Hamiltonian Dynamics of Monodromy of the maximal degenerate family of CY manifolds 4
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\[ f : \mathbb{P}^3 \rightarrow (\mathbb{P}^1)^3 \]

\[ f = \frac{z_0^4 z_1 z_2 z_3}{z_0^4 + z_1^4 + z_2^4 + z_3^4} \]

\[ H = |f| \]

\[ \Psi_e : \mathbb{P}^1 \rightarrow M_e \to M_e \]

Poincaré map
\( \mathcal{D}_0 : \text{ null of } z_0 = 0 \rightarrow \mathbb{P} \)

\[ V = \frac{z_0}{z_1}, \quad z = \frac{z_2}{z_1}, \quad w = \frac{z_3}{z_1} \]

\[ H = H_{00} H_{01} + \text{ small perturbation} \]

\[ H_{00} = |V| \]

\[ H_{01} = \frac{|zw| \sqrt{1 + |z|^2 + |w|^2}}{1 + z^4 + w^4} = \frac{1}{h(zw)} \]
\[ \mathbb{P}^2 \setminus \{z = 0, w = 0\} \leq \mathbb{P}^1 \]

basic locus

\[ (1 + z^\alpha - w^\alpha = 0) \]

\( \xi \) is approximated by the

integration of

\[ H_{0,1} = h(z, w) \]
Lemma

\((z,w)\) is a critical point of \(h\)

\[
\iff \quad z, w \in \mathbb{H}, \pm i \gamma
\]

\[
\iff \quad z^4, w^4 \in \mathbb{R}
\]
It suffices to show

\[ \frac{\partial g}{\partial e} = 0 \quad \frac{\partial g}{\partial \sigma} = 0 \Rightarrow \mathbb{R}, \sigma \subset \mathbb{R} \]

at \((e, \sigma) = 0\):

\[ \frac{\partial g}{\partial e} (u, u) = 0 \Rightarrow \frac{u}{1 + w} \subset \mathbb{R} \]

\[ \frac{\partial g}{\partial \sigma} (u, v) = 0 \Rightarrow \frac{v}{1 + w} \subset \mathbb{R} \]
\[ z \mapsto ze^{i\phi} \mapsto ze^{i\sigma} \]

\[ |z | \sqrt{1 + |z|^2 + M^2} \text{ does not change} \]

\[ |1 + ze^{i\phi} + w^i| \text{ changes} \]

\[ z^a = z^b \quad w^a = w^b \]

\[ |1 + e^{i\phi} z^i + e^{i\sigma} w^i| = g^2 (\phi, \sigma) \]
\[
\frac{Z'}{1 + \lambda w}, \quad \frac{w}{1 + \lambda w} \quad (11) \rightarrow z', \quad w 
\]

(1) exercise.

\[
h(z', w) = \frac{1 - \lambda(w) \sqrt{1 - |z'|^2 - \lambda^2 |w|^2}}{1 + \lambda^2 w^2 + |w|^2}.
\]

\[
z \mapsto z', \quad \overline{z} \mapsto \overline{z'} \quad \text{symmetry of } h
\]

\[\theta\]
Thus by lemma it suffices to study the case

\[ x^2, y^2 > 0 \]  \( \text{g} \)

\[ x^2 > 0 \text{ if } y \neq 0 \]  \( \text{g} \)

\[ y < 0 \text{ if } x \neq 0 \]  \( \text{g} \)

(1) Lemma 2

\((x, y) \in \mathbb{R}^2, \) is a critical point \( \Rightarrow \)

\[ x_y \sqrt{1 + x^2 + y^2} \]

\[ \frac{1}{1 + x^2 + y^2} = h(x, y) \]  \( \text{g} \)
High school Math

\[
\left( \text{calculate } \frac{2b}{dx}, \frac{2b}{db} \right)
\]

Lemma 3

\[
\frac{x^2 \sqrt{1+x^2+c}}{1+x^2-5} \quad \text{has no critical point for } (1.5) \in \mathbb{R}_+^2.
\]
\[
\frac{df}{dy} > 0 \quad \text{for} \quad 0 < y < 1 + \nu y \\
\frac{df}{dy} < 0 \quad \text{for} \quad y > 1 + \nu y
\]

Direct Calculations
\[
\frac{\sqrt[5]{1+x^2+y^2}}{1-x^2-y^2} = \frac{1}{x^2} \left( \frac{\sqrt{1 + \frac{y^2}{x^2} + \frac{1}{x^2}}}{\frac{1}{x^2} - \frac{\sqrt{1 + \frac{y^2}{x^2}}}{} - 1} \right)
\]

\[
\text{point } \vec{w} \text{ of } \vec{R}_{123}
\]

\[Q\]
Thus we proved that the critical point of

\[ \{z, w \in \mathbb{C}^2 \mid \frac{z^4 + w^4}{(1 + z^2 + w^4)} \text{ are 16 points} \} \]
There are 4 faces: $z_0 = 0$, $z_1 = 0$, $z_2 = 0$, $z_3 = 0$

So, there are $4 \times 16 = 64$ fixed pts.

Near the face:

There are $6 \times 8 = 48$ fixed pts

Near the edge:

(No near the vertex)
It remains to study $\Phi$ near the basic locus.

$$B_0 \quad \Rightarrow \quad Z_0 = 0$$

$$z_1 + z_2 + z_3 = 0$$

We take a short cut by using symmetry.
\[ R = \frac{|ZW| \sqrt{1 + |Z|^2 + |W|^2}}{1 + |Z|^4 + |W|^4} \]

\( R \) is invariant by a group \( G \).

\[ 1 \rightarrow (\mathbb{Z}_4)^2 \rightarrow G \rightarrow S_3 \rightarrow 1 \]

\( \mathbb{Z} \rightarrow \mathbb{Z}, \pm i \mathbb{Z} \)

\( W \rightarrow \pm W, \pm i W \)
\[
\bar{\Sigma} = \left\{ (z, w) \mid 1 + z^k + w^k = 0 \right\}
\]

\[
\subseteq \\
\text{genus 3 curve}
\]

\[
\frac{\bar{\Sigma}}{G} = ?
\]

\[
\text{Fixed point of } G \text{ action}
\]
3 kinds

\begin{align*}
\text{a} & : \# I_a = 8 & [1:0:0] \\
\text{b} & : \# I_b = 2 & [1, x^{2/3}, x^{2/3}] \\
\text{c} & : \# I_c = 3 & [4, 3, 3^2] \\
\end{align*}

\begin{align*}
\chi & = e^{2\pi i/8} \\
3 & = e^{2\pi i/3}
\end{align*}
\[
\overline{z}_3 / g
\]

Check \( X(5^2 - 3\mu t) = -1 \)

\[
\begin{align*}
4g &= 96 \\
-96 + \frac{96}{8} + \frac{96}{2} + \frac{96}{3} &= -96 + 12 + 48 + 32 = -4 = X(\overline{23})
\end{align*}
\]
\[ Z_3 = 1 + z^4 + w^4 = 0 \]

12 pts.  \( a \)  \( \Rightarrow I_a = 8 \)
48 pts.  \( b \)  \( \Rightarrow I_b = 2 \)
32 pts.  \( c \)  \( \Rightarrow I_c = 3 \)

12 pts. \( c \) corresponds to \( D_4 \cap D_4 \)  \[ 1 = 1, 3, 3 \]
4 pts. \( \bigcirc \)
\[ z_2 = z_1 = 0 \]

\[ z_2' + z_3' = 0 \]

\[ 4 \mu_5 \]
H_{J_{\text{b}}} = 2

H_{J} \text{ symm} 

H_{Z_{\text{c}}} = 3
fixed pt \( \iff \) fixed pt of \( g \)

\( g \) acts on \( X \)

doubly connected core
\[ b \rightarrow \text{hyperbolic} \quad \frac{96}{2} = 48 \]
\[ c \rightarrow \text{elliptic} \quad \frac{96}{3} = 32 \]

\[ 16 \rightarrow \text{cnt pf of } h \]

\[ 16 + 48 + 32 = 96 \quad \text{fixed points near each face} \]
\[ 4 \quad \text{fixed points near each edge} \]
96 \times 4 + 4 \times 6 = 408

\text{Then } \, Y_2 : M_2 \rightarrow M_c \text{ has 408 fixed points}
A bit more explanation on the dynamics near the basic locus.

\[ \mathbb{P}^3 \rightarrow \mathbb{P}^3 \]

\[ \text{blow up} \quad \bigcup \]

49 pts

\[ B_0 \cap B_1 \cap B_2 \cap B_3 \]

disjoint

basic locus

\[ B_1 = \{ z_0 z_1 z_2 z_3 = 0 \} \]

\[ \cap \{ z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0 \} \]
\[ f = 0 \]

\[ D \quad z_0 z_1 z_2 z_3 = 0 \]

\[ f = 0 \]

\[ A \quad z_0^x + z_1^x + z_2^x + z_3^x = 0 \]

\[ f = \frac{z_0 z_1 z_2 z_3}{z_0^x + z_1^x + z_2^x + z_3^x} \]
Blow up \( B_i \):

\[ \hat{f} : \hat{B}_i \times \mathbb{P}^1 \to \mathbb{P}^1 \]

projection to 2nd factor

\( \hat{f} = 0 \)

\( D \)
\[ \hat{f}^{-1}(\hat{\Sigma}) \cap (\hat{B}_i \times \Pi) \cong \hat{B}_i \]

\[ \cong \hat{Z}_3 \]

\[ X_H \text{ is tangent to } \hat{X}_H \]

Poincaré map on \( X_H \) on \( \hat{B}_i \) is

\[ \psi_{\epsilon_i} : \hat{B}_i \rightarrow \hat{B}_i \] some slow dynamics

( I do not know the precise form )

\[ X_h : h \circ G \circ \text{inv.} \]
This is enough to show

\[ 3 \text{ type a fixed points} \]

\[ 48 \text{ type b fixed points} \]

\[ 32 \text{ type c fixed points} \]

on it.

It may be small but unlikely.

The dynamics on $\Sigma$ is likely

a 96-fold cover of this picture.
i.e. $Z_3$ is triangulated by 192 triangles

around a

16 triangles
Around b

4 triangles

Around c

6 triangles
Neighborhood of $\Sigma_3$

Disk bundle core $\Sigma_3$

the dynamics of fiber direct

$\Sigma_3$, direct $X_3$
Note at fixed points of type a two of $\Sigma_3$ intersect transversally.

$D_1$ \quad \quad D_0$

$\Sigma^0_3 \cap D_0$

$\Sigma^1_3 \cap D_0$

$\Sigma^0_3 \cap \Sigma^{-1}_3$, 4 pts of type a.
KAM holds in a mix of these $Z_3$'s.

...nearly... near $D_1$ or $B_1$.

KAM holds.

But the dynamics is likely becomes Caotic inside.
Possible generalization and open questions.

1) Quintic 3-fold?

\[ \mathbb{P}^4 \rightarrow \mathcal{C} \]

\[ \frac{\mathbb{Z}_7, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_9}{\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_3 - \mathbb{Z}_3} \]

\[ \text{\textit{hop}} \rightarrow \text{\textit{check}} \]

\[ \text{\textit{hop}} \rightarrow \text{\textit{check}} \]

\[ \text{\textit{check}} \]
\[ M_3 = T_3^{(3)} \]

\[ \mathbb{Z}_6 \mathbb{Z}_1 \mathbb{Z}_2 \mathbb{Z}_3 \mathbb{Z}_4 = (\mathbb{Z}_6^{(5)} + \mathbb{Z}_1^{(5)} + \mathbb{Z}_2^{(5)} + \mathbb{Z}_3^{(5)} + \mathbb{Z}_4^{(5)}) \]

Quintic 3-fold

\[ H_1 = H_1 \]

\[ \text{Problem: Find the number of fixed points} \]

\[ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \quad ? \]
More generally

Let $X$ be a toric manifold

$D(X)$ toric divisor

$m^{-1}(dP) = D$

$p = m(X)$

$n : X \to \mathbb{R}^n$

corresponding to $n$
Assume $L$ is effective

$s_0 : X \rightarrow L$ section \quad $s_0^*(0) = 1$

$s : X \rightarrow L$ another section

$j : \frac{s_0}{s} : X \rightarrow C$ monomorphism

$f : X \rightarrow (\cup V)$ blow up

appropriate blow up
$M_\mathcal{E} \leftarrow \hat{f}^{-1}(\mathcal{E})$  

CY hyperbanty

(This is a typical construction of CY moduli all appears in the study of Mirror Symmetry.)

$H = |f|$  

$\Phi: M_2 \rightarrow M_2$  

Poincaré map
Problem

Study $\gamma_a$ as Hamiltonian dynamics

de calculate the number of fixed pts.

Rem

$Z_a Z_b Z_c$ $\overline{Z_a Z_b Z_c}$
$3(3^3 - Z_b Z_c Z_3)$$\overline{3(3^3 - Z_b Z_c Z_3)}$

S: homogeneous polynomial of order 4

is an example
We discussed the case

$$\omega = \omega_0 + \omega_1 + \omega_2 + \omega_3$$

But there are many other cases.

Some part of the argument applies.

But the short cut may symmetry does not work.