO(2n) \to BSO(2n)$ to $BU(n)$, i.e. the obstruction to finding a section of $BU(n) \to BSO(2n)$. This obstruction lies in $H^3(BSO(2n); \mathbb{Z})$.

By the Universal Coefficient Theorem,

$$H^3(BSO(2n); \mathbb{Z}) \cong \text{Hom}(H_3(BSO(2n); \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_2(BSO(2n); \mathbb{Z}), \mathbb{Z}).$$

As $H_3(BSO(2n); \mathbb{Q}) \cong H^3(BSO(2n); \mathbb{Q}) = 0$, we see that $H_3(BSO(2n); \mathbb{Z})$ is torsion, so the first summand is zero. On the other hand, $\pi_1(BSO(2n)) = \pi_0(SO(2n)) = 0$, and $\pi_2(BSO(2n)) = \pi_1(SO(2n)) = \mathbb{Z}_2$ as $n > 1$, so by Hurewicz, $H_2(BSO(2n); \mathbb{Z}) \cong \mathbb{Z}_2$. So $H^3(BSO(2n); \mathbb{Z}) \cong \mathbb{Z}_2$. What is the non-zero element?

Consider the short exact sequence of abelian groups $0 \to \mathbb{Z} \xrightarrow{x_2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$. This induces a long exact sequence in cohomology

$$\cdots \to H^2(BSO(2n); \mathbb{Z}) \xrightarrow{x_2^*} H^2(BSO(2n); \mathbb{Z}) \xrightarrow{\rho} H^2(BSO(2n); \mathbb{Z}_2) \xrightarrow{\beta} H^3(BSO(2n); \mathbb{Z}) \to \cdots$$

where $\rho$ is reduction modulo 2, and $\beta$ is the coboundary map which is called the Bockstein associated to the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{x_2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$. By exactness, $x \in H^2(BSO(2n); \mathbb{Z})$ satisfies $\beta(x) = 0$ if and only if there is $y \in H^2(BSO(2n); \mathbb{Z})$ such that $\rho(y) = x$; we usually write $y \equiv x \mod 2$ and say $y$ an integral lift for $x$. Recall, $w_2 \in H^2(BSO(2n); \mathbb{Z})$ is non-zero and $H^2(BSO(2n); \mathbb{Z}) \cong \text{Hom}(H_2(BSO(2n); \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_1(BSO(2n); \mathbb{Z}), \mathbb{Z}) = \text{Hom}(\mathbb{Z}_2, \mathbb{Z}) \oplus \text{Ext}(0, \mathbb{Z}) = 0$ so $w_2$ has no integral lift, and therefore $W_3 := \beta(w_2) \neq 0$ and hence must be the non-zero element of $H^3(BSO(2n); \mathbb{Z})$.

It turns out that the first obstruction to the existence of a section of $BU(n) \to BSO(2n)$ is $W_3$, the argument will be given later (see the section on the six-dimensional case). Therefore, the first obstruction to the existence of an almost complex structure on an orientable real rank 2n vector bundle $E$ is $f^*W_3$ where $f : B \to BSO(2n)$ is any classifying map. As the Bockstein is natural, $f^*W_3 = f^*\beta(w_2) = \beta(f^*w_2) = \beta(w_2(E)) = W_3(E)$. Note that $W_3(E) = 0$ if and only if $w_2(E)$ has an integral lift. Note, this shouldn’t be completely surprising as $c_1(E) \equiv w_2(E) \mod 2$ (so $W_3(E) = 0$ is clearly a necessary condition). What wasn’t clear from the beginning is that this is all that’s required to lift $B^{(3)} \to BSO(2n)$ to $B^{(3)} \to BU(n)$, there could have been other conditions.

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1Note, if $G$ is a topological group, $\pi_2(G)$ is not necessarily zero. For example, $\Omega X$ has the homotopy type of a topological group for any space $X$ and $\pi_2(\Omega X) = \pi_3(X)$ which can be arbitrary.
Theorem. Let $M^{2n}$ be an orientable smooth manifold with $n > 1$. The first obstruction to $M$ admitting an almost complex structure is $W_3(M)$.

Note, if $g : B^{(3)} \to BU(n)$ is defined, then $c := g^*c_1 \in H^2(B^{(3)}; \mathbb{Z}) \cong H^2(B; \mathbb{Z})$. This is important as further obstructions will be phrased in terms of $c$. In particular, if $g : B \to BU(n)$ can be defined, then $c$ will be the first Chern class of the corresponding complex vector bundle.

One might predict that the other obstructions will just be the necessary conditions $w_{2i+1}(E) = 0$ and $W_2(E) = 0$ (i.e. $w_2(E)$ has an integral lift). However, these are not sufficient. For example, they are satisfied by $E = TS^{2n}$ for every $n$, but the only spheres which admit almost complex structures are $S^2$ and $S^6$.

Now let's stick to a smooth manifold $M$ and let $f$ classify its tangent bundle.

FOUR-DIMENSIONAL CASE

In this case, $SO(4)/U(2) = S^2$. So there is one more potential obstruction in $H^4(M; \pi_3(S^2)) = H^4(M; \mathbb{Z})$. As $M$ is assumed to be oriented, this group is zero if $M$ is not closed, otherwise it is $\mathbb{Z}$ if it is closed. So, if $M$ is a non-compact, orientable four-manifold, it admits an almost complex structure if and only if $W_3(M) = 0$.

If $M$ is closed, then there is a genuine second obstruction. It is $c_1^2 - (2c(M) + p_1(M))$. Said another way, $c$ must satisfy $\int_M c^2 = 2\chi(M) + 3\tau(M)$. Again, it is not hard to see that this condition is necessary using the Hirzebruch signature theorem.

Note, in the closed case, the first obstruction always vanishes ($M$ is spin), so you can always find $c$ with $c \equiv w_2(M) \mod 2$, however, it may not be possible to choose one such that the second obstruction vanishes. This is the case for $M = S^4$ for example: $c$ must be 0, so $\int_M c^2 = 0$ while $2\chi(S^4) + 3\tau(S^4) = 4$.

Theorem. (Wu) Let $M$ be a closed oriented smooth four-manifold. Then $M$ admits an almost complex structure with $c_1(M) = 0$ if and only if

1. $c \equiv w_2(M) \mod 2$
2. $\int_M c^2 = 2\chi(M) + 3\tau(M)$.

SIX-DIMENSIONAL CASE

In this case $SO(6)/U(3) = \mathbb{CP}^3$. From the fibration $S^1 \to S^7 \to \mathbb{CP}^3$, we see that $\pi_i(\mathbb{CP}^3) = \pi_i(S^7) = 0$ for $i = 3, 4, 5, 6$. So there are no further obstructions.

Theorem. Let $M$ be an orientable six-manifold. Then $M$ admits an almost complex structure if and only if $W_3(M) = 0$.

Unlike in the four-dimensional case, the vanishing of $W_3$ is not automatic in six-dimensions. One example is $S^1 \times (SU(3)/SO(3))$; the manifold $SU(3)/SO(3)$ is known as the Wu manifold.

Now we can finally justify why the first obstruction to the existence of a section of $BU(n) \to BSO(2n)$ is $W_3$. If it weren't, the obstruction would vanish and hence every orientable six-manifold would admit an almost complex structure, including $S^1 \times (SU(3)/SO(3))$. But then $w_2(S^1 \times (SU(3)/SO(3)))$ would have an integral lift (given by the first Chern class), but this is impossible.

One example where the obstruction vanishes is $S^6$. This is one explanation for the existence of an almost complex structure on $S^6$.

The primary obstruction always vanishes for spheres (i.e. $S^{2n}$ is spin), but only $S^2$ and $S^6$ admit almost complex structures, so we see that in dimensions other than 2 and 6, there are always additional obstructions.
REFERENCES


