THE ICONIC WALL

Simons Center for Geometry and Physics

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Dedication May 8, 2015

Simons Center for Geometry and Physics

Stony Brook University, Stony Brook, NY

The Iconic Wall
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Stony Brook University, New York

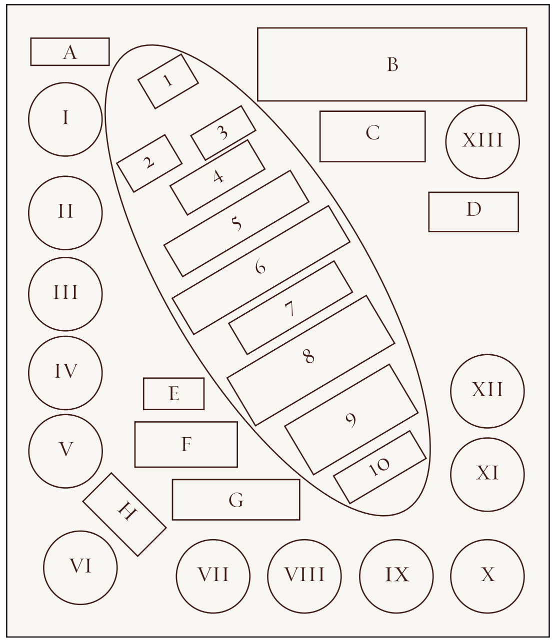
FOREWORD

The *Iconic Wall* is a site-specific artwork displaying significant equations and diagrams in mathematics and physics. Originally carved in stone, the work is permanently installed in the Simons Center for Geometry and Physics at Stony Brook University in New York. The final choice of items to display was the fruit of long deliberations among scientists at the university; their collective insight combines with an extraordinary artistic endeavor to create a remarkable work of art. Measuring twenty-three feet high and twenty feet wide with a two-inch depth, the *Iconic Wall* gracefully fills a vertical space measuring 465 square feet in the Simons Center lobby.

The initial concept of an iconic wall was proposed by Dr. Nina Douglas in 2010. Once the equations and diagrams had been selected, the overall design was developed by Dr. Douglas and Stony Brook University mathematician Dr. Anthony Phillips. That design was printed at full scale in Fall, 2010, in time for the opening of the Simons Center, and displayed in the lobby until a few weeks ago. The project was next adapted for sculpture and realized by the artist Christian White in the time-honored medium of hand carved stone. Mr. White, with three assistants, carved thirty-two equations and diagrams into sixty-nine slabs of Indiana limestone. A mold was made from each of the original carved limestone slabs and used to produce a lightweight handmade cast for installation. The work was completed in Spring, 2015.

Splendidly designed and masterfully wrought, the *Iconic Wall* harnesses art to look back at great past achievements in mathematics and physics, and to point towards discoveries to come.

JOHN MORGAN and LORRAINE WALSH



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INTRODUCTION

BY ROBERT P. CREASE

In an unforgettable passage of the American writer Sylvia Plath's novel *The Bell Jar,* the protagonist, a writer named Esther, describes how panic-struck she became in her college physics class.

> What I couldn't stand was this shrinking everything into letters and numbers. Instead of leaf shapes and enlarged diagrams of the holes the leaves breath through and fascinating words like carotene and xanthophyll on the blackboard, there were these hideous, cramped, scorpion-lettered formulas in Mr. Manzi's special red chalk.[1]

Esther missed out. Many artists have found beauty and mystery in formulas and equations. *Shakespearean Equations,* a series of paintings by the American painter Man Ray, incorporates mathematical models and occasionally formulas. Not long ago, the Simons Center Gallery exhibited a hand-bound, limited edition book—*Equations,* by the British artist-designer Jacqueline Thomas, of the Stanley Picker Gallery at Kingston University—which consisted entirely of designs inspired by specific graphical representations of equations.

Someone with even a rudimentary ability to speak the language of mathematics realizes that equations are among the most powerful forms of human communication. Scientists and engineers, students and educators use them as tools for simplifying, organizing, and unifying features of our world. Equations are highly effective for this purpose because they condense volumes of information with precision. They can allow us to see deep into nature, far beyond ourselves, and to grasp otherwise inaccessible truths concisely and efficiently.

Over centuries, a small number of equations have done their job so excellently that they have become iconic—symbols in themselves. Some of these equations revolutionized the scientific fields from which they sprang, others transformed the wider world. Most such equations emerged in quiet locations, such as studies and libraries, removed from distractions and encroachments. The Scottish physicist and mathematician James Clerk Maxwell wrote down his world-transforming equations in his study, while the German physicist Heisenberg began to piece together his on an isolated island in the North Sea. A few equations had more dramatic origins. The German-Jewish mathematician and astronomer Karl Schwarzschild wrote down the first exact solution to Albert Einstein's equations of general relativity—before Einstein himself—as a diversion while fighting as a German soldier on the Russian front during World War I; a few weeks later, he contracted a rare disease and soon died. These and other iconic equations achieved a special presence not only in science but also in culture and history, where they materialize in art, literature, and other media.

Everyone will recognize at least several of the iconic equations on this wall. Einstein's celebrated mass-energy expression at the upper left–which the Dalai Lama says is "the only scientific equation I know"—has appeared in literature, plays, films, poems, and art—and on the cover of *Time* magazine—since it was born (though with different symbols) in 1905. It was the title of a 1948 play by Hallie Flanagan, the American playwright who had headed the Federal Theatre Project during the Depression, and of a pop album by Mariah Carey (2008). It appears in popular fiction (Dan Brown's *Angels & Demons*), movies (*School of Rock*), and cartoons and video games, as well as far more serious scholarly contexts.

Down near the lower right is the familiar wordless proof of the Pythagorean Theorem, consisting simply of two diagrams. To the far bottom left are the five Platonic solids (cube, octahedron, tetrahedron, icosahedron, dodecahedron), which the ancient Greek philosopher Plato in his dialogue the *Timaeus* associated with the four elements earth, air, fire, and water; he saw the fifth, the dodecahedron, as representing the broader architecture of the heavens.

Crossing the wall from upper left to bottom right is a figure that demarcates an internal and external space on the wall. Many people will recognize this as an elongated ellipse with a Sun like object at one focus, which is how Johannes Kepler, the German mathematician and astronomer, characterized planetary motion. The wavy lines emanating from that Sun suggest light and other radiation; coursing through the wall, these waves seem to draw its spaces back together. At the center of the ellipse is the wave equation of the Austrian physicist Erwin Schrödinger, one of the equations that gave birth to quantum mechanics; Schrödinger crafted it while secluded with a mistress in a quiet villa nestled in the Swiss mountains during the Christmas holidays of 1925-6. Just above it is Einstein's equation for general relativity linking the curvature of space (the expressions to the left of the equals sign) and the distribution of mass-energy (the expression to the right). Physicists like to summarize its message as follows: "Space-time tells matter how to move, matter tells space-time how to curve."

To the right of Einstein's equation at the upper left is a set of four equations, each beginning with the symbol ▽, whose discovery had a far greater impact on human history than Einstein's and predated it by about 40 years. These equations, which provide a complete description of electromagnetism, were first compiled by Maxwell in a different and far less transparent form, then revised into their modern version by the self-taught English polymath Oliver Heaviside. In his famous Lectures on Physics, the American physicist Richard Feynman wrote that:

> From a long view of the history of mankind—seen from, say, ten thousand years from now—there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade.[2]

Though Feynman was known for his jokes and outrageous remarks, this was not one. Maxwell's equations described electromagnetism completely, and the resulting understanding helped transform electromagnetism from a curiosity into a structural foundation of the modern era. They gave birth to new technology that is behind any device based on electromagnetic waves, from radio, television, and microwave devices to computers and today's social media. Maxwell's equations affected human beings—how we live and interact with each other and with the world—far more profoundly than any war ever did, or could.

Another equation whose impact stretched far beyond science is one that appears in the ellipse, just underneath and a little to the right of the Sun. It asserts that gravity exists in all bodies universally, and its strength between two bodies depends on their masses and inversely as the square of the distance between their centers. (When Newton wrote out this conclusion, in 1687, he did not do so as an equation but in words; it was transformed into the familiar equation by which we know it only decades later.) Feynman called this idea "one of the most far reaching generalizations of the human mind." In the next century, it strongly influenced political theory and the modern conception of democracy through its promotion of the idea of universal law. It remains a symbol of the achievement of knowledge and rationality. In George Orwell's novel 1984, the final sign that the protagonist Winston Smith (after accepting that 2 + 2 = 5) has fully capitulated to the thought police—has been thoroughly broken and ceased to think—is that he denies the law of gravity.

Several equations here have special significance for Stony Brook in particular. The pair of equations that is second to last in the ellipse is the Yang Mills equation, the first of whose authors is C. N. Yang, the Nobel Prize—winning physicist who came to Stony Brook in 1965. The Yang Mills equations have a fascinating history. When proposed in 1954, they had a show-stopping flaw: a key term in them had to be zero in principle, but had to be non-zero if the theory were to have any application to the world. The Yang-Mills equations therefore seemed doomed to remain only a mathematical curiosity. Then a series of discoveries unexpectedly opened the door to its application, allowing these equations to become the foundation for modern elementary particle physics.

If you cross all the way to the left of the wall from the Yang-Mills equations you will see a diagram that looks something like a trampoline would if a cannonball were placed at its center. This is an image of Schwarzschild's discovery, the "dent" in space-time made by a single mass according to Einstein's equations, illustrating the Schwarzschild Radius that he calculated. Just to the right of that is another equation, the Heisenberg uncertainty principle, that transformed our understanding of the world far beyond the particular field in which it was discovered. Though its name suggests that it describes an irreducible squishiness in the world around us, the effect is precisely the opposite. The uncertainty principle (along with the Pauli exclusion principle) explains why all atoms of a particular species are absolutely alike and structured the way they are, and therefore the solidity of matter. Not only that, this principle describes the persistence and development of life itself, in accounting for the fact that DNA molecules for the most part hold together firmly even in difficult environments—but that these molecules are also occasionally vulnerable to change, thus making evolution possible.

Yes, Esther definitely missed out. Shrinking things into letters and numbers can vastly enlarge our grip on the world. The iconography on this wall shows the many ingenious ways—worthy of our fascination and awe—that human beings, for thousands of years, have devised to do this.

* * *

Robert P. Crease is a professor in the Department of Philosophy and author, among other books, of *The Great Equations: Breakthroughs in Science from Pythagoras to Heisenberg* (Norton 2008).

1. Sylvia Plath, *The Bell Jar* (Cutchogue, NY: Harper Perennial Modern Classics; 1 edition, 2005), 29.
2. Richard P. Feynman, Robert B. Leighton, and Mathew Sands, *The Feynman Lectures on Physics* (Pasadena, CA: California Institute of Technology; Third Printing Edition, 1965), 1-1.

The Babylonian Tablet YBC 7289

The tablet YBC 7289 in the Yale Babylonian Collection dates back to around 1800 BCE and is therefore one of the oldest texts in the history of mathematics. The tablet itself is 2 ½ inches in diameter; it shows a square with its two diagonals. Cuneiform numbers are written along one side of the square, along its diagonal, and below the diagonal.

Old Babylonians recorded numbers between 1 and 59 using two symbols: Y = 1 and < = 10. Two was YY, five was YYYYY, usually bunched in two rows (this can be seen in two places on the bottom line), and so forth up to 9. Similarly << was written for our 20, … up to <<<<< for our 50. Outside of this range, the system used place-values, just as ours does, except with base 60 and without the equivalent of our decimal point. This means that by itself Y could mean 1, 60, 3600, … or 1/60, 1/3600, … .

The number written along the diagonal is Y <<YYYY <<<<<Y < = 1 24 51 10. With a "decimal point" after the first Y, this is 1 + 24/60 + 51/3600 + 10/216000 = 1.41421296… , an excellent approximation to $\sqrt{2}$ = 1.4142135… . For the side length shown (<<< = 30), the length of the diagonal is 30$\sqrt{2}$; the number written below the diagonal is <<<<YY <<YYYYY <<<YYYYY = 42 25 35 giving 42.42638…, again a very good approximation.

YBC 7289 shows us that the Old Babylonians both understood how to calculate the diagonal of a square (and so had at least a practical knowledge of the matter 1000 years before Pythagoras) and had the ability, using their place-value system, to carry out the numerical computation with great accuracy.

ANTHONY PHILLIPS

Related items: The Pythagorean Theorem, p. 6
Illustration: from photographs by William Casselman

THE BABYLONIAN TABLET YBC 7289 1800-1600 BCE

The Pythagorean Theorem

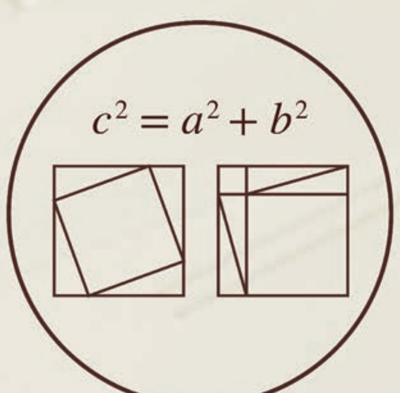
This beautiful theorem, stating that in a triangle with a 90-degree angle the sum of the squares of the lengths of the two short sides equals the square of the length of the long side, is known to all high school students and was likely known to the ancient Babylonians. There are dozens of proofs of this theorem; the one depicted may be the simplest, because it only involves repositioning the four copies of the triangle.

The Pythagorean theorem and its many offshoots have ubiquitous use throughout all branches of mathematics, statistics, physics, and science in general. For example, the theorem generalizes to the formula that the distance between two points in *n*-dimensional space is the square root of the sum of the squares of the differences of their respective coordinates. A very early use may have been in the construction of a perfect 90-degree angle: one merely needs to construct a triangle with sides in the ratio 3 to 4 to 5.

JAMES SIMONS

Related items: The Babylonian tablet YBC 7289, p. 5

THE PYTHAGOREAN THEOREM 550 BCE



The Golden Ratio

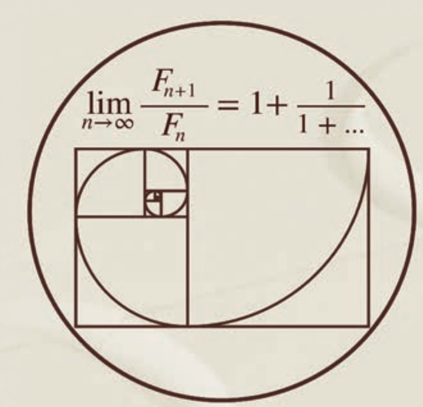
After π = 3.14159... and *e* = 2.71828..., perhaps the most famous irrational number is Φ = 1.61803..., called the "golden ratio." This number dates back to Euclid and has the property that a 1 × Φ rectangle can be divided into a square and smaller rectangle with the same proportions as the original. Subdividing the smaller rectangle gives a second (smaller) square and an even smaller rectangle similar to the first two. The process can be continued, giving the infinite sequence of squares suggested on the wall. For the rectangle to have this subdivision property, Φ must satisfy the condition that the ratio of Φ to 1 is equal to the ratio of 1 to Φ -1,meaning that the side length ratios of the larger and smaller rectangles agree. Writing Φ = 1/(Φ -1) leads to $\Phi^2 - \Phi - 1 = 0$, which can be solved to give Φ = 1.61803... .

The equation can also be rewritten as Φ = 1 + 1/Φ. Then repeatedly replacing the Φ on the right by 1 + 1/ Φ leads to the representation of Φ by the infinite continued fraction shown in the medallion.

The golden ratio is also related to the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ... where each number is the sum of the previous two. Calculating the ratios, each Fibonacci number divided by the one before it, leads to 1, 2, 3/2, 5/3, 8/5, 13/8, ... These ratios tend to a limit, which turns out to be exactly Φ. Writing F_n for the *n*th Fibonacci number, that statement becomes: the limit as *n* goes to infinity of F_{n+1}/F_n is equal to Φ. This is the equation on the wall, where Φ has been replaced by its representation as an infinite continued fraction.

CHRISTOPHER BISHOP

THE GOLDEN RATIO 550 BCE



The Five Platonic Solids

Known since antiquity, the Platonic solids have been the subject of writings by Plato, Euclid, Kepler, and many others. It is their symmetry that makes them special: each vertex (corner point) appears to be the same as every other; the same number of edges (segments joining pairs of vertices) touches each vertex, and the angles between each consecutive pair of edges are all the same. Each face is a regular polygon, having equal-length edges and equal angles at all corners. In fact, the requirement that a convex solid be bounded by a set of identical regular polygons, with each vertex having the same degree (number of incident edges) implies that the solid must be one of these five special Platonic solids—the tetrahedron (with 4 faces, each an equilateral triangle), the cube (with 6 faces, each a square), the octahedron (with 8 faces, each an equilateral triangle), the dodecahedron (with 12 faces, each a regular pentagon), and the icosahedron (with 20 faces, each an equilateral triangle).

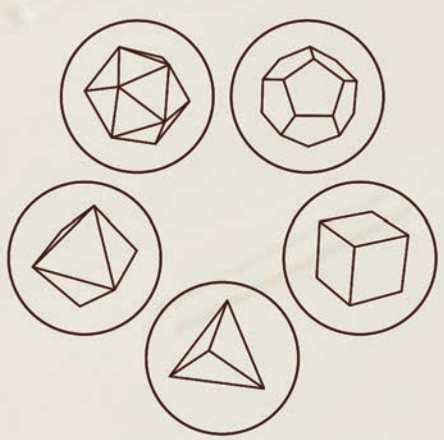
Switching the notion of "face" and "vertex," each Platonic solid has an associated dual solid among the Platonic solids: the octahedron and cube are duals of each other, the dodecahedron and icosahedron are duals of each other, and the tetrahedron is dual to itself.

Beyond mathematics, the tetrahedron, cube, and octahedron each show up naturally in crystal structures, and many viruses have icosahedral shells.

Mathematicians study the remarkable symmetries associated with the Platonic solids, and go on to consider highly regular convex solids in other dimensions. In two dimensions, there are regular polygons having any number of sides. In four dimensions, there are exactly six such solids, while in dimensions higher than four, there are exactly three—the simplex (higher-dimensional tetrahedron), the hypercube and its dual, the cross-polytope (a generalized octahedron).

JOSEPH MITCHELL

THE FIVE PLATONIC SOLIDS 360 BCE



Archimedes' calculation of the volume and area of the sphere

Archimedes (c.287-212 BCE) is one of the greatest mathematicians of all time. Today he is best known for his "Eureka!" (I found it!) discovery of the principle of buoyancy, but his own favorite achievement was the calculation of the area of the sphere and the volume it encloses: he proved that each of them is 2/3 of the corresponding number for a cylinder tangent to the sphere along the equator and at the two poles.

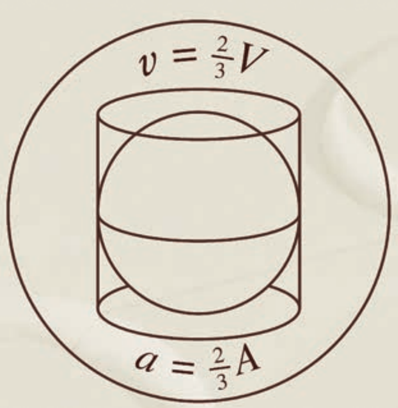
We know he was proud of these relations because he directed that the figure of a sphere inscribed in a cylinder, just as shown here, be engraved on his tombstone. And we know his wishes were carried out because the Roman statesman and orator Cicero tracked down his grave in 75 BCE and reported seeing sphere and cylinder.

Today any second-semester calculus student should be able to calculate the volume enclosed by a sphere of radius r: it is $(4/3)\ \pi\ r^3$. Its area, $4\pi\ r^2$, is only slightly more difficult. Euclid (around 300 BCE) had proved that the volume enclosed by a cylinder was equal to its height times the area of the base, and that the volume of a cone was 1/3 of the product of base-area and height. Archimedes used these two results, along with an extremely ingenious argument involving levers (explained in his *Method*) to show that the volume of the sphere was 2/3 the volume of the circumscribed cylinder. Since that cylinder's base is a circle of radius r (and therefore area $\pi\ r^2$) and its height is $2r$, its volume is $2\ \pi\ r^3$. Two thirds of that volume is exactly $(4/3)\ \pi\ r^3$.

Later Archimedes published *On the Sphere and Cylinder,* with a different, more formal proof of his volume calculation, and with the area result as well.

ANTHONY PHILLIPS

ARCHIMEDES' CALCULATION OF THE VOLUME AND AREA OF THE SPHERE 225 BCE



Kepler's Laws

Johannes Kepler (1571-1630) served as assistant to the great Danish astronomer Tycho Brahe. From Brahe's precise observations of planetary motions as seen from Earth, Kepler abstracted fundamental mathematical regularities of the orbits of planets around the Sun as they would be seen from outside the Solar System.

The first of Kepler's laws states that planetary orbits are ellipses, with the Sun at one of the foci. As the planet goes around its orbit, its distance to each focus (and hence to the Sun) changes, but the sum of the distances stays the same; it is always equal to the length of the "long side" of the ellipse, its major axis.

The eccentricity of an ellipse is the distance between the foci, divided by the length of the major axis. An ellipse with zero eccentricity (the two foci coincide) is a circle. The Earth's orbit, with eccentricity less than two percent, is very close to circular, while the orbit of Mars has an eccentricity of nearly ten percent. (Mars's eccentricity is still much less than the eccentricity of the ellipse on the wall, which is more like seventy percent). In fact, the motions of Mars played an important role in Kepler's discoveries. Kepler's laws apply to comets as well; the celebrated Halley's Comet has an eccentricity of nearly ninety-seven percent.

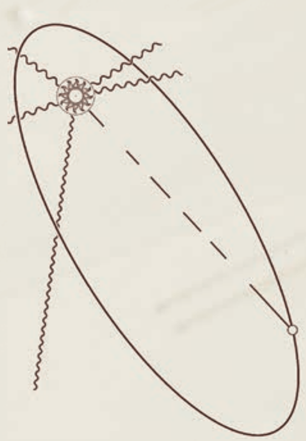
Kepler's Second Law requires us to look down on the elliptical orbits of planets "from above." We imagine a straight line from the Sun (at one focus of the ellipse) extending to a planet, as on the wall. As the planet traverses its orbit, this line sweeps out an area that grows with time. The Second Law states that the area grows at the same rate no matter where the planet is on its orbit. This implies that any planet moves more rapidly when it is near the Sun, and more slowly when it is far.

Kepler's Third Law, like his second, describes how a planet's elliptical orbit is traced out in time. This law gives a relation between a planet's average distance *a* to the Sun, and the time *T* it takes for it to go around the Sun (the length of that planet's "year"): the ratio of the square of *T* to the cube of *a* is the same for every planet in the Solar System. So planets that are further from the Sun move more slowly on average; the Third Law gives the exact relation.

All three of Kepler's laws can be derived from Newton's law relating force and acceleration, and his law expressing the force of gravity between two bodies in terms of their masses and the distance between them.

GEORGE STERMAN

KEPLER'S FIRST LAW 1609



KEPLER'S SECOND LAW 1609

$$\frac{d\theta}{dt} \propto \frac{1}{r^2}$$

KEPLER'S THIRD LAW 1619

$$T^2 \propto a^3$$

Newton's Second Law of Motion

The equation $F = ma$, traditionally called Newton's Second Law, is the starting point of all classical mechanics. The law states that a force F acting on a particle of mass m accelerates that particle (changes its velocity) by an amount $a=F/m$. The velocity, the force and the acceleration are vectors; that is, they have direction as well as magnitude, whereas m is a number with no direction associated with it. So the Second Law implies in particular that a particle will accelerate in the direction of the force that is applied to it. The law doesn't tell us how forces produce this effect, simply that they do. Indeed, forces can be identified by their action on different massive bodies.

The vector ma is the rate of change with time of the momentum vector (mass times velocity) of the particle on which the force acts. Newton's two other laws of motion are statements about momentum. The First Law (really a special case of the Second) holds that a body on which no force acts continues in uniform motion: it experiences zero acceleration; equivalently, its momentum is unchanged. On the other hand, if a force acts, the momentum of at least one particle must change. The Third Law states that for every "action" (force) there is an equal and opposite reaction, ensuring that even if the momentum of one particle changes, other particles experience just the right force or forces so that the sum of all the momenta stays the same.

GEORGE STERMAN

NEWTON'S SECOND LAW OF MOTION 1687

$$\mathbf{F} = m\mathbf{a}$$

Newton's Law of Gravitation

In 1687 Isaac Newton published his famous Principia *(Philosophiae Naturalis Principia Mathematica)*. This treatise gave a theoretical basis for the study of forces between bodies, and of how bodies move; it has become the basis for modern science. Only in the twentieth century were extensions (not corrections) found necessary for very large velocities (Special Relativity in 1905 and General Relativity in 1915) or very small distances (Quantum Mechanics in 1925 and 1926 for particles inside atoms).

Copernicus (1543) had rediscovered that the Sun, and not the Earth, is at the center of the Solar System. Then the careful astronomical measurements of Tycho Brahe had been summarized by Kepler (1609, 1619) into his 3 laws (also shown on the wall). In his study of gravitation, Newton made two key discoveries: Kepler's First and Second Laws (elliptical orbits, Sun at focus, equal areas in equal times) imply that the forces acting on planets are directed towards the Sun. Taking the special case of circular orbits of planets, Newton could derive Kepler's Third Law (periods proportional to the 3/2 power of distance from Sun) by assuming that the force of gravity is inversely proportional to the square of distance. He generalized these insights into a "universal" law of gravitation, a force acting between any two masses, and proportional to each*. This brought the laws of planetary motion down to earth, and the laws of motion to the heavens, an essential step toward the unity of science.

PETER VAN NIEUWENHUIZEN

*To learn more about how Newton achieved all this, we recommend Feynman's Lost Lecture, by D. Goodstein and J.R. Goodstein (W.W. Norton, 1999).

Related items: Kepler's Laws, p. 10; General Relativity, p. 22

NEWTON'S LAW OF GRAVITATION 1687

$$F = \frac{Gm_1m_2}{r^2}$$

The Riemann Zeta Function

The equation relates two functions of a variable *s*. The left side of the equation is an infinite sum, which starts $1+1/2^s+1/3^s+1/4^s+\ldots$. The right side is an infinite product: *p* ranges over all the prime numbers (numbers larger than 1 with no divisors besides 1 and themselves). The first few factors are $[1/(1-1/2^s)]$ $[1/(1-1/3^s)]$ $[1/(1-1/5^s)]$ $[1/(1-1/7^s)]$ … . Equality between the two sides was proven by Leonhard Euler in 1737 for $s > 1$ using the fact that every positive integer has a unique factorization into prime numbers. The common value of the two expressions is usually written as $\zeta(s)$.

Using calculus or a more elementary argument one can show that $\zeta(1) = \infty$, which implies that the product on the right must have infinitely many terms; that is, there must exist infinitely many prime numbers (there are easier proofs of this fact).

ζ is now called the Riemann zeta function because in 1859 Bernhard Riemann showed how to define $\zeta(s)$ for complex values of *s*; he then showed how deep results about the distribution of prime numbers follow from detailed knowledge about the location of the zeros of ζ; these are the points *s* in the complex plane where $\zeta(s) = 0$. It can be shown that $\zeta(s)$ has some zeros on the negative real axis (these are called the trivial zeros); the Riemann hypothesis conjectures that all the other zeros (the "non-trivial zeros") have real part exactly 1/2. This has been verified for about 10 trillion non-trivial zeros, but is still open in general. The Riemann hypothesis was part of Hilbert's eighth problem in 1900. It remains of central importance at the heart of mathematics and is one of the 7 "Millennium Problems" for which the Clay Institute offered a million dollar prize in 2000.

CHRISTOPHER BISHOP

THE RIEMANN ZETA FUNCTION 1737

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1-\frac{1}{p^s}}$$

The Gauss-Bonnet Theorem

The Gauss-Bonnet theorem describes a remarkable link between the curvature of a surface and its topology.

The curvature *K* of a smooth surface *M* is a function that depends on the intrinsic geometry of *M*. If the surface sits in 3-space, with its usual Euclidean geometry, its curvature will be positive at points where the surface is convex or concave, and negative at points where the surface looks like a saddle. However, the curvature may vary if the surface is stretched or twisted.

By contrast, topology describes properties that are unchanged by deforming an object without tearing it. The Euler characteristic $\chi(M)$ is an example of a topological invariant. If the surface is partitioned into polygons (triangles, for example), we can think of it as being like a polyhedron, with vertices, edges, and polygonal faces. Suppose there are *V* vertices, *E* edges, and *F* faces. While these numbers depend on the way we've partitioned the surface, the combination

$$\chi(M) = V - E + F$$

depends only on *M*. For example, the surface of a sphere has $\chi = 2$. Familiar special cases of partitions of the surface of a sphere are given by the regular polyhedra; for example the cube has 8 vertices, 12 edges and 6 faces, while the dodecahedron has 20 vertices, 30 edges and 12 faces; both satisfy $V - E + F = 2$. On the other hand, the surface of a doughnut has $\chi = 0$.

If *M* is a surface without boundary, the Gauss-Bonnet formula

$$2\pi\chi = \int_M K\,dA$$

implies that the average value of the curvature, multiplied by the area of the surface, is 2π times the Euler characteristic. For example, if *M* is the surface of a doughnut, the average value of its curvature will always be zero, even if the doughnut is quite irregular; regions where the surface is convex are always exactly cancelled by regions where the surface looks like a saddle!

CLAUDE LEBRUN

THE GAUSS-BONNET THEOREM 1750

The Navier-Stokes Equations

The Navier-Stokes Equations for the motion of a homogeneous incompressible fluid in three dimensions express, at each point in space, two properties of the flow related to mass and momentum. The three interlocking equations are too hard to be solved explicitly, but their solutions can be simulated on a computer: the image shows how a fairly coarse approximation can give, remarkably, a quite reasonable picture of the flow pattern around a cylindrical obstacle.

These equations are also used to design shapes of airplanes, to study the flow of blood through the heart and to model weather. They are magical equations in their simple beauty and their power to describe a large array of phenomena.

The Navier-Stokes Equations specify first that at any point the mass of the fluid in an infinitesimally small ball about that point must be conserved. And second, that the rate of change of momentum at that point equals the difference between the amount of momentum that leaves the ball and the amount of momentum that enters the ball, minus a frictional effect depending on ν, the viscosity of the fluid.

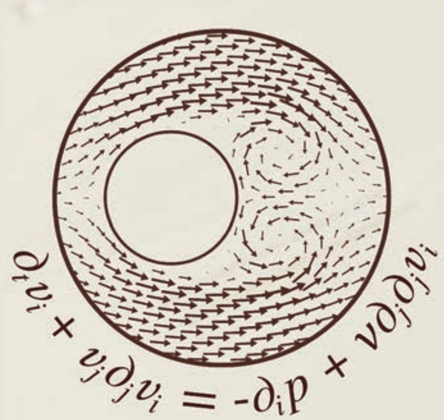
The terms controlling momentum and mass are quadratic functions of the field of velocity vectors. This quadratic non-linearity has created a century-old puzzle for mathematicians, physicists and engineers: to find methods for approximating solutions when solutions are known to exist, as in 2 dimensions, and to find useful computational models even when, as in 3 dimensions, the existence problem has not been settled. Remarkably, even though these equations are widely used in simulations, their widest range of applicability is still not known. Attempts to find it have led to the formulation of large parts of today's theory of partial differential equations.

The swirls behind the obstacle are vortices: they are the geometrical soul of this evolution. In three dimensions vortices twist and turn like tornadoes. They are terrifying but at the same time they provide the hope for eventually understanding the mathematics of the Navier-Stokes Equations.

DENNIS SULLIVAN

Image credits: Thanks to Zhen Gao, JoungDong Kim, Xiaolin Li and Qiangqiang Shi, Department of Applied Mathematics and Statistics

THE NAVIER-STOKES EQUATIONS 1845



Maxwell's Equations

The four Maxwell equations apply to the electric (**E**) and magnetic (**B**) fields in free space, that is, in the absence of matter with electric or magnetic charges. Both electric and magnetic fields are vector fields, each with three components at each point of space and each instant of time; there are thus six separate components between the two fields. The top two vector equations, which are almost but not quite symmetric between **E** and **B**, relate the change in time of each of the two fields to how the other one is changing in space. The lower two equations are conditions on the fields that follow from the absence of electric or magnetic charges.

Maxwell's publication of these equations in 1865 was the culmination of a half-century of progress in the study of electricity and magnetism. Taken together, they predict that electric and magnetic fields travel together in waves. The speed of these waves could be predicted on the basis of experiments that had already been done by Maxwell's time. Evaluating this speed, Maxwell found it to be equal to the speed of light, leading to the realization that light itself is an electromagnetic phenomenon. Maxwell's Equations led directly, over the next half-century, to the creation of radio, radar and related technologies that define modern life. Today we understand Maxwell theory as the first of the gauge theories, and that the "non-Abelian gauge theories" that were discovered by Yang and Mills in 1954, and that are the basis of the current Standard Model of elementary particles and forces, have an predecessor in Maxwell's work.

GEORGE STERMAN

Related items: The Yang-Mills Equations, p. 27

MAXWELL'S EQUATIONS 1861-1884

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{E} = 0$$

Stokes' Theorem

Stokes' theorem is the modern generalization of Newton's Fundamental Theorem of Calculus, which states that the integral from *a* to *b* of the derivative of a function *f* is equal to *f(b) – f(a).* Stokes' theorem includes earlier generalizations such as Stokes' (original) Theorem, Green's Theorem and the Divergence Theorem.

In modern parlance the theorem states that the integral of the exterior differential of a differential form over the interior of a chain is equal to the integral of the form itself over the boundary of the chain. All of this is taking place inside of an n-dimensional manifold. A differential form is a tensor of a type suitable for integration, and a chain can be thought of as a polyhedral subset of the manifold. For the two sides of the equation to be non-zero, the degree of the form must be one less than the dimension of the chain.

In the Fundamental Theorem of Calculus, the manifold is the real line, the polyhedron is the 1-dimensional segment from *a* to *b*, with boundary the 0-dimensional chain: endpoint *b* minus endpoint *a*. The "differential form" is the function *f*; integrating it over the boundary gives *f(b) – f(a).* The exterior differential of *f* is simply the derivative of *f* followed by *dx,* and the equality of the two integrals is the Fundamental Theorem.

Stokes' Theorem has had a myriad of applications across a wide range of fields in mathematics, including geometry, topology and ordinary and partial differential equations. It has had similarly pervasive applications in physics, from classical mechanics through electromagnetism, and in many branches of modern physics as well.

JAMES SIMONS

STOKES' THEOREM 1850

$$\int_M d\omega = \int_{\partial M} \omega$$

The Boundary of a Boundary is Zero

The very short equation ∂∂=0 is fundamental in almost every part of mathematics and theoretical physics. In its most basic interpretation, the equation represents a geometrical truism obvious to any child: For any solid figure *F*, its boundary ∂*F* (which is the surface of the solid) is a special kind of surface. It is topologically different from surfaces like very thin carpets or T-shirts or pairs of pants. Each of those surfaces *S* has its own boundary ∂*S*: the edge of the carpet, the neck, armholes and waist of the T-shirt , the cuffs and waist of the pair of pants. But the surface boundary of the solid is a surface without boundary! Namely the boundary of the surface of a solid is nothing at all: ∂∂=0.

The phenomenon continues if we look again at the surfaces mentioned. Each of their boundaries is a set of one dimensional objects, each one of which taken by itself is topologically a circle, with no boundary: ∂∂=0.

This geometrical or topological seed, planted in the domain of algebra, has grown luxuriantly. Many structural equations in mathematics can be formulated as *dd*=0. A potentially curved space is flat if and only if the calculus differential *d* in the space satisfies *dd*=0. A binary operation on entities is associative if and only if a certain operator on linear combinations of collections of these entities satisfies *dd*=0. Often all of the meaningful or most intrinsic information in a mathematical model can be computed from an operator *d* (in the model) satisfying *dd*=0, by what is now a standard algebraic procedure called "calculating the *d*-homology."

DENNIS SULLIVAN

THE BOUNDARY OF A BOUNDARY IS ZERO 1860-1900

$$\partial\partial = 0$$

E=mc²

Perhaps the most famous scientific formula of the past century, Einstein's mass-energy relation expresses the equivalence between energy E and mass m, two quantities previously thought to be of completely different natures; the proportionality constant c^2 is the square of the speed of light. The mass-energy relation is a consequence of a more general relation between the mass, the energy and the momentum of any object:

$m^2c^4 = E^2 - p^2c^2$

where p is the object's momentum. The rest mass is then an "invariant mass," which is defined for any object, whether stationary or moving, once its energy and momentum are known. The formula on the wall gives an expression for the energy content of any object when it is at rest.

The mass-energy relation follows from Einstein's Special Relativity, which showed there is no absolute measure of time. That is, if two observers are moving relative to each other, they will measure different time intervals between the same pair of events. In a separate paper, Einstein considered how such a pair of observers would see a massive object that radiates light, and showed that conservation of energy would only be possible if each observer saw the mass of the object decrease by an amount (Energy radiated)/c^2. In most situations, this is a tiny effect. In the following years, however, it was realized that the transformation of mass into energy through the fusion of light atomic nuclei is the primary source of the energy of the Sun and most other stars. On the other hand, the fission of heavy nuclei is the source of nuclear energy and the threatening power of nuclear weapons.

GEORGE STERMAN

Related items: General Relativity, p. 22

Schrödinger's Equation

The Schrödinger equation was discovered by Erwin Schrödinger in 1925, soon after the creation of matrix quantum mechanics by Werner Heisenberg and Max Born. It describes the evolution of the quantum state ψ of a physical system with time. In this equation, the two terms on the right-hand side constitute a linear operator on the space of states, called the Hamiltonian. The amazing thing about the Schrödinger equation that it is absolutely universal: the evolution of any system, from a set of atoms to the whole Universe, obeys this equation, provided one chooses the correct Hamiltonian. In his original paper Schrödinger wrote a concrete form of his equation suitable for describing a single particle, but it was quickly realized that its abstract form is completely general.

The Schrödinger equation can be simultaneously described as one of the deepest laws of Nature, a tautology, and a mystery. It is a deep law since it encapsulates the relation between the homogeneity of time and conservation of energy. This principle is a special case of Emmy Noether's theorem which states that conservation laws correspond to symmetries of equations of motion, and it is valid in both classical and quantum theories. It is a tautology because it is more or less equivalent to the statement that time evolution is represented by a unitary operator, and unitary transformations are the most general transformations of the wave-function which are compatible with its probabilistic interpretation. That is, the Schrödinger equation is forced on us by the requirement that the sum of the probabilities of all possible outcomes is always one, for all times. The Schrödinger equation is a mystery, because it allows and even forces normal-looking states to evolve into strange states which contradict our intuition, such as the notorious "Schrödinger's cat" state which describes a cat in a closed box which is neither alive nor dead.

ANTON KAPUSTIN

E=mC² 1905

$$E_0 = mc^2$$

SCHRÖDINGER'S EQUATION 1925

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

General Relativity

After Albert Einstein had written down his 1905 theory of Special Relativity, which describes what happens when observers are moving at large velocities with respect to each other, he started wondering what would happen if the velocities were not constant. The solution took him 10 years, but when it was finished, it turned out not only to describe the physics of arbitrary velocities in arbitrary coordinate systems, but also the force of gravity. Thus by 1915 he had extended Newton and Leibniz' seventeenth-century theory of gravity to a fully relativistic theory. His calculations predicted that perihelion of the planet Mercury would precess as Mercury orbits around the Sun (the orbit is an ellipse, and the axis of that ellipse itself rotates about the Sun, although very slowly). When Einstein realized that his calculation agreed with the observations of astronomers, "he had heart palpitations."

General Relativity is summarized by the famous equation carved on the wall, and shown below. The left-hand side describes the geometry of curved space (R with two subscripts is the Ricci curvature tensor, R alone is the scalar curvature, and g is the metric tensor). Einstein referred to these ingredients as "fine marble." The right-hand side describes the matter in curved space; Einstein called this part "low-grade wood." In later years he tried to find a geometric description of matter as well, so he could move the right-hand side to the left-hand side. Currently Supergravity and Superstring theory aim to achieve his goal.

Since its discovery, General Relativity has been tested in many experiments; it has always turned out to be in complete agreement with the observations. Today it is the foundation on which studies of our expanding universe are based. The theory is also very interesting (and much studied) from the mathematical point of view, which is perhaps not surprising since it explains the physical force of gravity as being due to the geometry of curved space.

PETER VAN NIEUWENHUIZEN

Related items: Newton's Law of Gravitation, p. 13; Supergravity, p. 34

GENERAL RELATIVITY 1915

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$$

The Schwarzschild Black Hole

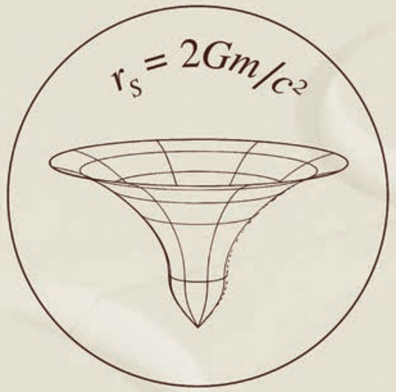
This diagram represents the simplest nontrivial solution of the Einstein vacuum equations, namely the Schwarzschild black hole. The illustration provides a 2-dimensional representation of the 3-dimensional spatial geometry of the black hole at a specific instant of time. In particular, each circle actually represents a sphere. The circle of zero circumference, the point at the bottom of the picture, is the singularity at the center of the black hole, where a certain amount of mass m is stored and the geometry is infinitely curved. The circle of circumference $2\pi\ r_s$, where r_s, $=2Gm/c^2$ with G the universal gravitational constant, m the mass and c the speed of light, serves as the event horizon of the black hole. This is a point of no return, where the gravitational pull becomes so great as to make escape impossible. In fact, not even light can break free from this gravitational confinement, making the event horizon appear black to an outside observer.

The Schwarzschild solution, named after its discoverer Karl Schwarzschild, was found in 1916 shortly after the publication of Einstein's theory of General Relativity. Not only does this solution play a special role in theoretical considerations, but it is also used in many experimental tests of General Relativity, including the classical ones: deflection of light, precession of perihelia, and gravitational redshift. Finally, the Schwarzschild black hole is useful as a model for approximating general relativistic effects due to slowly rotating astronomical objects such as many stars and planets, including the Earth and the Sun.

MARCUS KHURI

Related items: General Relativity, p. 22

THE SCHWARZSCHILD BLACK HOLE 1915



Heisenberg's Uncertainty Principle

The "Heisenberg uncertainty principle" captures one of the most essential yet counter-intuitive properties of quantum mechanics: wave-particle duality. In classical mechanics, a particle with a given mass is completely characterized at any given time by its position and its momentum (equivalently, its velocity). If both of these quantities are fixed at some time, and the forces acting on the particle are specified, the future position and momentum of the particle can be predicted to arbitrary accuracy for all time to come. All we need to do is apply Newton's laws of motion and calculate sufficiently carefully.

In quantum mechanics, however, the most complete knowledge we can have of the same particle is contained in its "wave function", which is usually denoted $\psi(x,t)$. The wave function satisfies the Schrödinger equation, also given on the wall, and contains information on a whole range of possible positions (the points where ψ is nonzero) and on a range of possible momenta (roughly, determined by the distances between peaks and valleys in the graph of ψ).

To measure where a particle is, we must constrain it to be within some small volume, but to do so we must change the wave function, distorting its pattern of peaks and valleys; this spreads out its range of momenta. The uncertainty principle quantifies this effect. The smaller the specification of position (Δx) we demand, the larger the spread in momenta (Δp) we are forced to accept, and vice-versa. Their product must be no smaller than (1/2)*h*-bar; this is equal to (1/4π) times Planck's constant, Nature's measure of the quantum scale. One extraordinary consequence of the uncertainty relation is a certain "restlessness" of matter in its coldest, lowest-energy states. Even at absolute zero temperature, the atoms in a crystal vibrate with what is called "zero-point energy." This phenomenon and other related ones associated with the uncertainty principle have profound importance in science and technology, from elementary particles to electronics.

GEORGE STERMAN and
ALFRED SCHARFF GOLDHABER

Related items: Schrödinger's Equation, p. 21

HEISENBERG'S UNCERTAINTY PRINCIPLE 1927

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

The Dirac Equation

Paul Dirac discovered this equation in 1928. The history of the Dirac equation illustrates perfectly how a pursuit of mathematical consistency can lead one to a deeper understanding of Nature. Dirac's goal was to find an analogue of the Schrödinger equation, which would take into account that electrons cannot travel faster than the speed of light. Dirac regarded a prior proposal in this direction (the Klein-Gordon equation) as unsatisfactory because it appeared to predict negative probabilities. Dirac's elegant equation solved this problem, but this was only the beginning of a new chapter in physics. Dirac showed that his equation predicts the correct magnetic moment for the electron. But the equation also seemed to predict the existence of electrons with negative energy. In 1932 Dirac made the brilliant proposal that the vacuum is a sea of electrons with negative energies, and that the absence of one these electrons is a particle with a positive energy and charge. The logic of the equation forced Dirac to conclude that this was a new particle, which would be called a positron. Soon after, the positron was experimentally discovered, and Dirac's theory became accepted. Dirac's equation applies not only to electrons and positrons, but to all known fermions (particles with half-integer spin). The Dirac equation is fundamental both to the Standard Model of elementary particles and to Supersymmetry. Dirac's equation also plays an important role in the Atiyah-Singer Index Theorem.

ANTON KAPUSTIN

Related items: Schrödinger's Equation, p. 21; Supersymmetry, p. 32; The Atiyah-Singer Index Theorem, p. 29

THE DIRAC EQUATION 1928

$$(i\not{\partial} - m)\psi = 0$$

Feynman Diagrams

Crisscrossing the entire wall are wavy lines emanating from the Sun, one of which meets a solid line near the lower left corner. These lines are the world-lines of photons, the particles of light. Instead of thinking of a particle as tracing a path in space, in quantum physics one thinks of the entire history of the particle as a line moving back and forth through space-time, its world-line. The solid line represents the world-line of an electron, and each of the two points where it bends represents an elementary interaction, in which the electron emits or absorbs a photon.

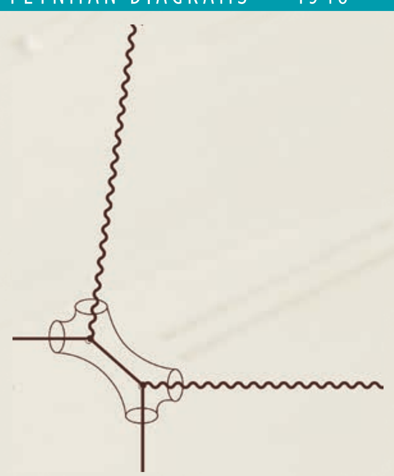
These diagrams of solid and wavy lines are known as Feynman diagrams, after the physicist Richard Feynman (1918-1988). He proposed the diagrams both as a simple way to envision the basic interactions of fundamental physics, and as a set of rules to calculate the amplitudes of quantum processes. The results of these calculations for the interactions of photons and electrons are among the most precisely tested predictions in all of physics.

Looking more closely at the two interaction vertices, one sees that they are surrounded by a faint pattern of circles connected by curves. This is a string diagram, representing the interactions of superstring theory. According to superstring theory, each of the particles we see, each electron and photon, is actually a tiny loop of "string." Each of the world-lines entering or exiting the interaction region is replaced by the tube swept out by the corresponding string; in the interaction the tubes join to form a two-dimensional surface, the "world-sheet" of the string, representing its history in space-time.

The relation between the Feynman diagram and the string diagram is the following. Each interaction point of particles becomes a string interaction, which can be pictured as a "pair of pants," with two cylinders (the legs) joining into a single cylinder (the waist). By sewing two pairs of pants together at the waists, leaving four holes for the four legs, one realizes the string diagram depicted here. In string theory, interactions are not special points at which particles are created or destroyed; rather they follow from a continuous change in the topology of the world-sheet.

MICHAEL R. DOUGLAS

FEYNMAN DIAGRAMS 1948



The Yang-Mills Equations

The Yang-Mills equations were written down in 1954 by physicists C.N. Yang and R. Mills, in a study of the strong interaction in particle physics. The equations are generalizations of Maxwell's equations for electro-magnetism, which also appear on the wall. Mathematically, the relevant concept is that of a connection A on a fibre bundle, a notion in differential geometry which crystallized at about the same time (although these developments were initially independent, and it was some years before the parallels were understood). One key feature is the introduction of an internal symmetry group G, the gauge group of the theory. In geometric language this is the structure group of the fibre bundle.

We imagine having at a point an object which can be transported along a path in space (or space-time). The connection A determines how the state of the object changes during the transport; when the path is a loop, the state at the end may differ from what it was initially, but the difference is given by the action of the group G. In classical differential geometry the "space" is a curved surface and the objects are tangent vectors to the surface, but in general one considers bundles made up of more abstract objects. In the displayed equations, the term A denotes the connection, and F is its curvature. If the group G is not commutative then the relation between A and F is non-linear, with a quadratic term $A \wedge A$. The non-linearity means that the connection cannot be eliminated from the equations, as happens in electromagnetism (where the relevant group is the circle, which is commutative and hence $A \wedge A$ vanishes.) The other equation of the pair is the Euler-Lagrange equation for the Yang-Mills action functional. These equations have had a profound impact on many developments in geometry over the past half-century and the ideas are a crucial part of the standard model in elementary particle physics.

SIMON DONALDSON

Related items: Maxwell's Equations, p. 17

THE YANG-MILLS EQUATIONS 1954

$$F_A = dA + A \wedge A$$

$$D_A^* F_A = 0$$

The Aharonov-Bohm Effect

The arrows in the center of this diagram represent lines of magnetic flux, completely shielded from the outside world by a metal tube. The tube is surrounded by two bands, each representing a group of charged particles passing on one side (c_1) or the other (c_2).

According to classical physics, since in this experiment the electromagnetic field is totally contained inside the tube, it should not matter which path the particles take.

But in fact, if a beam of electrons is split, with one half following path c_1 and the other half following c_2, when the two halves are brought together again they produce an interference pattern. This is the Aharonov-Bohm effect, first confirmed experimentally in 1959. It can only be explained using quantum mechanics.

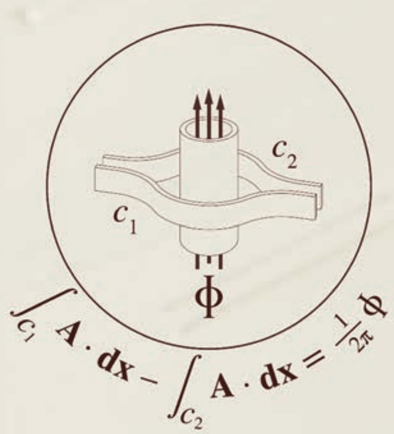
In quantum mechanics, besides the electric and magnetic fields, one must take into account the magnetic vector potential. What happens in this experiment is that the magnetic vector potential, which is non-zero even outside the tube, interacts with the phases of charged particles that traverse it. In the language of gauge fields, the magnetic vector potential acts as a connection in the bundle of phases. This means that when a particle moves along a path, the magnetic vector potential can cause its phase to advance or retard. In this experiment the potential retards phases along path c_1 and advances them along c_2. At any point where the two beams are brought together, electrons in one beam and electrons in the other may be in or out of phase, according to the exact lengths of the paths they have followed: this forms the interference pattern that allows the effect to be detected. This phenomenon is part of the physics that would make a quantum computer more powerful than a classical computer.

The formula below the diagram represents the difference in the phases associated with the two paths: it is proportional to the total flux in the tube, here denoted by Φ.

MICHAEL R. DOUGLAS

Related items: The Yang-Mills Equations, p. 27; Maxwell's Equations, p. 17

THE AHARONOV-BOHM EFFECT 1959



The Atiyah-Singer Index Theorem

The Atiyah-Singer Index Theorem (1963) forms a bridge between topology and analysis. The formula uses topology to compute the index of the operator associated to an elliptic partial differential equation, for example. This index is the "formal" dimension of the space of solutions to the equation in the sense that it is the dimension of the space of solutions minus the dimension of the obstruction space associated with the equation. In good cases, the latter vanishes and the topological formula gives the actual dimension of the space of solutions. The topological nature of the formula means that in many situations, the dimension of the space of solutions can be computed directly even before specific solutions have been found. The theorem applies to many equations and operators appearing naturally in geometry and physics and therefore has important applications in both these fields.

One of the first uses of the Atiyah-Singer theorem in theoretical physics (1978) was to compute the dimension of the moduli space of anti-self-dual connections; these are the lowest-energy solutions of the Yang-Mills equations, and include the special case of instantons in 4-dimensional space-time. This was one of the first direct connections between topology and modern theoretical high-energy physics. It was a significant early step in the resurgence of interactions between these fields that continues to play an important role in each of them today.

JOHN MORGAN

Related items: The Yang-Mills Equations, p. 27

THE ATIYAH-SINGER INDEX THEOREM 1963

$$\dim \ker \not{D}_E - \dim \operatorname{coker} \not{D}_E = \int_M \hat{A}(M) \cdot ch(E)$$

The Lorenz Attractor

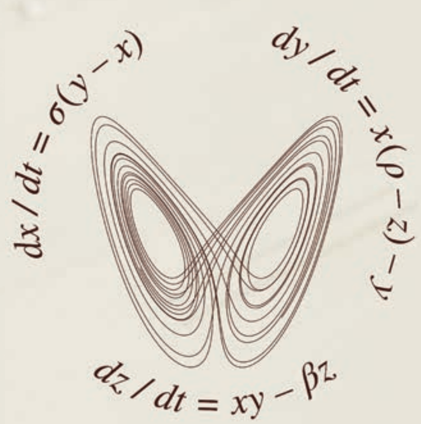
The “Lorenz butterfly” has become the best-known image of a chaotic dynamical system. It was created in the 1960s when Edward Lorenz, a meteorologist at MIT, studied this system of three innocent-looking differential equations for the motion of a particle in space as a “toy model” for atmospheric circulation; that is, for the behavior of the weather. Computer simulation showed that the behavior of this system is extremely complex. Each curve in the image is an orbit of this dynamical system: the path traced out by a particle moving according to the Lorenz equations, starting at some initial data point.

As shown, the structure of the set of orbits is quite intricate: the orbits accumulate onto what is now called the Lorenz attractor, with an overall butterfly shape and a complicated fractal structure on small scales. Moreover, the dynamics of the system is highly unstable, in the sense that a tiny change in initial data can lead to a totally different outcome: two orbits, initially almost indistinguishable, can diverge wildly as time evolves and can eventually become completely independent. The overall behavior can be adequately described only in statistical terms (which gives a reason why a reliable long-term weather forecast is inconceivable). This picture, and the story it tells, sparked great interest in chaos theory and fractals in mathematics, physics and other branches of natural science.

MIKHAIL LYUBICH

Image credits: Thanks to Scott Sutherland

THE LORENZ ATTRACTOR 1963



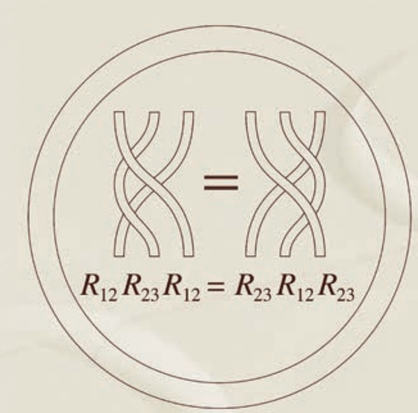
The Yang-Baxter Equation

The Yang-Baxter equation is one of the most remarkable developments in mathematical physics of the 20th century. This compact formula describes the equality of two ways of scattering three particles and also the equivalence of two ways in which three strands of string may be braided. The Yang-Baxter Equation is a set of equations between sums of products of the entries of the matrices describing local interactions of various states. In general there will be many more equations than there are matrix entries in the interaction matrices (which are the unknowns). It is a miracle that solutions to the matrix equations exist, but under appropriate conditions they do, and each such solution gives rise to a many-body system that can be explicitly solved for an arbitrary number of particles. These solvable models form the basis of much of our physical intuition of many-body problems in diverse fields of physics.

In 1967, C.N. Yang discovered that solutions to this equation lead to systems where the multiple scattering of many particles is computable from simple two-body scattering. In 1971, Rodney Baxter discovered that this equation is also the key ingredient for solving problems in statistical mechanics with infinite numbers of particles. Since then many solutions have been discovered, which describe, for example, quantum magnetic spin systems, phase transitions, the theory of knots and quantum field theory. From these very special solvable systems, approximate models have been formulated, which have been applied with amazing success to real-world experimental systems. In pure mathematics, the search for efficient ways to solve the Yang-Baxter equation has led to the invention by Drinfeld and Jimbo of *quantum groups*, a field of algebra that was previously completely unknown. The Yang-Baxter equation is an outstanding example of the intimate connection between physics and mathematics.

BARRY MCCOY

THE YANG-BAXTER EQUATION 1968



Supersymmetry

In nature there are two kinds of fundamental particles: fermions and bosons. Fermions (for example, electrons) obey the Pauli exclusion principle, meaning that two of them cannot be forced to be simultaneously in the same state. Bosons (for example, photons) prefer in contrast to behave coherently and hence can give rise to long-range forces.

In 1971, Yu. A. Golfand and E. P. Likhtman proposed that under special circumstances, there could be a new sort of symmetry, *supersymmetry*, that relates these two different kinds of particles. The equation $\{Q,Q\}=P$ is an equation for the operator Q generating supersymmetry. The equation expresses a remarkable property of the operator Q : its square, when applied to a particle, gives the momentum and energy of that particle, represented by the operator P.

Supersymmetry has had far-ranging applications in mathematics and physics. In some situations, it allows exact calculations of path integrals that otherwise are difficult to define, let alone calculate. In other circumstances, one can prove important positivity conditions because momentum and energy is the square of Q.

Supersymmetry was upgraded to a gauge theory by S. Ferrara, D. Z. Freedman, and P. van Nieuwenhuizen in 1976; the resulting theory, Supergravity, became an important extension of Einstein's theory of General Relativity.

Consequences of the fundamental relation $\{Q,Q\}=P$ are still being explored today.

MARTIN ROCEK

Related items: General Relativity, p. 22; Supergravity, p. 34

SUPERSYMMETRY 1974

$$\{Q,Q\} = P$$

The Jones Polynomial

Given two knots, i.e. closed loops in three dimensional space which do not intersect themselves, one would like to know whether one can be smoothly deformed, without self-intersection, into the other. In particular, can a given knot (for example, the trefoil in the picture) be unknotted: deformed into a circle lying in a plane? To distinguish topologically distinct knots mathematicians look for invariants; these are quantities which can be assigned to every knot, and which do not change under smooth deformations without self-intersection. One such invariant was introduced by Vaughan Jones in 1984; it assigns to each knot a polynomial in one variable.

Edward Witten made the remarkable discovery that the Jones polynomial of a knot, evaluated at a certain complex number z, can be calculated from gauge theory. Gauge theories play a prominent role throughout physics; for example, the standard model of physics is expressed as a particular gauge theory, an elaboration of Yang-Mills theory. Witten's insight was that another type of gauge theory could be used to produce topological invariants of knots in ordinary 3-dimensional space and in other, more exotic 3-dimensional spaces. Here is a sketch of the procedure, essentially an analysis of the equation written under the trefoil. A fiber bundle with group SU(2) is constructed over the space containing the knot. For every connection A in that bundle, the trace is calculated for the SU(2) matrix representing parallel transport by A once around the knot. These traces are averaged with each connection assigned a weight calculated using z and $CS(A)$, the Chern-Simons class of the connection. The Chern-Simons class, used in this calculation, is defined by a geometrically motivated construction due to the mathematicians S. S. Chern and James Simons.

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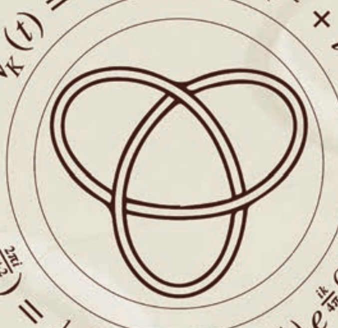
Related items: The Yang-Mills Equations, p. 27

THE JONES POLYNOMIAL 1984

$$V_K(t) = (t + t^3 - t^4)(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$$



$$V_K(e^{\frac{2\pi i}{k+2}}) = \frac{1}{Z}\int (\mathrm{Tr}\, \mathrm{Pexp}\oint_K A)\, e^{\frac{ik}{4\pi}CS(A)}\mathcal{D}A$$

Supergravity

Supergravity is an extension of Einstein's theory of General Relativity, but it is also the gauge theory of Supersymmetry. It was discovered at Stony Brook in 1976 as a field theory; later, physicists realized that it is also the low-energy limit of Superstring theory. Supergravity is now intensively studied all over the world. Yet, if this theory arises in Nature, new particles should exist: every known particle should have a supersymmetric partner particle. These particles are being looked for using the Large Hadron Collider at CERN. If they are found, we shall have another revolution in physics.

Three earlier revolutions in physics — Maxwell's theory of electromagnetism in the nineteenth century, Einstein's 1905 theory of Special Relativity, and Quantum Mechanics of the 1920's— describe the non-gravitational world very well. However the fourth revolution, General Relativity (1915), is a classical theory, whose predictions have been confirmed in detail, but whose quantization has defied the efforts of many physicists. Supergravity unifies General Relativity with the non-gravitational interactions, while Superstring theory solves the problem of quantum gravity.

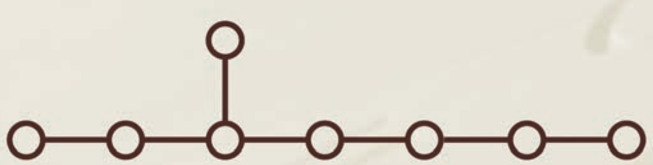
The displayed equation gives the Supergravity Lagrangian as the sum of two terms, one with *R* and one with Greek letters. The first corresponds to the left-hand side of Einstein's General Relativity equation, also on the wall (Ricci tensor, metric tensor, scalar curvature, all describing geometry), while the second corresponds to the right-hand side of that equation (the tensor *T*, describing matter). In Supergravity it is as if the right-hand side of Einstein's equation had been moved to the left-hand side; what Einstein called "low grade wood" has been promoted to "fine marble," and matter has been understood as part of geometry.

Supergravity contains Yang-Mills theory, the Dirac equation and General Relativity. These theories have classical symmetries which can be destroyed by quantum effects, in which case they become inconsistent. In Supergravity this is avoided if the Yang-Mills symmetry group is one of the two 496-dimensional Lie groups E(8)×E(8) or SO(32). The exceptional Lie group E(8), one of the most intricate objects in algebra, can be represented by its Dynkin diagram, shown here.

PETER VAN NIEUWENHUIZEN

Related items: Maxwell's Equations, p. 17; General Relativity, p. 22; Supersymmetry, p. 32;
The Yang-Mills Equations, p. 27; The Dirac Equation, p. 25

SUPERGRAVITY 1976



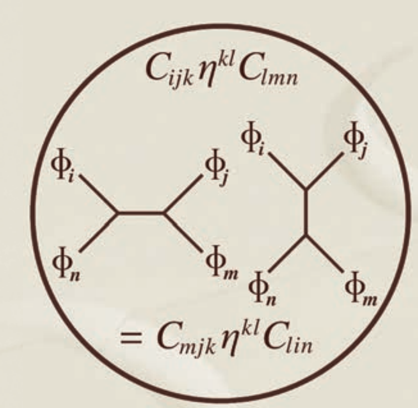
$$\mathcal{L} = R - \bar{\psi}_\mu \gamma^{\mu\rho\sigma} D_\rho \psi_\sigma$$

Associativity in Quantum Field Theory

We are all used to associativity in real life: (a + b) + c = a + (b + c). It amounts to the correctness of the notion that you can add three things at the same time: you can pick any two of the three, add them together, and then add the one left over—the result does not depend on the way you chose which ones to add first. The same applies to multiplication (a × b) × c = a × (b × c). Even when the result of the multiplication depends on the order, when a × b ≠ b × a (which is the case, for example, when you multiply, i.e. compose, rotations of three-dimensional space), the result of multiplication of three objects taken in a definite order, say a × b × c, is independent of whether you first multiply b × c and then insert a × on the left, or if you multiply a × b and then insert × c on the right. In quantum field theory, the physical theory describing the world at the scale of elementary particles, the role of a, b and c is played by the "observables," operators creating or annihilating particles or their composites at some point in space and at some moment of time. The requirement that the product of these observables be associative is a very powerful technique in analyzing possible quantum field theories.

NIKITA NEKRASOV

ASSOCIATIVITY IN QUANTUM FIELD THEORY 1970-1975



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Lorraine Walsh
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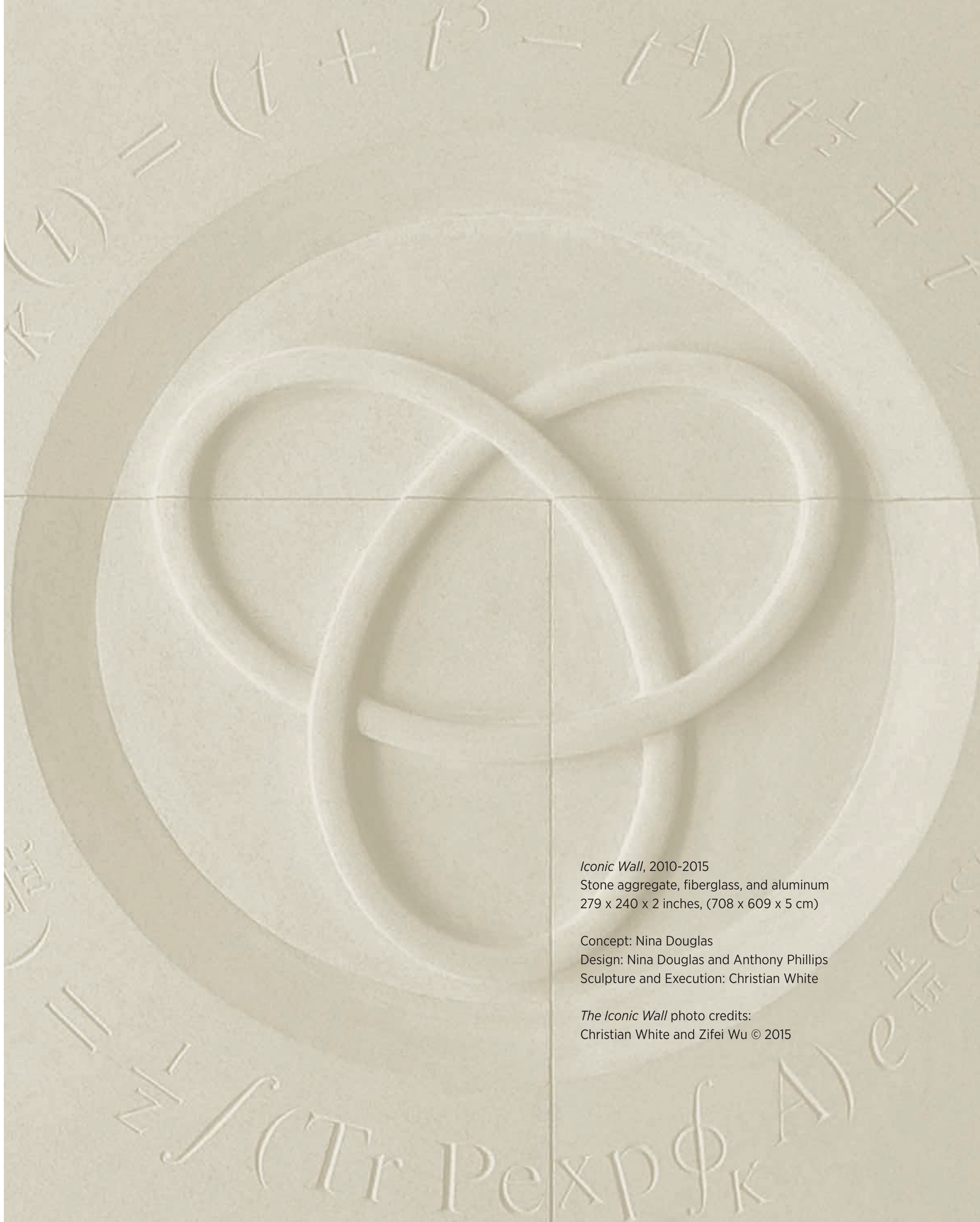
Simons Center for Geometry and Physics



Iconic Wall, 2010-2015
Stone aggregate, fiberglass, and aluminum
279 x 240 x 2 inches, (708 x 609 x 5 cm)

Concept: Nina Douglas
Design: Nina Douglas and Anthony Phillips
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