2019 Oswald Veblen Prize in Geometry
Citation for
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The 2019 Oswald Veblen Prize in Geometry is award to Xiuxiong Chen, Simon Donaldson, and Song Sun for the three-part series entitled "Kähler-Einstein metrics on Fano manifolds, I, II and III" published in 2015 in the *Journal of the American Mathematical Society,* in which Chen, Donaldson and Sun proved a remarkable nonlinear Fredholm alternative for the Kähler-Einstein equations on Fano manifolds. They show this fully nonlinear PDE can be solved if and only if a certain stability condition involving only finite-dimensional algebro-geometric data holds.

In 1982 Shing-Tung Yau received the Fields Medal in part for his 1978 proof of the so-called Calabi Conjecture. In particular Yau proved that if the first Chern class of a compact Kähler manifold vanishes (respectively, is negative), then it admits a Kahler-Einstein metric, i.e. there is a unique Kähler metric in the same class with vanishing (respectively, negative) Ricci curvature.

Yau later conjectured that a solution in the case of Fano manifolds, i.e., those with positive first Chern class, would necessarily involve an algebro-geometric notion of stability. Seminal work of Gang Tian and then Donaldson clarified and generalised this idea. The resulting Yau-Tian-Donaldson conjecture – that a Fano manifold admits a Kähler-Einstein metric if and only if it is K-stable – became one of the most active topics in geometry. Tian established the necessity of a stability condition, introduced the notion of K-stability used in the cited papers, and demonstrated that there are Fano manifolds with trivial automorphism group which do not admit Kähler-Einstein metrics.

Proving such a result had long been understood to involve a vast amalgam of ideas from symplectic and complex geometry, infinite dimensional Hamiltonian reduction and geometric analysis. All methods involved some kind of continuity method; in 2011 Donaldson proposed one involving Kähler-Einstein metrics with cone singularities (published by Springer in *Essays in mathematics and its applications* in 2012).

The main technical obstacle was then how to control certain limits of sequences of Kähler metrics on Fano manifolds (equivalently, how to obtain the "partial C^0-estimate"). One can take the so-called Gromov-Hausdorff limit, but *a priori* this could be a metric space with horrible non-algebro-geometric properties (such as being odd dimensional!)

It was a huge breakthrough when, in 2012, Donaldson and Sun managed to use Bergman kernels to put the structure of a normal projective algebraic variety on the Gromov-Hausdorff limit of a non-collapsing sequence of Fano varieties (published in *Acta Mathematica* in 2014).

Chen, Donaldson and Sun gave a complete solution of the Yau-Tian-Donaldson conjecture for Fano manifolds a few months later. The announcement was published in *International Mathematics Research Notices* in 2014 and full proofs followed in "Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities," "Kähler-Einstein metrics on Fano manifolds. II: limits with cone angle less than 2π," and "Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof," all published in 2015 in the *Journal of the AMS*.

According to one nominator, "This is perhaps the biggest breakthrough in differential geometry since Perelman's work on the Poincaré conjecture. It is certainly the biggest result in Kähler geometry since Yau's solution of the Calabi conjecture 35 years earlier. It is already having a huge impact that will only grow with time."