Lecture 1: Obstruction Theory, Eilenberg-Maclane spaces, Postnikov towers

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1 Obstruction Theory

Fix a simply connected space Y with basepoint y_0.

Lemma 1.1. *For any $n \geq 2$ the homotopy classes of maps of S^n to Y, denoted $[S^n, Y]$ are in natural bijective correspondence with $\pi_n(Y, y_0)$ Let $\{D_i\}_i$ be a finite set of disjoint, closed $(n+1)$-balls in S^{n+1} and denote by D their union. If $f\colon S^{n+1} \setminus \mathrm{int}(D) \to Y$, then the sum of the homotopy elements represented by $f|_{\partial D_i}$ is the trivial element in $\pi_n(Y, y_0)$.*

Let X be a CW complex and denote by $X^{(k)}$ its k-skeleton. Suppose that Y is a simply connected space with basepoint y_0 and suppose that $f\colon X^{(k-1)} \to Y$ is a continuous map. Then the association to each oriented k-cell e^k of the element in $\pi_{k-1}(Y, y_0)$ corresponding to the composition

$$S^{k-1} \xrightarrow{\eta_{e^k}} X^{(k-1)} \xrightarrow{f} Y,$$

where η_{e^k} is the attaching map for e^k, defines a CW k-cochain on X with coefficients in $\pi_{k-1}(Y, y_0)$, called the obstruction cochain for extending f to the k-skeleton. It is denoted $\mathcal{O}(f)$.

Lemma 1.2. *$\mathcal{O}(f)$ is a cocycle. If $g\colon X^{(k-1)} \to Y$ has the property that $f|_{X^{(k-2)}} = g|_{X^{(k-2)}}$ then $\mathcal{O}(f) - \mathcal{O}(g)$ is a coboundary. Conversely, given any coboundary, dc, there is a map $g\colon X^{(k-1)} \to Y$ agreeing with f on $X^{(k-2)}$ such that $\mathcal{O}(f) - \mathcal{O}(g) = dc$.*

Proof. To show that $\mathcal{O}(f)$ is a cocycle is to show that $\mathcal{O}(f)$ vanishes on the relative homology class in $H_k(X^{(k)}, X^{(k-1)})$ represented by the attaching map $\varphi\colon S^k \to X^{(k)}$ for each $(k+1)$-cell of X. By a homotopy of the attaching map $S^k \to X^{(k)}$ we can assume that there are a finite number of disjoint closed disks $\{D_i\}$ in S^k and under the attaching map the interior of each disk maps homeomorphically onto an open k-cell of X. Furthermore, the complement of the interiors of these disks maps to $X^{(k-1)}$. The relative homology class represented by the sphere is a sum of the k-handles each counted with the multiplicity given by the number of disks in the sphere mapping onto it, counted with sign. Thus, the evaluation of $\mathcal{O}(f)$ on this chain is $\alpha = \sum_i [f \circ \iota_i|_{\partial D_i}] \in \pi_k(Y)$ where ι_i is the inclusion of D_i to S^k. But the $\cup_i \partial D_i$ is the boundary of $S^k \setminus (\cup_i \mathrm{int}(D_i)$ which is mapped by the attaching map to $X^{(k-1)}$. It follows from Lemma 1.1 that α is the zero element of $\pi_{k-1}(Y, y_0)$. This shows that $\mathcal{O}(f)$ vanishes on the boundary of each generator of $C_{k+1}(X)$. Hence it vanishes on the image of $\partial\colon C_{k+1}(X) \to C_k(X)$, which is the definition of its being a cocycle.

If f and g agree on the $(k-2)$-skeleton, then for each $(k-1)$-cell we have an element of $\pi_{k-1}(Y, y_o)$, given by the difference of the restriction of f and g to the cell. [For each $(k-1)$-cell we have the composition of the map of the closed cell into $X^{(k-1)}$ followed by f and followed by g give us two maps of the $(k-1)$-disk into Y that agree on the boundary. Thus, their difference is the map on the sphere agreeing with f on the upper hemisphere and g on the lower hemisphere. This function on the generators produces a homomorphism $C_{k-1}(X) \to \pi_{k-1}(Y, y_0)$.] It follows easily that $\mathcal{O}(f) - \mathcal{O}(g)$ is equal to the coboundary of this element.

Conversely, given $f\colon X^{(k-1)} \to Y$ and a homomorphism from $c\colon C_{k-1}(X) \to \pi_{k-1}(Y, y_0)$, we define a map g that agrees with f off of a small disk in the interior of each $(k-1)$-cell and with the property that the difference (as defined above) of the restrictions of f and g to this small disk is c evaluated on the cochain represented by oriented cell. This produces a $g\colon X^{(k-1)} \to Y$ agreeing with f on $X^{(k-2)}$ and such that $\mathcal{O}(f) - \mathcal{O}(g) = dc$. $\square$

In view of the above lemma, we see that the cohomology class $[\mathcal{O}(f)] \in H^k(X; \pi_{k-1}(Y, y_0))$ is the obstruction to extending $f\big|_{X^{(k-2)}}$ over $X^{(k)}$, in the sense that the cohomology class $[\mathcal{O}(f)]$ vanishes if and only if there is such an extension.

There is a relative version: If (X, X_0) is a pair consisting of a CW complex and a subcomplex and if we have a map $f_0\colon X_0 \to Y$ and an extension $f\colon X^{(k-1)} \cup X_0 \to Y$, then there is an obstruction cocycle $\mathcal{O}(f) \in C^k(X, X_0; \pi_{k-1}(Y, y_0))$ whose cohomology class is the obstruction to extending $f\big|_{(X^{(k-2)} \cup X_0)}$ to a map of $X^k \cup X_0 \to Y$.

Applying this to $(X \times I, X \times \partial I)$ we see that given two maps f_0, f_1 from X to Y and a homotopy between their restrictions to $X^{(k-1)}$. The obstruction to extending the restriction of this homotopy to $X^{(k-2)}$ over $X^{(k)}$ is an obstruction class lying in

$$H^k(X\times I, X\times\partial I;\pi_{k-1}(Y,y_0)) = H^{k-1}(X;\pi_{k-1}(Y,y_0)).$$

Theorem 1.3. *(Whitehead's Theorem) Suppose that X and Y are connected CW complexes and $f\colon X\to Y$ is a continuous map inducing an isomorphism on all homotopy groups Then f is a homotopy equivalence.*

Proof. (Sketch) First suppose that X is a sub CW complex of Y and the inclusion of $X\subset Y$ induces an isomorphism on homotopy groups. We show that there is a deformation retraction of Y to X. This is proved by induction on dimension. Suppose by induction we have a deformation $(X\cup Y^{(k-1)})\times I\to Y$ from the inclusion to a map to X, a deformation that is the trivial homotopy on X, meaning that it is a *strong* deformation retract. Consider an k-cell e of Y. If it is contained in X, then the homotopy is already defined on it. Otherwise, consider the k-disk $e\times\{0\}\cup(\partial(e)\times I)$. The already defined deformation is defined on ∂e and maps $\partial e\times I$ to a homotopy from the inclusion of $\partial e\subset Y$ to a map of ∂e to X. Thus, the union of the image of $\partial e\times I$ under the deformation and the inclusion of e into Y represents a relative homotopy class in $\pi_k(Y,X)$. By the homotopy exact sequence, this homotopy group is trivial. Hence there is a map $e\times I\to Y$ extending the given map on $e\times\{0\}\cup\partial e\times I$. This map is a homotopy from the inclusion of $e\subset Y$ to a map of $e\to X$ extending the homotopy on ∂e. We can do this simultaneously on all k-cells at once giving an extension of the strong deformation retract from $X\cup Y^{(k-1)}$ to $X\cup Y^{(k)}$. We continue by induction over k to produce the strong deformation retraction of Y to X.

The general case follows by first deforming f until, for each k, it maps the k-skeleton of X to the k-skeleton of Y (which is possible by obstruction theory since $\pi_i(Y,Y^{(k)})$ is trivial for all $i\leq k$) and then replacing Y by the mapping cylinder M_f. The mapping cylinder has a cell decomposition whose cells are those of X, the product of those of X with I, and those of Y. Applying the special case to $X\subset M_f$ gives a strong deformation retraction of the mapping cylinder to X. Restricting the retraction to Y produces a map $Y\to X$ that is a homotopy inverse to f. $\square$

2 Eilenberg-MacLane spaces

2.1 The definition and basic properties

Fix an abelian group π and an integer $n\geq 1$.

Lemma 2.1.

- *There is a connected CW complex X with $\pi_i(X, x_0) = 0$ for $i \neq n$ and with $\pi_n(X, x_0)$ isomorphism to π.*
- *If we fix two such, X and X' and choose identifications of the n^{th} homotopy group of each with π, then there is a homotopy equivalence $f \colon X \to X'$ inducing an isomorphism $\pi_n(X, x_0) \to \pi_n(X', x_0')$ compatible with these identifications. Any two such f are homotopic.*

Proof. First we construct a CW complex with the required homotopy groups. We define $X^{(n-1)} = \{x_0\}$. Take a presentation

$$\oplus_J \mathbb{Z} \xrightarrow{a} \oplus_I \mathbb{Z} \to \pi \to 0$$

and define $X^{(n)}$ be the adjunction space obtained from $\{x_0\} \coprod (\coprod_I D^n)$ using the collapsing map $\partial(\coprod_I D^n) \to \{x_0\}$. Clearly, $\pi_n(X^{(n)})$ is identified with $\oplus_I \mathbb{Z}$. Now let $X^{(n+1)}$ be the adjunction space obtained from $X^{(n)} \coprod (\coprod_J D^{n+1})$ by the map that for each $j \in J$ restricts to a map $\partial(D_j) \to X^{(n)}$ representing the element $a(j) \in \oplus_I \mathbb{Z}$. The n^{th} homotopy group of this space is identified with the cokernel of $a \colon \oplus_J \mathbb{Z} \to \oplus_I \mathbb{Z}^I$, i.e., to π. Suppose for some $k \geq n+1$ we have a CW complex $X^{(k)}$ extending the complex $X^{(n+1)}$ with the following properties:

- $X^{(n+1)}$ is the $(n+1)$-skeleton of $X^{(k)}$.
- The inclusion $X^{(n+1)} \subset X^{(k)}$ induces an isomorphism on π_n.
- $\pi_i(X^{(k)}) = 0$ for all $n+1 \leq i < k$.

Choose a generating set $\{\alpha_r\}_{r \in R}$ for $\pi_k(X^{(k)})$ and attach $(k+1)$-cells $\coprod_{r \in R} D_r^{k+1}$ by a map with the property that its restriction to ∂D_r^{k+1} represents the homotopy element α_r. The result is defined to be $X^{(k+1)}$ and it satisfies the same three conditions with k replaced by $k+1$.

This establishes the existence of a space X with the required homotopy groups. Suppose that X' is another such space. We construct a map $f \colon X \to X'$ inducing an isomorphism on π_n compatible with the given identifications with π. First we send $X^{(n-1)}$ (which recall is a point) to a 0-cell in X'. Now each n cell of X represents an element of $\pi_n(X) = \pi$. We define the map f_n on this n-cell to represent the corresponding element in $\pi_n(X') = \pi$. This map extends to a map f_{n+1} over the $(n+1)$-skeleton of X since the attaching map for any $(n+1)$-cell of X represents the trivial element in $\pi_n(X) = \pi$ and hence the composition of f_n with the attaching map for this cell is a homotopically trivial map of $S^n \to X'$. Given $f_{n+1} \colon X^{(n+1)} \to X'$ inducing an isomorphism on π_n we inductively extend it over all of X by obstruction theory, using the fact that $\pi_i(X') = 0$ for $i > n$. [Notice when extend the map $X^{(n+1)} \to X'$ to $X^{(n+2)}$ we do not have to vary it over the $(n+1)$-skeleton since $\pi_{n+1}(X') = 0$.] By Whitehead's theorem, the resulting map $X \to X'$ is a homotopy equivalence.

The proof of the uniqueness of the map f up to homotopy is left as an exercise to the reader.

$\square$

This means that such a space with an identification of its n^{th} homotopy group with π is unique up to homotopy equivalence, itself unique up to homotopy. That is to say in the homotopy category there is a unique such object up to canonical isomorphism with only one non-trivial homotopy group, that group being in dimension n and identified with π. Such a space together with an identification of its non-zero homotopy group with a fixed group π is denoted $K(\pi, n)$. It is an *Eilenberg-MacLane space*.

The Hurewicz Theorem and the Universal Coefficient Theorem imply:

Lemma 2.2.

$$\widetilde{H}_*(K(\pi, n); \mathbb{Z}) = \begin{cases} 0 & \textit{if } * < n \\ \pi & \textit{if } * = n \end{cases}.$$

There is a distinguished element $\iota_{\pi,n} \in H^n(K(\pi, n); \pi)$ corresponding to the identity homomorphism of π to itself

There is a generalization of the uniqueness argument given above.

Lemma 2.3. *For any CW complex X, the set of homotopy classes of maps $[X, K(\pi, n)]$ is canonically identified with the elements of the group $H^n(X; \pi)$. The bijection is induced by the natural correspondence $f\colon X \to K(\pi, n)$ maps to $f^*(\iota_{\pi,n})$.*

Proof. Let X be a CW complex let $\alpha \in H^n(X; \pi)$ be a class. We construct a map $f\colon X \to K(\pi, n)$ with $f^*\iota_n = \alpha$. We begin with the map $X^{(n-1)} \to \{x_0\} \in K(\pi, n)$. Now fix a cocycle $\widetilde{\alpha} \in C^n(X; \pi)$ representing α. Then for each n-cell e, orienting the cell gives an element $[e] \in C_n(X)$. We define the map f_n such that its restriction to e represents in the element in $\langle \widetilde{\alpha}, [e] \rangle \in \pi = \pi_n(K(\pi, n))$. There is no obstruction to extending the map over $X^{(n+1)}$ since for every $(n+1)$-cell its attaching map gives a sphere representing a cycle in $C_n(X)$ and $\widetilde{\alpha}$ evaluates on this sphere to give the value of α on the homology class of the sphere in X. But the homology class of the sphere is trivial since the sphere bounds a disk in X. Once we have f_{n+1}, the fact that the higher obstructions vanish means that we can extend this map to a map $f\colon X \to K(\pi, n)$. Clearly, $f^*\iota_n = \alpha$.

For maps from X to $K(\pi, n)$, denoted f and g, to be homotopic, it is necessary that $f^*(\iota_n) = g^*(\iota_n)$. Conversely, suppose we have maps f and g from X to $K(\pi, n)$ with $f^*(\iota) = g^*(\iota)$. Since the homotopy groups of $K(\pi, n)$ vanish in dimensions less than n, we can assume that f and g both map $X^{(n-1)}$ to the basepoint. From there it is easy to see that the obstruction to extending the constant homotopy on the $(n-2)$-skeleton to a homotopy defined over the n-skeleton is $f^*(\iota) - g^*(\iota)$. Thus, if $f^*(\iota) = g^*(\iota)$, then the restrictions of f and g to $X^{(n)}$ are homotopic. The obstructions to extending the homotopy from $X^{()n)}$ to all of X vanish since all the higher obstruction groups vanish. $\square$

This suggests that $K(\pi, n)$ is a group object in the homotopy category, i.e., an H-space, and indeed the H-space structure is the map (well-defined up to homotopy) $K(\pi, n) \times K(\pi, n) \to K(\pi, n)$ that corresponds to

$$\iota_{\pi,n} \otimes 1 + 1 \otimes \iota_{\pi,n} \in H^n(K(\pi, n) \times K(\pi, n); \pi).$$

The induced group structure on $[X, K(\pi, n)]$ is identified with the usual addition on $H^n(X; \pi)$. All of this is summarized in the statement: *The Eilenberg-MacLane space $K(\pi, n)$ is the classifying space for the homotopy functor $H^n(\cdot; \pi)$.*

2.2 Hurewicz and Serre fibrations

Lemma 2.4. *Let $P \to X$ be a Hurewicz fibration with fiber over the base point x_0 being F. Fix a basepoint f_0 for F. Then there is a long exact sequence of homotopy groups:*

$$\cdots \longrightarrow \pi_n(F, f_0) \longrightarrow \pi_n(P, f_0) \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_{n-1}(F, f_0) \longrightarrow \cdots.$$

Proof. This is immediate from the homotopy lifting property. $\square$

Remark 2.5. Since one only needs the homotopy lifting property for maps of disks and spheres, in fact the same result holds for Serre fibrations.

If $p\colon E \to B$ is a Hurewicz (Serre) fibration and $f\colon X \to B$ is a continuous map, we define $f^*E \subset X \times E$ to be the set of pairs (x, e) with $f(x) = p(e)$. It is an easy exercise in the definitions to see that the natural map $f^*E \to X$ induced by projection onto the first coordinate is a

Hurewicz (Serre) fibration. Also, for any $\in X$, the fiber of $f^*E \to X$ over x is canonically identified with the fiber of $p\colon E \to B$ over $f(x)$.

Let X be a path-connected space with base point x_0. Define the path space $\mathcal{P}(X, x_0)$ to be the set of paths $\omega\colon [0,1] \to X$ with the property that $\omega(0) = x_0$. (The topology is the compact open topology.) The map $p_1\colon \mathcal{P}(X, x_0) \to X$ given by $\omega \mapsto \omega(1)$ is continuous and is a Hurewicz fibration. The fiber of $\mathcal{P}(X, x_0) \to X$ over the base point x_0 is called the based loop space of X based at x_0 and denoted $\Omega(X, x_0)$. Since the path space is contractible, we have

Corollary 2.6. *Let $\omega_0 \in \Omega(X, x_0)$ be the constant loop at x_0. There is a natural isomorphism*

$$\pi_n(\Omega(X, x_0), \omega_0) \cong \pi_{n+1}(X, x_0).$$

Corollary 2.7. *The fiber of $p\colon \mathcal{P}(K\pi, n)) \to K(\pi, n)$ is $K(\pi, n-1)$. That is to say there is a fibration*

$$K(\pi, n-1) \to \mathcal{P}(K(\pi, n)) \to K(\pi, n).$$

2.3 Action of the fundamental group of the base on the (co)homology of the fiber

Let us define the action of the fundamental group of the base on the homology and cohomology of the fiber of a Serre fibration.

Let $\pi\colon E \to B$ be a Serre fibration. Fix a basepoint $b_0 \in B$ and let $F_0 = \pi^{-1}(b_0)$ be the fiber over the basepoint. Then there is an action of $\pi_1(B, b_0)$ on $H_*(F_0)$ defined as follows. Let $h\colon \Sigma^k \to F_0$ be a singular cycle. That is to say Σ is an n dimensional simplicial complex and $H_n(\Sigma) = \mathbb{Z}$ with generator $[\Sigma]$. Fix a loop γ based at b_0. Using the homotopy lifting property for Serre fibrations we inductively extend the map $\Sigma \to F_0$ to a map $H\colon \Sigma \times I \to E$ whose projection to B is the composition $\Sigma \times I \to I \xrightarrow{\gamma} B$. In particular the restriction of H to $\Sigma \times \{1\}$ is a map $\Sigma \to F_0$ representing a homology class, which is by definition the action of γ on the class $h_*([\Sigma])$. An analogous argument shows that the action depends only on the image of γ in $\pi_1(B, b_0)$ and the homology class of Σ, not the representing cycle. This defines

$$\pi_1(B, b_0) \times H_*(F_0) \to H_*(F_0).$$

The action on cohomology is the dual action. [To be precise an integral cohomology class is determined by its evaluation of integral homology classes and its evaluation in $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{Z}/n\mathbb{Z}$ homology for all n. Thus, we also need to define the action (in the same way) on $\mathbb{Z}/n\mathbb{Z}$-homology and show (as is obvious) that these actions are compatible with change of coefficient maps.] Then we can define the action on integral cohomology as the 'dual' to all of these actions.

Provided that the fiber is path connected, there is a similarly defined action of the fundamental group of the base on the homotopy groups of the fiber.

Remark 2.8. If $\pi\colon E \to B$ is a Hurewicz fibration, then covering any loop γ in the base based at b_0 there is a self-map of $\mu_\gamma\colon F_0 \to F_0$ well defined up to homotopy. Then the actions defined above for a Serre fibration are the action of μ_γ on homotopy, homology and cohomology of the fiber.

Lemma 2.9. *Let B be a path connected space and fix a basepoint $b_0 \in B$. If $\pi\colon E \to B$ is a Serre fibration and if the actions of $\pi_1(B, b_0)$ on the homology (cohomology, homotopy groups) of the fiber F_0 over b_0, then for any $b \in B$ denoting by F_b the fiber over b, there is a canonical identification of the homology (cohomology, homotopy groups) of F_b with those of F_0.*

In this case the homology (cohomology, homotopy groups) of the fiber forms a trivial local system over the base. In general, these homologies (cohomologies, homotopy groups) form a local system determined by the action of $\pi_1(B, b_0)$ on the homology (cohomology, homotopy groups) of F_0.

2.4 Obstruction theory for sections of fibrations

Lemma 2.10. *Suppose that $F \to P \to X$ is a fibration with X a simply connected CW complex and simply connected fiber. Suppose that the fundamental group of the base acts trivially on the homotopy groups of the fiber. Then the obstructions to a section $X \to P$ lie in $H^*(X; \pi_{*-1}(F))$.*

Proof. The statement means that given a section over $X^{(k-1)}$ the obstruction to extending the restriction of that section to $X^{(k-2)}$ to a section defined on $X^{(k)}$ is an element in $H^k(X; \pi_{k-1}(F))$. Suppose we have a section over $X^{(k-1)}$. For each n-cell e consider the pull back $f_e^*(E) \to D^k$ of the fibration by the natural map $f_e\colon D^k \to X^{(k-1)}$ (whose image is the closure of e). The section over $X^{(k-1)}$ determines a section of $f_e^*(E)$ defined over ∂D^k. From the homotopy lifting property we see that there is a map $F_0 \times D^k \to f_e^*(E)$ where F_0 is the fiber over a point $x_0 \in D^k$. This map is a homotopy equivalence. Thus, the section over the boundary followed by the homotopy inverse $f_e^*(E) \to F_0 \times D^k$ followed by the projection to F_0 determines a homotopy element in $\pi_{k-1}(F_0)$. Since the homotopy groups of all fibers are identified, this element can be viewed as an element in $\pi_{k-1}(F)$. This defines the obstruction cochain in $C^k(X; \pi_{k-1}(F))$, which vanishes if and only if the section over $X^{(k-1)}$ extends to $X^{(k)}$. Arguments analogous to the case of maps to spaces show that this cochain is a cocycle and by varying the section over the open $(k-1)$-cells we can vary the obstruction class by an arbitrary coboundary. Thus, the cohomology class of this cocycle is the obstruction to extending the restriction of the section to $X^{(k-2)}$ to a section of $X^{(k)}$. $\square$

Corollary 2.11. *In the previous result, if F is $K(\pi, n-1)$ there is only one obstruction to a section and it lies in $H^n(X; \pi_{n-1})$. This obstruction class is classified by a map $X \to K(\pi, n)$.*

A fibration with fiber $K(\pi, n)$ is said to be a *principal fibration* if the action of the fundamental group of the base on the non-trivial homotopy group of the fiber is the trivial action.

Theorem 2.12. *Let B a CW complex and let $K(\pi, n) \to E \to B$ be a principal fibration. The obstruction to a section is an element in $H^{n+1}(B; \pi)$ or equivalently is a homotopy class of maps $f\colon B \to K(\pi, n+1)$. The fibration*

$$K(\pi, n) \to E \to B$$

is equivalent in the homotopy category to the pull-back via f of the fibration

$$K(\pi, n) \to \mathcal{P}(K(\pi, n+1)) \to K(\pi, n+1).$$

The proof of this result is left as an exercise to the reader. (See the discussion after the proof of Theorem 3.4.

3 Postnikov towers

Definition 3.1. A *Postnikov tower* is a sequence of spaces

$$\frac{\{pt\} buildrel p_2}{\longleftarrow X_2 \xleftarrow{p_3} X_3 \xleftarrow{p_4} \cdots}$$

with each X_n having a basepoint x_n with $p_n(x_n) = x_{n-1}$, where $X_{n+1} \to X_n$ is homotopy equivalent to Hurewicz fibration with fiber $K(\pi_{n+1}, n+1)$ for some abelian group π_{n+1}.

Definition 3.2. Let X be a simply connected CW complex with base point x_0. A *Postnikov tower for X* is a Postnikov tower

$$\{pt\} = X_0 \xleftarrow{p_2} X_2 \xleftarrow{p_3} X_3 \longleftarrow \cdots$$

together with maps $f_n\colon (X,x_0) \to (X_n,x_n)$ such that the following hold for all $n \geq 2$

- $p_n \circ f_n$ is homotopic to f_{n-1}
- $(f_n)_*\colon \pi_k(X,x_0) \to \pi_k(X_n,x_n)$ induces an isomorphism for $k \leq n$.

Corollary 3.3. *In a Postnikov tower for X the composition of the projection mapping p_i define, for every $m \geq n$, maps $p_{m,n}\colon X_m \to X_n$ that induce isomorphisms on π_i for all $i \leq n$,*

Theorem 3.4. *Every simply connected CW complex has a Postnikov tower.*

Proof. For every $k \geq 2$ we define X_k as follows. Begin with X and inductively on $\ell \geq k+1$ attach $(\ell+1)$-cells to kill π_ℓ of the space created at the previous step. The map $X \to X_k$ is the natural inclusion. To define the map $X_{k+1} \to X_k$ we take the identity map from the copy of $X \subset X_{k+1}$ to the copy of $X \subset X_k$. We can extend this map over the other cells of X_{k+1} (all of which have dimensions at least $k+2$), since the homotopy groups of X_k vanish in degrees $\geq k+1$. This gives us a tower of spaces

$$\{pt\} \longleftarrow X_2 \longleftarrow X_3 \longleftarrow \cdots$$

with maps $X \to X_k$ making the triangles commute.

The last thing to check is that up to homotopy $X_k \to X_{k-1}$ is a fibration with fiber homotopy equivalent to a $K(\pi_k(X),k)$. We can assume that $X_k \to X_{k-1}$ is a fibration with fiber F. Then the homotopy lifting property implies that $\pi_*(F) = \pi_{*+1}(X_k, X_{k-1})$. By construction the only non-trivial relative homotopy group is π_{k+1}, which is identified with $\pi_k(X)$. It follows that the fiber has only one non-trivial homotopy group, that being in degree k and the group being identified with $\pi_k(X)$. Hence, the fiber is homotopy equivalent to $K(\pi_k(X),k)$. Since X_{k-1} is simply connected the fibration is a principal fibration with fiber $K(\pi_k(X),k)$. $\square$

There is another way to think about the last argument. We know that $\pi_*(X_k, X_{k-1})$ is trivial for $* < (k+1)$ and identified with $\pi_k(X)$ for $* = (k+1)$ Hence, $H_{k+1}(X_{k-1}, X_k) = \pi_k(X)$ and there is a distinguished element in $H^{k+1}((X_{k-1}, X_k); \pi_k(X))$ corresponding to the identity homomorphism. Let M_i be the mapping cylinder associated to the inclusion $X_k \subset X_{k-1}$. then the distinguished element determines a homotopy class of a map of pairs

$$(M_i, X_k) \to (K(\pi_k(X), k+1), p_0).$$

That is to say we have a map $\varphi\colon X_{k-1} \to K(\pi_k(X), k+1)$ and a homotopy from $\varphi|_{X_k}$ to the constant map to the basepoint. This gives a map $\psi\colon X_k \to \mathcal{P}(K(\pi_k(X), k+1))$ with the property that the composition of the projection to $K(\pi_k(X), k+1)$ following ψ is equal to the restriction of φ to X_k, and hence a lifting of the restriction of φ to X_k to the path space. This shows that in the homotopy category the map $X_k \to X_{k-1}$ is induced from the path space fibration over $K(\pi_k(X), k+1)$ by a map $X_{k-1} \to K(\pi_k(X), k+1)$.

Definition 3.5. The maps $X_k \to K(\pi_{k+1}(X), k+2)$ are equivalent to cohomology classes $H^{k+2}(X_k; \pi_{k+1}(X))$; the latter are called the *k-invariants* of X.

4 Exercises

1. Show that for a simply connected space and two basepoints y_0, y_1 the groups $\pi_n(Y, y_0)$ and $\pi_n(Y, y_1)$ are canonically identified.

2. Let Y be a path connected space with basepoint y_0. For any module M over the integral group ring $\mathbb{Z}[\pi_1(Y, y_0)]$ define $H^*(Y; M)$ by considering the chain complex $C_*(\widetilde{Y})$ of the universal covering $(\widetilde{Y}, \widetilde{y}_0)$. The free action of π on this chain complex makes it a chain complex in the category of (free) $\mathbb{Z}[\pi_1(Y, y_0)]$-modules. We form

$$C^*(Y, M) = \mathrm{Hom}_{\mathbb{Z}[\pi_1(Y,y_0)]}(C_*(\widetilde{Y}), M).$$

The cohomology of this cochain complex is $H^*(Y; M)$. Show that if M is $\mathbb{Z}$ with the trivial $\pi_1(Y, y_0)$-action, then the result is the usual integral cohomology of Y.

3. Using the cell decomposition of S^n with two cells in each degree, compute $H^*(\mathbb{R}P^n; \mathbb{Z})$. Let $\mathbb{Z}^t$ be the integers with the non-trivial action of $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$. Compute $H^*(\mathbb{R}P^n; \mathbb{Z}^t)$ [Here, we take $n \geq 2$.] Compute $H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$.

4.Compute $H^*(\mathbb{C}P^n; \mathbb{Z})$ from a 'small' cell decomposition.

5. Show that up to weak homotopy equivalence any $f\colon X \to Y$ is a Hurewicz fibration. [Hint: first show we can assume that f is an inclusion. Then use a relative path space construction to replace the inclusion by a fibration.

6. Show that for any not necessarily abelian, discrete group π there is a space $K(\pi, 1)$ whose fundamental group is π and whose higher homotopy groups are trivial. Show that any two such are homotopy equivalent by a homotopy equivalence computing with given identifications of the fundamental groups with π.

7. Show that Postnikov towers and CW complex structures are 'dual' in the following sense. If $\{X_m, f_m\}$ is a Postnikov tower for X and if $Y^{(n)}$ is a CW complex structure for Y then we have two decreasing filtrations on $[Y, X]$. The first F_n, is the homotopy classes $[f]$ whose restriction to $Y^{(n)}$ is a constant map, and the second F'_n is the homotopy classes of maps whose projection to X_n is trivial. Show that these filtrations agree.

8. Suppose that we have a finite tower

$$X = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = \{pt\}$$

where each $X_k \to X_{k-1}$ is homotopy equivalent to a Hurewicz fibration with fiber a $K(\pi_k, 1)$ with π_k an abelian group. Suppose in addition that for each k the action of $\pi_1(X_{k-1})$ on the homology of the fiber is trivial. Show that $\pi_1(X)$ is a nilpotent group with index of nilpotency at most $n+1$, meaning that all iterated commutators of length $n+1$ vanish.

9. Conversely, for any nilpotent group π show that there is a finite Postnikov tower as in 8 whose total space has π as fundamental group.

10. Prove the uniqueness up to canonical isomorphism in the homotopy category of the Postnikov tower of a simply connected space X.