Hamiltonian Dynamics of monodromy of the
maximal degenerate family of CY manifolds 1
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SCP
\[ \pi: X^{m+1} \rightarrow D^2 \quad \text{holomorphic & proper} \]

\[ C \subset D^2 \quad \Rightarrow \quad M_0^n := \pi^{-1}(C) \]

Assumption

1. \( M_c \) is smooth for \( c \neq 0 \)
2. \( M_0 \) is normal crossing divisor.

\[ C \cdot M_0 = 0 \quad \Rightarrow \quad \text{codim}_X \text{ intersection is transversal} \]

3. \( \exists p \in M_0 \) s.t. \( p \in \bigcap_{i=1}^{m+1} D_i \)
We call such \( \pi : X^{n+1} \to \mathbb{D}^2 \) the maximal degenerate family.

Let \( W \) be a Kähler fan of \( X \).

Define \( H : X \to \mathbb{R} \)

\[
H(x) = |\pi_c(x)|
\]

We consider \( H \) as a Hamiltonian.
Let $X_H$ be the Hamiltonian vector field generated by $H$.

\[ w(V, X_H) = V(H) \quad \forall \in TX. \]

It is well known $X_H(H) = 0$.

Let $\varepsilon \in \mathbb{R}_+$

We will define Poincaré map $\varepsilon : M \to M$ which is a simple diffeomorphism below.
For $\Re \alpha M_\alpha$ (or $\Re \alpha X$), let

$$\gamma_p : \mathbb{R} \rightarrow X \text{ the curve } \begin{cases} \frac{d\gamma_p}{dt} = X_H(\gamma_p) \\ \gamma_p(0) = p \end{cases}$$

Since $X_H(0) = 0$, $H(\gamma_p) = |\pi(\gamma_p)| \equiv \varepsilon$ (constant)

$t \rightarrow \pi(\gamma_p(t))$ looks like

$\varepsilon$
Therefore \( \exists \; \tau \; s.t. \)

\[
\tau = \inf \left\{ t > 0 \mid \gamma_{p}(t) \in M_{e} \right\}
\]

We put

\[
\nu_{e} : M_{e} \rightarrow M_{e} \quad \text{by} \quad \nu_{e}(x) = \gamma_{p}(\tau_{x}).
\]

This is the Poincaré map.
Lemma \( Y_e : M_e \rightarrow M_e \) is a symplectic diffeomorphism

e.g. put \( w = w \mid M_e \) then \( Y_e^* w = w \).

Proof: \( p \in M_e \), \( \tau_p > 0 \)
\( g = Y_e(p) = \gamma_p(\tau_p) \)

\( U_e \) mbd of \( p \) in \( M_e \).

Let \( V_\varepsilon = \{ x \in X \mid d(p, x) < \varepsilon, H(x) = 3 \} \)

\( n+1 \) dimensional

\[ R_v : V_\varepsilon \rightarrow U_e \] a mbd of \( g \) in \( M_e \)

\( v \mapsto y_e(p) \) \( \gamma \)
Here \( t \rightarrow y_x(t) \) satisfies

\[
\begin{aligned}
\frac{d y_x}{dA} &= -x_H \\
y_x(0) &= x.
\end{aligned}
\]

\[ M_x \]

\[ V_\varepsilon \rightarrow V_x \rightarrow y_x(t) \rightarrow V_\varepsilon \]

\[ X \rightarrow P_x(t) \]

The composition is \( P_\varepsilon \).
Note: \( x \rightarrow \varphi(x) \) is a closed sym

diffeomorphism. (from a nbhd of \( x \in X \) to a
neighborhood of \( z \in ) \)

It is also easy to see \( \omega|_{V_x} = R^*_x(\omega|_{V_z}) \)

They imply \( \varphi^*\omega = \omega \)

QED
We want to study:

**Problem**

Study the symplectic diffeomorphism

\[
\varphi_c : M_c \rightarrow M_c
\]

as a Hamiltonian dynamics.
Conj (Fukaya: Galois Symmetry on Floer Cohomology

Conjecture 4.3)

\[ \text{As } \varepsilon \to 0 \text{ } \Psi_\varepsilon \in \text{ converges to a complete integrable system. In particular (using KAM)} \]

\[ \exists \text{ } M^0 \subseteq C M_c \text{ s.t.} \]

1) \( M^0_c \) is \( \Psi_\varepsilon \) invariant and foliated by invariant tori. (by \( \Theta_M C M_c \))

2) \( \text{Vol} (M^0_c \setminus M^0) \to 0 \text{ as } \varepsilon \to 0 \)
I explained it as a conjecture again last year in the Colloquium.

Actually, as I will explain this week, the conjecture seems to be completely wrong.
In place of $\omega$ a kähler form on $X$ we might use certain kähler forms $\omega$ on $X > M$, which diverge at $M_0$.

The new Hamiltonian vector field $X_\omega$ by this symplectic structure.

The I believe the conjecture is correct. I will explain certain points related to it in the second half of this week.
In the first half I will work on Kähler form \( w \) on \( X \) and study \( \xi : M_\xi \to M \), via examples etc.
Examples

$\mathbb{P}^{m+1}$, $m+1$ dimensional complex projective space

$\cup$

$[z_0 : \cdots : z_{m+1}]$

$z_i \in \mathbb{C}$

$(z_0, \cdots, z_{m+1}) \neq (0, \cdots, 0)$

$[z_0 : \cdots : z_{m+1}] = [z_0' : \cdots : z_{m+1}']$

$\forall c \in \mathbb{C} \setminus \{0\}$

$z_i' = cz_i$

Define

$f : \mathbb{P}^{m+1} \rightarrow \mathbb{C}$

$f([z_0 : \cdots : z_{m+1}]) = \frac{z_0 z_1 \cdots z_{m+1}}{z_0^n + \cdots + z_{m+1}^n}$ \(\mathbb{C}^*)\)
$f$ is a meromorphic function

$f(B_\alpha - Z_{c+1})$ is well defined in $\mathbb{C}(\cup \mathbb{L})$

Unless

$z_0 - z_{c+1} - z_0^{a_{c+1}} - \ldots - z_0^{a_1} = 0$

There exists a blow up $\hat{1}_{\hat{\mathcal{P}}_{c+1}}$ of $1_{\mathcal{P}_{c+1}}$

So $f$ is well defined meromorphic map

$f: \hat{1}_{\hat{\mathcal{P}}_{c+1}} \rightarrow \mathbb{C}(\cup \mathbb{L})$

(This point will be explained more later)
$\exists \in C \setminus \{0\} | \varepsilon | \text{ is small}$

$M_\varepsilon$ is a hypersurface $C_\varepsilon$ defined by

an equation

$Z_0 \cdots Z_{m+1} = \varepsilon (Z_{m+2}^m + \cdots + Z_{m+2}^{m+2})$

$M_\varepsilon$ is a Calabi-Yau manifold and is its typical example

$m = 3 \implies$ quintic 3 fold.
This is a famous example in Mirror symmetry.

\[ D^2 \subset C(\nu_1 \nu_0) \text{ mod } \nu_1 = 0 \]

\[ X = f^{-1}(0') \subset \mathbb{P}^{n+1} \]

\[ f^{-1} : X \longrightarrow D^2 \]

We may take $D^2$ small so $\Omega^2(2) = M_2$ is smooth for $\nu \neq 0$.
\[ M_0 = \mathcal{K}^I(x) \text{ is given by} \]

\[ Z_0 - Z_{m+1} = 0 \quad (i.e. \text{ blow up actually}) \]

\[ D_i \rightarrow Z_i = 0 \]

\[ M_0 = D_0 \cup -uD_{m+1}, \quad \text{normal crossing divisor} \]

\[ \mathbb{P} \quad \text{for example} \quad p = (1, 0, \ldots, 0) \]

\[ \mathbb{P} \in D, u = uD_{m+1} \]
Thus \( \pi : X \rightarrow D^2 \) is a maximal degenerate family.

Take an appropriate fibre for a \( X \in P(n) \).

We are in the situation discussed before \( \Psi_e : M_e \rightarrow M_e \) is the induced map.
We want to study \( \varphi_2 : M_2 \to M_2 \) as a symplectic diffeomorphism.

We state a theorem in case \( n=2 \):

\[
M_2 \cong \mathbb{P}^3
\]

\[
\mathcal{Z}_0 \cdot \mathcal{Z}_3 = \varepsilon (z_0^4 + z_1^4 + z_2^4 + z_3^4)
\]

This is a quartic surface

(An example of a K3 surface.)
\[ f_\epsilon : M^2 \rightarrow M^2 \text{ has 408 fixed points for } \epsilon \neq 0 \text{ small.} \]

Remark: \( \text{rk } H(M^2, \mathcal{D}) = 24 = \chi(M^2) \)

408 is much bigger.
We start studying $\Psi_\varepsilon : M_\varepsilon \rightarrow M_\varepsilon^0$ via perturbation theory in Hamiltonian Dynamics.

Reference

*Encyclopedia of Math. Science*

*Dynamical Systems III* (ed. Arnold)

Chapter 5 (Perturbation theory of integrable systems)
We consider \( \varphi_c : M^n_\varepsilon \to M^m_\varepsilon \) (for maximal descent
\[ f_{\text{sing}} \times \to \mathbb{D}^2 \])

**Proposition 1.**

\[ \exists \ M^{00}_\varepsilon \subset M^m_\varepsilon \ \text{st.} \]

1. \( \varphi_c (M^{00}_\varepsilon) = M^{00}_\varepsilon \)
2. \( M^{00}_\varepsilon \) is foliated by the \( \varphi_c \) invariant Lagrangian tori.
3. \( \text{Vol} (M^{00}_\varepsilon) > 0 \).
Note \[ \text{Vol}(M_\varepsilon - M^\omega_\varepsilon) \rightarrow 0. \]

This proposition is actually an easy consequence of KAM theory.

Review of KAM

We begin with a definition of complete integrable system.

\[ T^m \rightarrow X \]

\[ \downarrow \]

B

(25)
$T^m \rightarrow M^2m$  
\[ \lambda \]
\[ \beta \] 

(M, w) symplectic

fiber bundle

fiber = m dim compact  
log. sub manifold \((w|_{fiber} = 0)\)

\[ \downarrow \] 

Liouville-Arnold

fiber \(\cong T^m = \mathbb{R}^m/\mathbb{Z}^m\)

Action angle coordinate \(\theta_1, ..., \theta_m, \varphi_1, ..., \varphi_m\)
\( \pi \left( \mathcal{N} \times \mathcal{E} \right) \)

\( q \) \( q \) \( \text{coordinate of } \mathcal{B} \)

\( \frac{\partial}{\partial r_i} = \frac{\partial}{\partial r_i} \text{ tangent to the fiber} \)

\( W = \sum \text{dr}_i \text{ col.} \)

\( T^a = \mathbb{R}^n / \mathbb{Z}^n \) \( \text{fiber} \)

\( T \) \( r \) \( r \) \( \text{coordinate of } \mathbb{R}^n \)

\( H : \mathcal{M}^m \to \mathbb{R} \) \( \text{Hamiltonian} \)

\( H_0 (\mathcal{E}) \) \( H_0 \) \( \text{depends only on } q \)
Namely \[ \nu \to H_0. \]

In other words, \[ \{ \Pi_i, H_0 \} = 0 \quad i = 1 \ldots m. \]

Note if \( \pi \) is Poisson bracket \( \{ f, \pi \} = \dot{x}_f(s) \).

\[ X_{H_0} \quad \text{Hamiltonian vector field} \]

on each fiber

\[ X_{H_0} = \sum \frac{\partial H_0}{\partial \pi_i} \partial_i \]

\( \nabla \)

\( \text{contract on a fiber} \quad \otimes \)

\( \text{on the fibers} \)
Theorem (KAM)

Let $H_0, M$ etc as above. $H_1 : M \to \mathbb{R}$

$H_{\varepsilon} = H_0 + \varepsilon H_1$, $X_{H_{\varepsilon}}$ its Hamiltonian vector field

We assume certain non-degeneracy conditions for $H_0$

described later

=) $\exists M_0^\varepsilon \subset M \text{ s.t.}$

1) $M_0^\varepsilon$ is $X_{H_{\varepsilon}}$ invariant

2) $M_0^\varepsilon$ is foliated by Lagrangian tori (close to the)

3) $\text{Vol}(M - M_0^\varepsilon) \to 0$ as $\varepsilon \to 0$
Non-degeneracy condition

\[ \vec{q} \in B \quad \chi_{H_0} = \sum w_i(\vec{q}) \frac{\partial}{\partial q_i} \]

\[ p_i = p_m \]

Non-degeneracy:

\[ \det \left( \frac{\partial w_i}{\partial q_j} \right) \neq 0. \]
We go back to the point of Proposition 1,

\[
\times
\]

\[
\downarrow
\]

\[
\Rightarrow
\]

\[
D^2
\]

Take \( p_0 \in \mathbb{R}^2 \) to be \( M_0 \), let \( p_x \in D_1 \cap -nD_{m+1} \).

\( D_1 \) irreducible component of \( M_0 \).

A neighborhood \( U \) of \( p \), \( UC \subset C^{m+1} \).

\( D_x \cap U \subset (x \times x \times 0 \times 0 \times 0 \times \ldots \times 0) \cap U \subset C^{m+1} \) \( p_d = \overline{3} \)

(3)
Rescale by $R = \frac{1}{3}$, replace $u \rightarrow u^3$

\[ W \text{ on 1 ball is} \]
\[ Z_i = x_i + N_i y_i, \]
\[ \sum_{i=1}^{n+1} \text{term of order } 3 \]

\[ H: X^{\text{out}} \rightarrow D \rightarrow \Pi \]

\[ H \text{ on } 1 \text{ ball (in shell before scale)} \Rightarrow \]

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\[ H = \left| Z_{1} \cdots Z_{n+1} \right| + \text{term of order } \varnothing \]

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\[ \text{Poincaré's theorem} \]

\[
\gamma \begin{pmatrix} x_1, y_1 \\ x, y \end{pmatrix} \rightarrow \begin{pmatrix} x_1, y_1 \end{pmatrix}
\]

\[ w = \sum \alpha x_1 / \text{gcd} x_1, y_1 \] 

\[ \left| \psi - \text{id} \right|_{\infty} < \varepsilon \]
In new coordinates

\[ H = \prod \sqrt{6x_i^2 + 5y_i^4} \quad + \text{terms of order } 8. \]

\[ \equiv H_0 \]

\[ r_i = \sqrt{x_i^2 + y_i^2} \quad \text{if } (r_i, r_j) = 0 \]

\[ H_0 = r_i - r_{n+1} \]

\[ \sum H_0, r_i = 0 \quad i=1-n+1 \]
$U \setminus M_0 \to \mathbb{R}^{n+1}$

fiber = Lagrangian tori.

So $H_0$ is a complete integrable system.

It satisfies somewhat weaker non-degeneracy condition.
Refena Dynamic System III page 183

\[ H_0 : M^{n} \rightarrow \mathbb{R} \text{ completely integrable as before} \]

\[ (M^{n} \rightarrow B) \]

\[ (a, b) \quad \text{is} \]

\[ w_{i}(3), \ldots, w_{i}(2) \quad \text{frequencies} \]

**Def:** Isoenergetically non-degenerate

\[ w_{i}(3) \neq 0 \quad \overline{w}_{i} = w_{i} / w_{i} \quad i = 3, \ldots, m \]

\[ r_{k} \left( \frac{\partial \overline{w}_{i}}{\partial q_{j}} \right) = n - 1 \]
KAM theorem still holds under this weaker condition.

(Theorem 13 of Dynamical Systems III, page 183)

Note: Why nonresonance condition?

If \( \mathbf{w}, \cdots, \mathbf{w}_n \) has rationally related

then \( \alpha \to \mathbf{r}_\alpha \in \mathbb{R} \) re

\[ \sum k_\alpha \mathbf{w}_\alpha = 0 \quad (\mathbf{r}_\alpha - \mathbf{r}_n) \neq 0 \]
Then the occur resonance which makes the system unstable.

(that w small perturbation destroy invariant tuvi.)

Non-degeneracy \Rightarrow \text{ Resonance occurs w measure 0 value of } \vec{E}.

For this "is energetically non-degenerate" is enough.
Going back to our situation

\[ H_0 = r_1 - r_{n+1} \]

\[ r_i = \sqrt{x_i^2 + y_i^2} \]

\[ T^{n+1} \subseteq X^{n+1} \rightarrow B \]

Fiber is given by

\[ f_1 = c_i \quad \text{and} \quad x_{n+1} = c_{n+1} \]

\[ X_{r_i} = \frac{1}{r_i} \frac{2}{\delta \theta} \quad r_i e^{2\pi i \theta} = x_i + i y_i \]
$\theta_i$'s are angle coordint (coordinate $\theta$
$T^{\mu \nu} = \mathcal{R}^{\mu + i} / 2^{n+1}$)

$W_i = \frac{r_i}{r_i - r_{i+1}}$

$r_i$ are action coordint

Lemmas

Our $H_0$ is isoelectronically non-degenerate

$\therefore \frac{W_i}{W_1} = \frac{r_i^2}{r_1^2} // 40$
The conclusion of KAM holds in a small neighborhood of $\mathcal{O}$.

(This is about $X^\alpha \to D^\alpha$ and $X^\beta$.)

The KAM tori for this system give a KAM tori of $\mathcal{O}$.

$M_c \xrightarrow{\psi_2} M_c$.  

Q.E.D.