Azumaya noncommutative geometry and D-branes
- an origin of the master nature of D-branes

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Abstract. In this lecture I review how a matrix/Azumaya-type noncommutative geometry arises for D-branes in string theory and how such a geometry serves as an origin of the master nature of D-branes; and then highlight an abundance conjecture on D0-brane resolutions of singularities that is extracted and purified from a work of Douglas and Moore in 1996. A conjectural relation of our setting with 'D-geometry' in the sense of Douglas is also given. The lecture is based on a series of works on D-branes with Shing-Tung Yau, and in part with Si Li and Ruifang Song.

DEDICATION. This lecture is dedicated to *Shiraz Minwalla*, *Mihnea Popa*, *Ling-Miao Chou*, who together made this project possible; and to my mentors (time-ordered): *Hai-Chau Chang*, *William Thurston*, *Orlando Alvarez*, *Philip Candelas*, *Shing-Tung Yau*, who together shaped my unexpected stringy/brany path.

Outline.

1. D-brane as a morphism from Azumaya noncommutative spaces with a fundamental module.
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   - · Reflection and a conjecture on D-geometry in the sense of Douglas:
     Douglas meeting Polchinski-Grothendieck.

2. Azumaya geometry as the origin of the master nature of D-branes.
   - · Azumaya noncommutative geometry as the origin of the master nature of D-branes.
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Notes and acknowledgements added after the workshop.
References.

1   D-brane as a morphism from Azumaya noncommutative spaces with a fundamental module.

My lecture today is based on three guiding questions:

Q.1 *What is a D-brane?*
Q.2 *What is a noncommutative geometry?*
Q.3 *How are the two related?*

To reflect the background of this lecture, I assume:

When: October, 1995; or, indeed, 1989.
Where: In the *geometric phase of* Wilson's theory-space $\mathcal{S}_{Wilson}^{d=2, CFTw/boundary}$ for $d = 2$ conformal field theory with boundary; // and with assumption that *open string tension is large enough* (so that *D-brane is soft* with respect to open strings).

The emergence of a matrix-/Azumaya-type noncommutativity.

• Let me begin with Polchinski's TASI lecture on D-branes in 1996 ...

· ... and first recall the very definition of a D-brane from string theory:

Definition 1.1. [**D-brane**]. A *Dirichlet-brane* (in brief *D-brane*) is a submanifold/cycle/locus in an open-string target space-time in which the boundary/end-points of an open string can lie.

· *Figure 1-1: Oriented open strings with end-points on D-branes.*

- $f : X \to Y$, where X is endowed with local coordinates $\xi := (\xi^a)_a$, Y local coordinates $(y^a; y^\mu)_{a,\mu}$, and f is given by $y^a = \xi^a$ and $y^\mu = f^\mu(\xi)$.

· This definition, though mathematically far from obvious at all as what it'll lead to, is very fundamental from physics point of view. // It says that all the fields on D-branes and the dynamical law that governs them are created by open strings.

· Open strings vibrate and its end-points create (both massless and massive) fields on the D-brane world-volume. // Massless fields are created by an open string with both ends on the same branes. // There are two complementary sets of these: One corresponds to vibrations of ends of the open string in the tangential directions along the D-brane. This creates an $u(1)$ gauge field on the branes. The other set corresponds to vibrations of ends of the open string in the normal directions to the D-brane. This creates a scalar field that describes fluctuations of the D-brane in space-time.

• When r-many D-branes coincide in space-time, something mysterious happens:

· One key feature of an open or closed string, compared to the usual mechanical string in our daily life, is that its tension is a constant in the theory; // and hence the mass of states or fields on D-branes created by open-strings are proportional to the length of the string. // Once r-many D-branes are brought to coincide in space-time, there are states/fields that were originally massive but now becomes massless. // (Continuing *Figure 1-1*.)

· In particular, the gauge fields A_a on the stacked D-brane is now enhanced to $u(r)$-valued // and the scalar field y^μ on the D-brane world-volume that describes the deformation of the brane is also $u(r)$-valued.

· For this, Polchinski made the following comment in his by-now-standard textbook for string theory:

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· ([Po2: vol. I, Sec. 8.7, p. 272].) (With mild notation change.)
  "For r-separated D-branes, the action is r copies of the action for a single D-brane. We have seen, however, that *when the D-branes are coincident, there are r^2 rather than r massless* vectors and *scalars on the brane*, and we would like to write down the effective action governing these. The fields $y^\mu(\xi)$ and $A_a(\xi)$ will now be $r \times r$ matrices. For the gauge field, the meaning is obvious – it becomes a non-Abelian $U(r)$ gauge field. *For the collective coordinates y^μ, however, the meaning is mysterious: the collective coordinates for the embedding of r D-branes in spacetime are now enlarged to $r \times r$ matrices.* This 'noncommutative geometry' has proven to play a key role in the dynamics of D-branes, and there are conjectures that it is an important *hint about the nature of spacetime*."

A naive/direct space-time interpretation of this noncommutativity.

● As y^μ are meant to be the coordinates for the open-string target-space-time Y, it is very natural for one to perceive that somehow there is something noncommutative about this space-time that is originally hidden from us before we let the D-branes collide. // And once we let the D-branes collide, this hidden feature of space-time reveals itself suddenly through a new geometry whose coordinates are matrix/Azumaya-algebra-valued. // It seems to me that this is what Polchinski reflects in the above comment and it turns out to be what the majority of stringy community think about as well.

A second look: What is a D-brane (mathematically)? - From Polchinski to Grothendieck.

● Re-think about the phenomenon *locally* and *from Grothendieck's construction* of modern algebraic geometry via the language schemes:

· Let $R(X)$ be the ring of local functions (e.g. $C^\infty(X)$ in real smooth category) of X and $R(Y)$ be the ring of local functions on Y (e.g. $C^\infty(Y)$). // Then $\xi^a \in R(X)$; $y^a, y^\mu \in R(Y)$; and f above is equivalently but contravariantly given by a ring-homomorphism $f^\sharp : R(Y) \to R(X)$ specified by

$$y^a \longmapsto \xi^a \qquad \text{and} \qquad y^\mu \longmapsto f^\mu(\xi),$$

  i.e. $f : X \to Y$ is determined how it *pulls back local functions from Y to X*.

· When r-many D-branes coincide, formally y^μ becomes matrix-valued. But y^μ takes values in the function ring of X under $f^\sharp$. // This suggest that the original $R(X)$ is now enhanced to $M_r(R(X))$ (or more precisely $M_r(R(X) \otimes_{\mathbb{R}} \mathbb{C}) = M_r(\mathbb{C}) \otimes_{\mathbb{R}} R(X)$). // In other words, *the D-brane world-volume becomes matrix/Azumaya noncommutatized!*

Remark 1.2. [*pure open-string effect*]. It is conceptually worth emphasizing that, from the above reasoning, one deduces also that *this fundamental noncommutativity on D-brane world-volume is a purely open-string induced effect*. // No B-field, supersymmetry, or any kind of quantization is involved.

Remark 1.3. [*Lie algebra vs. Azumaya/matrix-ring algebra*]. Acute string theorists may recall that in the original string-theory setting and in the world-volume field-theory language, this field y^μ is indeed an $u(r)$-adjoint scalar. So, *why didn't we take directly the Lie-algebra-enhancement $u(r) \otimes R(X)$ to the function ring $R(X)$ of the D-brane world-volume X?* // The answer comes from two sources:

(1) *For geometry reason*: Local function ring of a geometry has better to be associate and with an identity element 1. // Without the latter, one doesn't even know how to start for a notion of localization of the ring, a concept that is needed for a local-to-global gluing construction.

(2) *For field-theory reason*: The *kinetic term* is the action on D-brane world-volume involves matrix multiplication; it is not expressible in terms of Lie brackets alone.

Proto-Definition 1.4. [D-brane: Polchinski-Grothendieck]. A *D-brane* is *an Azumaya noncommutative space with a fundamental module*

$$(X^{Az}, \mathcal{E}) := (X, \mathcal{O}_X^{Az}, \mathcal{E}),$$

where $\mathcal{O}_X^{Az} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$. A *D-brane on Y* is a morphism

$$\varphi \; : \; (X^{Az}, \mathcal{E}) \; \longrightarrow \; Y$$

defined by

$$\varphi^{\sharp} \; : \; \mathcal{O}_Y \; \longrightarrow \; \mathcal{O}_X^{Az}$$

as an equivalence class of gluing systems of ring homomorphisms of local function rings from Y to X.

- Two reasons I call this a proto-definition for D-branes:

  (1) I *focus only on fields on D-branes that are relevant to the occurrence of the matrix/Azumaya type noncommutativity in question.*

  (2) I conceal *subtle local-to-global issues from the constructibility and nonconstructibility in noncommutative geometry*, which I need to explain and will come back ...

... but, to help casting away the possible doubt from string theorists as whether this proto-definition makes sense, let me give first a very simple, concrete, and yet deep enough example which we are now ready.

Example 1.5. [D0-brane on the complex line $\mathbb{A}_{\mathbb{C}}^1$ via Polchinki-Grothendieck]. An *Azumaya point/$\mathbb{C}$ with a fundamental module of rank r* is given by

$$(pt, End_{\mathbb{C}}(E), E),$$

where E is isomorphic to $\mathbb{C}^r$. This is our *D0-brane.* // To be explicit, let's fix an isomorphism $E \simeq \mathbb{C}^r$, which fixes also the $\mathbb{C}$-algebra isomorphism $End_{\mathbb{C}}(E) \simeq$ the $\mathbb{C}$-algebra $M_r(\mathbb{C})$ of $r \times r$ matrices. One should think of this as a noncommutative point

$$Space\,(M_r(\mathbb{C})),$$

whose function ring is given by $M_r(\mathbb{C})$, with a built-in module $\mathbb{C}^r$ of the function ring. // We take the complex line $\mathbb{A}_{\mathbb{C}}^1$ as an affine variety over $\mathbb{C}$, whose local rings is given the polynomial ring $\mathbb{C}[y]$ over $\mathbb{C}$ in one variables y. One could think of this y as a *coordinate function* on $\mathbb{A}_{\mathbb{C}}^1$. // In algebro-geometric notation (and with a few subtleties concealed),

$$\mathbb{A}_{\mathbb{C}}^1 \;=\; Spec\,(\mathbb{C}[y]).$$

Following the setting above, a *D0-brane on $\mathbb{A}_{\mathbb{C}}^1$* is then a *morphism*

$$\varphi \; : \; (Space\,(M_r(\mathbb{C})), \mathbb{C}^r) \; \longrightarrow \; \mathbb{A}_{\mathbb{C}}^1$$

defined by a $\mathbb{C}$-algebra homomorphism

$$\varphi^{\sharp} \; : \; \mathbb{C}[y] \; \longrightarrow \; M_r(\mathbb{C}).$$

This, in turn, is determined by an (arbitrary) specification

$$y \; \longmapsto \; m_{\varphi} \; \in \; M_r(\mathbb{C}).$$

Now comes the most essential question:

Q. *Does this match with how D-branes behave in string-theorists' mind?*

Let's now examine this by looking at two things:

  (1) the image 0-brane with Chan-Paton sheaf on $\mathbb{A}_{\mathbb{C}}^1$;

  (2) how do they vary when we vary φ.

Here, we adopt the standard set-up of Grothendieck's theory of (commutative) schemes:

  (1) The image 0-brane $Im\,\varphi$ on $\mathbb{A}_{\mathbb{C}}^1$:

- This is the subscheme of $\mathbb{A}^1_{\mathbb{C}}$ defined by the ideal $I_\varphi := Ker\,\varphi^\sharp = (\varphi^\sharp)^{-1}(0) \subset \mathbb{C}[y]$.

- Let $I_\varphi = ((y - c_1)^{n_1} \cdots (y - c_k)^{n_k})$. Then $(y - c_1)^{n_1} \cdots (y - c_k)^{n_k}$ is the minimal polynomial for m_φ. In particular, $n_1 + \cdots n_k \le r$ and, ignoring multiplicity, $\{c_1, \cdots, c_k\}$ is exactly the set of eigen-values of m_φ.

- In plain words, this says that $Im\,\varphi$ is a collection of fuzzy/thick points supported at points $c_1, \cdots, c_k$ in the complex line $\mathbb{C}$ with multiplicity of fuzziness $n_1, \cdots, n_k$ respectively.

· The Chan-Paton sheaf $\varphi_*(\mathbb{C}^r)$:

- Through the $\mathbb{C}$-algebra homomorphism $\varphi^\sharp : \mathbb{C}[y] \to M_r(\mathbb{C})$, the $M_r(\mathbb{C})$-module $\mathbb{C}^r$ becomes a $\mathbb{C}[y]$-module with $I_\varphi \cdot \mathbb{C}^r = 0$. // Thus, $\varphi_*(\mathbb{C}^r)$ is simply $\mathbb{C}^r$ as a $\mathbb{C}[y]/I_\varphi$-module.

- Geometrically, this says that $\varphi_*(\mathbb{C}^r)$ is a 0-dimensional coherent sheaf on $\mathbb{A}^1_{\mathbb{C}}$, supported on the 0-dimensional subscheme $Im\,\varphi$ of $\mathbb{A}^1_{\mathbb{C}}$.

(2) Deformations of φ are *defined by* deformations of the $\mathbb{C}$-algebra homomorphism $\varphi^\sharp$. // The corresponding $Im\,\varphi$ and $\varphi_*(\mathbb{C}^r)$ on $\mathbb{A}^1_{\mathbb{C}}$ vary accordingly.

These are illustrated in FIGURE 1-2. From this very explicit example/illustration, we see that:

· The notion of *Higgsing and un-Higgsing* of D-branes and of *recombinations* of D-branes are nothing but outcomes of *deformations of morphisms* from an Azumaya space with a fundamental module, as is defined in Proto-Definition 1.4.

In other words, our setting does indeed capture some key features of D-branes in string theory!

$\square$

Remark 1.6. [*D-brane world-volume vs. open-string target-space-time*]. Now we have two aspects of this matrix/Azumaya-type noncommutativity: one as part of a hidden structure of open-string target-space-time revealed through stacked D-branes, and the other as a fundamental structure on the D-brane world-volume when D-branes become coincident. // There are two fundamental reasons we favor the latter, rather than the former:

(1) *From the physical aspect/a comparison with quantum mechanics*: In quantum mechanics, when a particle moving in a space-time with spatial coordinates collectively denoted by x, x becomes operator-valued. // There we don't take the attitude that just because x becomes operator-valued, the nature of the space-time is changed. // Rather, we say that *the particle is quantized but the space-time remains classical*. // In other words, it is the nature of the particle that is changed, *not* the space-time. // Replacing the word '*quantized*' by '*matrix/Azumaya noncommutatized*', one concludes that this matrix/Azumaya-noncommutativity happens on D-branes, *not (immediately on) the space-time*.

(2) *From the mathematical/Grothendieck aspect*: The function ring R is more fundamental than the topological space $Space\,(R)$, if definable. A morphism

$$\varphi \;:\; Space\,(R) \;\longrightarrow\; Space\,(S)$$

is specified *contravariantly* by a ring-homomorphism

$$\varphi^\sharp \;:\; S \;\longrightarrow\; R\,.$$

If the function ring R of the domain space $Space\,(R)$ is commutative, then $\varphi^\sharp$ factors through a ring-homomorphism $\bar{\varphi}^\sharp : S/[S, S] \to R$,

$$
\begin{array}{ccc}
R & \xleftarrow{\;\varphi^\sharp\;} & S \\
& \nwarrow_{\bar{\varphi}^\sharp} \quad \circ & \downarrow{\scriptstyle \pi_{S/[S,S]}} \\
& S/[S, S] &
\end{array}
\qquad .
$$

Here, $[S, S]$, the *commutator* of S, is the bi-ideal of S generated by elements of the form $s_1 s_2 - s_2 s_1$ for some s_1, $s_2 \in S$; and $S/[S, S]$ is the commutatization of S. It follows that

$$\begin{array}{ccc}
Space\,(R) & \xrightarrow{\;\;\varphi\;\;} & Space\,(S) \\
 & {}_{\bar{\varphi}}\searrow \quad {}^{\circ} & \uparrow{}_{\iota} \\
 & & Space\,(S/[S, S])
\end{array} \qquad .$$

In other words,

  · *if the function ring on the D-brane world-volume is only commutative, then it won't be able to detect the noncommutativity, if any, of the open-string target-space!*

Cf. FIGURE 1-3.

Example 1.7. [*implicit examples in string theory literature*]. Once accepting the above aspect from Grothendieck's viewpoint of geometry, one immediately recognizes that there are many local examples hidden implicitly in the string theory literature. For instance, the *commuting variety/scheme*

$$\{(m_1, \cdots, m_l) \,:\, m_i \in M_r(\mathbb{C})\,,\; [m_i, m_j] = 0\,,\; 1 \le i, j \le l\,\}$$

that appears in the description of the D-brane ground states in the Coulomb branch/phase of the supersymmetric gauge theory coupled with matter on the D-brane world-volume is exactly the moduli space of morphisms from the *fixed* Azumaya point-with-a-fundamental module $(Spec\,\mathbb{C}, M_r(\mathbb{C}), \mathbb{C}^r)$ to the affine space $\mathbb{A}^l_{\mathbb{C}} := Spec\,(\mathbb{C}[y_1, \cdots, y_l])$. This moduli space in general is quite complicated, having many nonreduced irreducible components as a scheme. It is indeed canonically isomorphic to the Quot-scheme $Quot(\mathcal{O}^{\oplus r}_{\mathbb{A}^l_{\mathbb{C}}}, r)$ of 0-dimensional coherent $\mathcal{O}_{\mathbb{A}^l_{\mathbb{C}}}$-module of length r on $\mathbb{A}^l_{\mathbb{C}}$. After modding out the global symmetry $GL_r(\mathbb{C})$, which corresponds to the change of basis of $\mathbb{C}^r$, one obtains the stack

$$\mathfrak{M}^{0^{Az f}}(\mathbb{A}^l) \;\simeq\; [Quot(\mathcal{O}^{\oplus r}_{\mathbb{A}^l_{\mathbb{C}}}, r)/GL_r(\mathbb{C})]$$

of D0-branes of length r on $\mathbb{A}^l$.

For another instance, whenever one sees a ring-homomorphism or an *algebra representation*

$$\rho \;:\; A \;\longrightarrow\; M_r(B)\,,$$

where A is a (possibly noncommutative) associative, unital ring – for example, a quiver algebra – and B is a (usually-commutative-but-not-required-so) ring, one is indeed looking at a morphism from an Azumaya space with a fundamental module

$$\varphi_\rho \;:\; (Space\,(B), M_r(B), B^{\oplus r}) \;\longrightarrow\; Space\,(A)$$

defined by ρ, i.e. a *D-brane on $Space\,(A)$*!

What is a noncommutative (algebraic) geometry? - Looking for a D-brane-sensible/motivated settlement in an inperfect noncommutative world.

● Morphisms between ringed spaces: first attempt.

  · Taking Grothedineck's path: (local/affine picture; all rings assumed associative and unital)

    noncommutative ring $R \implies$ topological space $Spec\,R \implies$ ringed space $(Spec\,R, R)\,.$

  · A morphism from $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a pair $(f, f^\sharp)$, where $f : X \to Y$ is a continuous map between topological spaces and $f^\sharp : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ is a map of sheaves of rings on Y.

  · Leaving aside the issue of localizations, the starting point $R \Rightarrow Spec\,R$ already imposes challenges; there are subtle issues on the notion/construction of $Spec\,R$ in the case of general noncommutative rings. This remains an ongoing issue for the current and the future noncommutative algebraic geometers.

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- Another path via the category of quasi-coherent sheaves.

  · A fundamental work [Ro] of Alexander Rosenberg (1998): *The spectrum of abelian categories and reconstruction of schemes.*

  · Instead of constructing noncommutative algebraic geometry from noncommutative rings R, construct noncommutative geometry from the category Mod_R of R-modules!

  · An unfortunate fact: *Non-isomorphic noncommutative rings may have equivalent categories of modules*; cf. *Morita equivalence.* That is,

    · *in general, Mod_R does not contain all the information of R when R is noncommutative.*

  Indeed, the two $\mathbb{C}$-algebras, $M_r(\mathbb{C})$ and $\mathbb{C}$, are Morita equivalent. More generally:

    · Let $(X, \mathcal{O}_X)$ be a (commutative) scheme and $\mathcal{E}$ be a locally free sheaf on X. Then the two sheaves of algebras, $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ and $\mathcal{O}_X$, are Morita equivalent.

- Re-examine Example 1.5.

  · Any existing way in noncommutative algebraic geometry to define the topological space $Space(M_r(\mathbb{C}))$ for the ring $M_r(\mathbb{C})$ implies that $Space(M_r(\mathbb{C})) = \{pt\} = Spec\,\mathbb{C}$, if one really wants to define $Space(M_r(\mathbb{C}^r))$ honestly.

  · One is thus supposed to define a morphism from the ringed space $(Spec\,\mathbb{C}, M_r(\mathbb{C}))$ to $(\mathbb{A}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}})$ by a pair $(f, f^\sharp)$, where $f : Spec\,\mathbb{C} \to \mathbb{A}^1_{\mathbb{C}} = Spec\,(\mathbb{C}[y])$ and $f^\sharp : \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}} \to f_*(M_r(\mathbb{C}))$.

  · Since $f_*(M_r(\mathbb{C}))$ is a skyscraper sheaf at $f(pt)$, the data $(f, f^\sharp)$ is the same as the data of a $\mathbb{C}$-algebra homomorphism
  $$h : \mathbb{C}[y] \to M_r(\mathbb{C})$$
  such that $Ker\,h = h^{-1}(0) \subset \mathbb{C}[y]$ is the ideal associated to a fuzzy point supported at $f(pt) \in \mathbb{A}^1_{\mathbb{C}}$. This is a subclass of morphisms in Example 1.5 which assume the additional constraint that $I_\varphi = ((y-c)^n)$ for some $c \in \mathbb{C}$ and $1 \le n \le r$.

  · Mathematically, there is nothing wrong with this. // But, for our purpose even just to describe D0-branes on the complex line $\mathbb{A}^1_{\mathbb{C}}$, this is *too restrictive.* // In particular, we won't be able to reproduce the Higgsing/un-Higgsing nor the D-brane recombination phenomenon if we confine ourselves to this traditional definition of morphisms between ringed spaces.

- Morphisms between ringed spaces: second attempt guided by D-branes.

  · *Forget*(!) the topological space; *keep only the rings.*

  · A "*morphism*" $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *defined contravariantly by* a "*morphism*" $\varphi^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$ *in the sense of an equivalence class of gluing systems of ring-homomorphisms, when the latter can be defined.*

  · In the commutative case, this recovers the usual definition of morphisms between (commutative) schemes since in that case $\varphi^\sharp$, in the sense above, truly defines a compatible continuous map (with respect to the Zariski topology) $\varphi : X \to Y$ *and* a sheaf homomorphism $\mathcal{O}_Y \to \varphi_* \mathcal{O}_X$, the usual $\varphi^\sharp$ in the theory of (commutative) schemes.

- A major issue: *localization of an (associative, unital) noncommutative ring.*

  · We are thinking of a 'space', whatever that means, *contravariantly* as an equivalence class of *gluing systems of rings* related by localizations of rings.

  · An unfortunate fact: The notion of *localization of an (associative, unital) noncommutative ring* begins in 1931 in a work of Ore and is much more subtle than in the commutative case.

· Various techniques were developed, e.g. Gabriel's filter construction. This is an ongoing issue for the current and the future ring-theorists.

• A D-brane-sensible/motivated settlement in the inperfect noncommutative world:
re-reading Proto-Definition 1.4.

· Keep track only of and glue rings only through *central localizations*;
i.e. localizations only by elements that are in the center of a ring.

· $(X, \mathcal{O}_X^{nc})$, where X is a topological space with a commutative structure sheaf $\mathcal{O}_X$
that lies in the center of $\mathcal{O}_X^{nc}$,
= an equivalence class of gluing system of rings in which the localization uses elements in $\mathcal{O}_X$.

· The topological space X is only auxiliary and for this purpose.
Truly, we are thinking the space $Space(\mathcal{O}_X^{nc})$, though we never define it!
This explains basic noncommutative geometry on the D-brane world-volume.

· For the target-space-time Y, take any class of commutative or noncommutative spaces as long as they have a presentation as a class of gluing system of rings.

· A morphism $(X, \mathcal{O}_X^{nc}) \to Y$ is defined contravariantly as an equivalence class of gluing systems of ring-homomorphisms, exactly as one does for schemes.

• A shift of perspective: a comparison with functor of points:

· In commutative algebraic geometry, we are very used to the concept that a space can also be defined by how others spaces are mapped into it. // Here, we are taking a reverse perspective. As indicated by Example 1.5, we are actually using how a "space" can be mapped to other (more understood) spaces to feel this hidden-behind-the-veil "space".

**Reflection and a conjcture on D-geometry in the sense of Douglas:
Douglas meeting Polchinski-Grothendieck.**

Before leaving this section, for the conceptual completeness of the lecture, let me give also some reflection on the notion of '*D-geometry*' in the sense of Michael Douglas [Do]. For any $r \in \mathbb{N}$, this is meant to be a certain noncommutative Kähler geometry on the moduli/configuration space $\mathcal{X}_r$ of D-brane for r-many D-branes on a Kähler manifold; see [Do] and [D-K-O] for a more detailed description. Let me recall first some basic facts from [L-L-L-Y] (D(2)) and [L-Y7] (D(6)).

Lemma 1.8. [special role of D0-brane moduli stack]. ([L-L-L-Y: Sec. 3.1] (D(2)) and [L-Y7: Sec. 2.2] (D(6)).) *Let Y be a (commutative) scheme over $\mathbb{C}$ and $\mathfrak{M}_r^{0^{Az\,f}}(Y)$ be the moduli stack of D0-branes of rank/type r on Y in the sense of Proto-Definition 1.4. Then, a morphism*

$$\varphi \,:\, (X, \mathcal{O}_X^{Az}, \mathbb{C}^r) \,\longrightarrow\, Y\,,$$

as defined in Proto-Definition 1.4 is specified by a morphism

$$\tilde{\varphi} \,:\, X \,\longrightarrow\, \mathfrak{M}_r^{0^{Az\,f}}(Y)\,;$$

and vice versa.

Note that the universal family of D0-branes on Y over $\mathfrak{M}_r^{0^{Az\,f}}(Y)$ defines an Azumaya structure sheaf $\mathcal{O}_{\mathfrak{M}_r^{0^{Az\,f}}(Y)}$ with a fundamental module $\mathcal{E}_{\mathfrak{M}_r^{0^{Az\,f}}(Y)}$ on $\mathfrak{M}_r^{0^{Az\,f}}(Y)$, realizing it canonically as an Azumaya (Artin/algebraic) stack with a fundamental module. A comparison of the space-time aspect – cf. Aspect (2) in FIGURE 1-3, the setting of [Do] and [D-K-O], and the above lemma leads one then to the following conjecture, which brings Douglas' D-geometry into our setting:

Conjecture 1.9. [D-geometry: Douglas meeting Polchinski-Grothendieck]. *An atlas for the Azumaya stack with a fundamental module*

$$(\,\mathfrak{M}_r^{0^{Az\,f}}(Y)\,,\,\mathcal{O}_{\mathfrak{M}_r^{0\,Az\,f}(Y)}\,,\,\mathcal{E}_{\mathfrak{M}_r^{0\,Az\,f}(Y)}\,)\;:=\;Y_r^{nc}$$

corresponds to the configuration space $\mathcal{X}_r$ of D-branes in the work of Douglas [Do]. For Y a Kähler manifold, there exists an associated formal Kähler geometry on the irreducible component of Y_r^{nc} that contains all the 0-dimensional $\mathcal{O}_Y$-module of length r whose support are r distinct points on Y. This associated formal Kähler geometry can be made to satisfy the mass conditions of [Do] and [D-K-O] if and only if the Kähler manifold Y is Ricci flat.

2 Azumaya geometry as the origin of the master nature of D-branes.

• In Sec. 1, we see that the matrix/Azumaya-type noncommutativity on D-brane world-volume occur in a very fundamental - almost the lowest - level. // We also see in Example 1.5 that thinking of D-branes on an open-string target-space-time Y as morphisms from such Azumaya-type noncommutative space with a fundamental module does reproduce some features of D-branes in string theory.

• If the setting is truly correct from string-theory point of view, we should be able to see what string-theorists see in quantum-field-theory language solely by our formulation. In particular,

    · **Q. [QFT vs. maps]**
         Can we reconstruct the geometric object that arises in a quantum-field-theoretical study
         of D-branes through morphisms from Azumaya noncommutative spaces?

This is the guiding question for this section.

Azumaya noncommutative geometry as the origin of the master nature of D-branes.

• During the decade I was struggling to understand D-branes, I read through quite a few string-theorists's work with various level of understanding. However, there is one thing I failed to come by at that time:

    Q. *For those D-brane works that carry a strong flavor of geometry, what exactly is going on*
       geometrically?

For that reason, for the scattered small pieces about D-branes I felt I understood something, I remained missing a real crucial piece to link them. For that reason, I didn't truly understand what D-brane really is. I asked several string theorists, including *Joe Polchinski* in TASI 1996, *Jeffrey Harvey* in TASI 1999, *Ashoke Sen* in TASI 2003, *Paul Aspinwall*'s TASI 2003 lectures and after-lecture discussions with participants, and *Cumrun Vafa* in a few occasions in and outside his courses at Harvard. Each one gave me an answer. That means each of these experts has his own working definition of D-branes strong and encompassing enough to create lots of significant works. Yet, I wasn't able to fit their answer coherently together even to the picture I obtained when I read these experts' work. // Then came a completely unexpected twist in the end of 2006. A train of communications with *Duiliu-Emanuel Diaconescu* on a vanishing lemma of open Gromov-Witten invariants derived from [L-Y1] and [L-Y2] and his joint work with Florea [D-F] on open-string world-sheet instantons from the large N duality of compact Calabi-Yau threefolds drove me back to re-understand D-branes. After leaving this project for four years, in this another attempt I came up with the understanding that there is a very fundamental noncommutativity on the D-brane world-volume and D-branes can be thought of as morphisms from such spaces, if this notion of morphism is defined "correctly". Then, I re-looked at some of the works that influenced me but I had failed to understand the true geometry behind. At last, these pieces settle down coherently by one single notion: namely, *morphisms from Azumaya spaces*!
    Below are a few examples.

• *For B-branes*: (Cf. [L-Y7: Sec. 2.4] (D(6)).)

(1) Bershadsky-Sadov-Vafa: Classical and quantum moduli space of D0-branes.
    (*Bershadsky-Sadov-Vafa vs. Polchinski-Grothendieck*; [B-V-S1], [B-V-S2], [Va] (1995).)

The moduli stack $\mathfrak{M}_\bullet^{0^{Az^f}}(Y)$ of morphisms from Azumaya point with a fundamental module to a *smooth variety Y of complex dimension* 2 contains various substacks with different coarse moduli space. One choice of such gives rise to the symmetric product $S^\bullet(Y)$ of Y while another choice gives rise to the Hilbert scheme $Y^{[\bullet]}$ of points on Y. The former play the role of the classical moduli and the latter quantum moduli space of D0-branes studied in [Va] and in [B-V-S1], [B-V-S2].

See [L-Y3: Sec. 4.4] (D(1)), theme: 'A comparison with the moduli problem of gas of D0-branes in [Va] of Vafa' for more discussions.

(2) Douglas-Moore and Johnson-Myers:
D-brane probe to an ADE surface singularity.
(*Douglas-Moore/Johnson-Myers vs. Polchisnki-Grothendiecek*; [D-M] (1996), [J-M] (1996).)

Here, we are compared with the setting of Douglas-Moore [Do-M]. The notion of 'morphisms from an Azumaya scheme with a fundamental module' can be formulated as well when the target Y is a stack. In the current case, Y is the *orbifold associated to an ADE surface singularity.* It is a *smooth Deligne-Mumford stack.* Again, the stack $\mathfrak{M}_\bullet^{0^{Az^f}}(Y)$ of morphisms from Azumaya points with a fundamental module to the orbifold Y contains various substacks with different coarse moduli space. An appropriate choice of such gives rise to the resolution of ADE surface singularity.

See [L-Y4] (D(3)) for a brief highlight of [D-M], details of the Azumaya geometry involved, and more references. In Sec. 3 of this lecture, we will present an abundance conjecture extracted and purified from the study initiated by [D-M].

(3) Klebanov-Strassler-Witten: D-brane probe to a conifold.
(*Klebanov-Strassler-Witten vs. Polchinski-Grothendieck*; [K-W] (1998), [K-S] (2000).)

Here, the problem is related to the moduli stack $\mathfrak{M}_\bullet^{0^{Az^f}}(Y)$ of morphisms from Azumaya points with a fundamental module to a local conifold Y, a singularity Calabi-Yau 3-fold, whose complex structure is given by $Y = Spec\,(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4))$. Again, different resolutions of the conifold singularity of Y can be obtained by choices of substacks from $\mathfrak{M}_\bullet^{0^{Az^f}}(Y)$, as in Tests (1) and (2). Such a resolution corresponds to a low-energy effective geometry "observed" by a stacked D-brane probe to Y when there are no fractional/trapped brane sitting at the singularity $\mathbf{0}$ of Y.

New phenomenon arises when there are fractional/trapped D-branes sitting at $\mathbf{0}$. Instead of resolutions of the conifold singularity of Y, a low-energy effective geometry "observed" by a D-brane probe is a complex deformation of Y with topology T^*S^3 (the cotangent bundle of 3-sphere). From the Azumaya geometry point of view, two things happen:

 · Taking *both* the (stacked-or-not) D-brane probe and the trapped brane(s) into account, the Azumaya geometry on the D-brane world-volume remains.
 · A noncommutative-geometric enhancement of Y occurs via morphisms

$$
\begin{array}{c}
\Xi = Space\,R_\Xi \\
\downarrow \pi^\Xi \\
Y \hookrightarrow \mathbb{A}^4
\end{array}
\qquad .
$$

Here, $\mathbb{A}^4 = Spec\,(\mathbb{C}[z_1, z_2, z_3, z_4])$,

$$
R_\Xi = \frac{\mathbb{C}\langle\, \xi_1, \xi_2, \xi_3, \xi_4 \,\rangle}{([\xi_1 \xi_3, \xi_2 \xi_4]\,,\ [\xi_1 \xi_3, \xi_1 \xi_4]\,,\ [\xi_1 \xi_3, \xi_2 \xi_3]\,,\ [\xi_2 \xi_4, \xi_1 \xi_4]\,,\ [\xi_2 \xi_4, \xi_2 \xi_3]\,,\ [\xi_1 \xi_4, \xi_2 \xi_3])}
$$

with $\mathbb{C}\langle\, \xi_1, \xi_2, \xi_3, \xi_4 \,\rangle$ being the associative (unital) $\mathbb{C}$-algebra generated by ξ_1, ξ_2, ξ_3, ξ_4 and $[\bullet, \bullet']$ being the commutator, $Y \hookrightarrow \mathbb{A}^4$ via the definition of Y above, and π^Ξ is specified by the $\mathbb{C}$-algebra homomorphism

$$
\begin{array}{rcll}
\pi^{\Xi,\sharp} \ : & \mathbb{C}[z_1, z_2, z_3, z_4] & \longrightarrow & R_\Xi \\
& z_1 & \longmapsto & \xi_1 \xi_3 \\
& z_2 & \longmapsto & \xi_2 \xi_4 \\
& z_3 & \longmapsto & \xi_1 \xi_4 \\
& z_4 & \longmapsto & \xi_2 \xi_3 \quad .
\end{array}
$$

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One is thus promoted to studying the stack $\mathfrak{M}_{\bullet}^{0^{Az^f}}(Space\, R_\Xi)$, of morphisms from Azumaya points with a fundamental module to $Space\, R_\Xi$.

To proceed, we need the following notion:

Definition 2.3.1. [superficially infinitesimal deformation]. Given associative (unital) rings, $R = \langle r_1, \ldots, r_m \rangle / \sim$ and S, that are finitely-presentable and a ring-homomorphism $h : R \to S$. A *superficially infinitesimal deformation of h with respect to the generators* $\{r_1, \ldots, r_m\}$ of R is a ring-homomorphism $h_\varepsilon : R \to S$ such that $h_\varepsilon(r_i) = h(r_i) + \varepsilon_i$ with $\varepsilon_i^2 = 0$, for $i = 1, \ldots, m$.

When S is commutative, a superficially infinitesimal deformation of $h_\varepsilon : R \to S$ is an infinitesimal deformation of h in the sense that $h_\varepsilon(r) = h(r) + \varepsilon_r$ with $(\varepsilon_r)^2 = 0$, for all $r \in R$. This is no longer true for general noncommutative S. The S plays the role of the Azumaya algebra $M_\bullet(\mathbb{C})$ in our current test. It turns out that a morphism $\varphi : pt^{Az} \to Space\, R_\Xi$ that projects by π^Ξ to the conifold singularity $\mathbf{0} \in Y$ can have superficially infinitesimal deformations φ' such that the image $(\pi^\Xi \circ \varphi')(pt^{Az})$ contains not only $\mathbf{0}$ but also points in $\mathbb{A}^4 - Y$. Indeed there are abundant such superficially infinitesimal deformations. Thus, beginning with a substack $\mathcal{Y}$ of $\mathfrak{M}_{\bullet}^{0^{Az^f}}(Space\, R_\Xi)$, that projects onto Y via $\varphi \mapsto Im\,(\pi^\Xi \circ \varphi)$, one could use a 1-parameter family of superficially infinitesimal deformations of $\varphi \in \mathcal{Y}$ to drive $\mathcal{Y}$ to a new substack $\mathcal{Y}'$ that projects to $\mathbf{0} \cup Y' \subset \mathbb{A}^4$, where Y' is smooth (i.e. a deformed conifold). It is in this way that a deformed conifold Y' is detected by the D-brane probe via the Azumaya structure on the common world-volume of the probe and the trapped brane(s).

See [L-Y5] (D(4)) for a brief highlight of [K-W] and [K-S], details of the Azumaya geometry involved, and more references.

(4) Gómez-Sharpe: Information-preserving geometry, schemes, and D-branes.
(*Gómez-Sharpe vs. Polchisnki-Grothendieck*; [G-S] (2000).)

Among the various groups who studied the foundation of D-branes, this is a work that is very close to us in spirit. There, Gómez and Sharpe began with the quest: [G-S: Sec. 1]

> "*As is well-known, on N coincident D-branes, $U(1)$ gauge symmetries are enhanced to $U(N)$ gauge symmetries, and scalars that formerly described normal motions of the branes become $U(N)$ adjoints. People have often asked what the deep reason for this behavior is – what does this tell us about the geometry seen by D-branes?* ",

like us. They observed by comparing colliding D-branes with colliding torsion sheaves in algebraic geometry that it is very probable that

> *coincident D-branes should carry some fuzzy structure – perhaps a nonreduced scheme structure*

though the latter may carry more information than D-branes do physically. Further study on such nilpotent structure was done in [D-K-S]; cf. [L-Y7: Sec. 4.2: theme 'The generically filtered structure on the Chan-Patan bundle over a special Lagrangian cycle on a Calabi-Yau torus'] (D(6)).

From our perspective,

> *the (commutative) scheme/nilpotent structure Gómez and Sharpe proposed/ observed on a stacked D-brane is the manifestation/residual of the Azumaya (noncommutative) structure on an Azumaya space with a fundamental module when the latter forces itself into a commutative space/scheme via a morphism.*

This connects our work to [G-S].

(5) Sharpe: B-field, gerbes, and D-brane bundles.
(*Sharpe vs. Polchinski-Grothendieck*; [Sh] (2001).)

Recall that a B-field on the target space(-time) Y specifies a gerbe $\mathcal{Y}_B$ over Y associated to an $\alpha_B \in \check{C}^2_{\acute{e}t}(Y, \mathcal{O}_Y^*)$ determined by the B-field. A morphism $\varphi : (X^{Az}, \mathcal{E}) \to (Y, \alpha_B)$ from a general Azumaya scheme with a twisted fundamental module to (Y, α_B) can be lifted to a morphism $\check{\varphi} : (\mathcal{X}^{Az}, \mathcal{F}) \to \mathcal{Y}_B$ from an Azumaya $\mathcal{O}_X^*$-gerbe with a fundamental module to the gerbe $\mathcal{Y}_B$. In this way, our setting is linked to Sharpe's picture of gerbes and D-brane bundles in a B-field background.

See [L-Y6: Sec. 2.2] (D(5)) theme: 'The description in term of morphisms from Azumaya gerbes with a fundamental module to a target gerbe' for details of the construction.

(6) Dijkgraaf-Hollands-Sułkowski-Vafa: Quantum spectral curves.
(*Dijkgraaf-Hollands-Sułkowski-Vafa vs. Polchisnki-Grothendieck*;
[D-H-S-V] (2007), [D-H-S] (2008).)

Here we focus on a particular theme in these works: the notion of *quantum spectral curves from the viewpoint of D-branes*. Let C be a smooth curve, $\mathcal{L}$ an invertible sheaf on C, $\mathcal{E}$ a coherent locally-free $\mathcal{O}_C$-module, and $\boldsymbol{\mathcal{L}} = \mathbf{Spec}\,(Sym^\bullet(\mathcal{L}^\vee))$ be the total space of $\mathcal{L}$. Here, $\mathcal{L}^\vee$ is the dual $\mathcal{O}_C$-module of $\mathcal{L}$. Then one has the following canonical one-to-one correspondence:

$$\left\{ \begin{array}{l} \mathcal{O}_C\text{-module homomorphisms} \\ \phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{morphisms } \varphi : (C^{Az}, \mathcal{E}) \to \boldsymbol{\mathcal{L}} \\ \text{as spaces over } C \end{array} \right\}$$

induced by the canonical isomorphisms

$$Hom_{\mathcal{O}_C}(\mathcal{E}, \mathcal{E} \otimes \mathcal{L}) \simeq \Gamma(\mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{L}) \simeq Hom_{\mathcal{O}_C}(\mathcal{L}^\vee, \mathcal{E}nd_{\mathcal{O}_C}(\mathcal{E})).$$

Let $\Sigma_{(\mathcal{E}, \phi)} \subset \boldsymbol{\mathcal{L}}$ be the (classical) spectral curve associated to the Higgs/spectral pair $(\mathcal{E}, \phi)$; cf. e.g. [B-N-R], [Hi], and [Ox]. Then, for φ corresponding to ϕ, $Im\,\varphi \subset \Sigma_{(\mathcal{E}, \phi)}$. Furthermore, if $\Sigma_{(\mathcal{E}, \phi)}$ is smooth, then $Im\,\varphi = \Sigma_{(\mathcal{E}, \phi)}$. This gives a *morphism-from-Azumaya-space interpretation of spectral curves*.

To address the notion of 'quantum spectral curve', let $\mathcal{L}$ be the sheaf Ω_C of differentials on C. Then the total space $\boldsymbol{\Omega}_C$ of Ω_C admits a canonical $\mathbb{A}^1$-family $Q_{\mathbb{A}^1}\boldsymbol{\Omega}_C$ of deformation quantizations with the central fiber $Q_0\boldsymbol{\Omega}_C = \boldsymbol{\Omega}_C$. Let $(\mathcal{E}, \phi : \mathcal{E} \to \mathcal{E} \otimes \Omega_C)$ be a spectral pair and $\varphi : (C^{Az}, \mathcal{E}) \to \boldsymbol{\Omega}_C$ be the corresponding morphism. Denote the fiber of $Q_{\mathbb{A}^1}\boldsymbol{\Omega}_C$ over $\lambda \in \mathbb{A}^1$ by $Q_\lambda\boldsymbol{\Omega}_C$. Then, due to the fact that the Weyl algebras are simple algebras, the spectral curve $\Sigma_{(\mathcal{E}, \phi)}$ in $\boldsymbol{\Omega}_C$ in general may not have a direct deformation quantization into $Q_\lambda\boldsymbol{\Omega}_C$ by the ideal sheaf of $\Sigma_{(\mathcal{E}, \phi)}$ in $\mathcal{O}_{\boldsymbol{\Omega}_C}$ since this will only give $\mathcal{O}_{Q_\lambda\boldsymbol{\Omega}_C}$, which corresponds to the empty subspace of $Q_\lambda\boldsymbol{\Omega}_C$. However, one can still construct an $\mathbb{A}^1$-family $(Q_{\mathbb{A}^1}C^{Az}, Q_{\mathbb{A}^1}\mathcal{E})$ of Azumaya quantum curves with a fundamental module out of $(C^{Az}, \mathcal{E})$ and a morphism $\varphi_{\mathbb{A}^1} : (Q_{\mathbb{A}^1}C^{Az}, Q_{\mathbb{A}^1}\mathcal{E}) \to Q_{\mathbb{A}^1}\boldsymbol{\Omega}_C$ as spaces over $\mathbb{A}^1$, using the notion of 'λ-connections' and 'λ-connection deformations of ϕ', such that

· $\varphi_0 := \varphi_{\mathbb{A}^1}|_{\lambda=0}$ is the composition $(Q_0C^{Az}, Q_0\mathcal{E}) \longrightarrow (C^{Az}, \mathcal{E}) \xrightarrow{\varphi} \boldsymbol{\Omega}_C$, where $(Q_0C^{Az}, Q_0\mathcal{E}) \to (C^{Az}, \mathcal{E})$ is a built-in dominant morphism from the construction;

· $\varphi_\lambda := \varphi_{\mathbb{A}^1}|_\lambda : (Q_\lambda C^{Az}, Q_\lambda\mathcal{E}) \longrightarrow Q_\lambda\boldsymbol{\Omega}_C$, for $\lambda \in \mathbb{A}^1 - \{0\}$, is a morphism of Azumaya quantum curves with a fundamental module to the deformation-quantized noncommutative space $Q_\lambda\boldsymbol{\Omega}_C$.

In other words, we *replace the notion of 'quantum spectral curves' by 'quantum deformation φ_λ of the morphism φ'*. In this way, both notions of classical and quantum spectral curves are covered in the notion of morphisms from Azumaya spaces.

See [L-Y6: Sec. 5.2] (D(5)) for more general discussions, details, and more references.

• *For A-branes*:

(7) Denef: (Dis)assembling of A-branes under a split attractor flow.
(*Denef-Joyce meeting Polchisnki-Grothendieck*; [De] (2001), [Joy1] (1999), [Joy2] (2002–2003).)

(Dis)assembling of A-branes under a split attractor flow is realizable as Morse cobordisms of morphisms from Azumaya spaces with a fundamental module into the family of Calabi-Yau 3-folds associated to the flow in the complex moduli space of the Calabi-Yau. Cf. [L-Y8: Sec. 3.2] (D(7)).

(8) Cecotti-Cordava-Vafa: Recombination of A-branes under RG-flow.
(*Cecotti-Cordova-Vafa meeting Polchisnki-Grothendieck*; [C-C-V] (2011).)

The renormalization group flow (RG-flow) in their setting specifies a flow on the moduli stack of morphisms from an Azumaya 3-sphere with a fundamental module to the Calabi-Yau 3-folds in question. The associated deformation family of morphisms corresponds the their brane recombinations. Cf. [L-Y8: Sec. 2.3] (D(7)), [L-Y9] (D(8.1)), and work in progress.

These and many more examples together motivate the next theme.

Azumaya noncommutative algebraic geometry as the master geometry for commutative algebraic geometry.

- A surprising picture emerges:

  · [**unity in geometry vs. unity in string theory**]

the master nature of morphisms from Azumaya-type noncommutative spaces with a fundamental module in geometry	in parallel to	*the master nature of D-branes in superstring theory*

This strongly suggests that

  · *Azumaya noncommutative algebraic geometry could play the role as the master geometry for commutative algebraic geometry.*

Details remain to be understood.

3 D-brane resolution of singularities - an abundance conjecture.

Beginning with Douglas and Moore: D-brane resolution of singularities.

- For this third part of the lecture, let me begin with the work of Douglas and Moore [D-M].

  · Let $\Gamma \simeq \mathbb{Z}_r \subset SU(2)$ acting on $\mathbb{C}^2$, with the standard Calabi-Yau 2-fold structure, by automorphisms in the standard way. Consider the open and closed string target-space-time of the product form $\mathbb{R}^{5+1} \times [\mathbb{C}^2/\Gamma]$ and an effective-space-time-filling D-brane world-volume supported by the locus $\mathbb{R}^{5+1} \times \mathbf{0}$, where $\mathbf{0}$ is the singular point of $\mathbb{C}^2/\Gamma$.

  · The action of the supersymmetric QFT on the D-brane world-volume has various sectors arising from both open and closed strings. It involves, among other multiplets, vector multiplets and hypermultiplets.

  · The potential energy function V of hypermultiplets can be obtain by integrating out the Fayet-Iliopoulos D-term in the vector multiplets from the action. The result involves scalar fields $\vec{\phi}_\bullet$ from NS-NS twisted sectors.

  · From this, by taking $V^{-1}(0)/global\ symmtry$, one obtains the moduli space $\mathcal{M}_{\vec{\zeta}_\bullet}$ of D-brane ground states. It depends on the vacuum expectation value $\vec{\zeta}_\bullet$ of the scalar fields $\vec{\phi}_\bullet$.

  · For appropriate choices of $\vec{\zeta}_\bullet$, $\mathcal{M}_{\vec{\zeta}_\bullet}$ gives a resolution of the singularity of $\mathbb{C}^2/\Gamma$.

The richness and complexity of Azumaya noncommutative space.

- There are lots of contents hidden in the Azumaya cloud $\mathcal{O}_X^{Az}$ of an Azumaya space $(X, \mathcal{O}_X^{Az}, \mathcal{E})$; cf. FIGURE 3-1. // This is already revealed by how an Azumaya point pt^{Az} can be mapped to other spaces in the sense of Proto-Definition 1.4 and is the origin of D-brane resolution of singularities, from our point of view; cf. FIGURE 3-2.

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An abundance conjecture.

Definition 3.1. [punctual D0-brane]. (Cf. [L-Y10: Definition 1.4] (D(9.1)).) Let Y be a variety over $\mathbb{C}$. By a *punctual* 0-dimensional $\mathcal{O}_Y$-module, we mean a 0-dimensional $\mathcal{O}_Y$-module $\mathcal{F}$ whose $Supp\,(\mathcal{F})$ is a single point (with structure sheaf an Artin local ring). A *punctual D0-brane on Y* of rank r is a morphism $\varphi : (Spec\,\mathbb{C}, End\,(E), E) \to Y$, where $E \simeq \mathbb{C}^r$, such that $\varphi_* E$ is a (0-dimensional) punctual $\mathcal{O}_Y$-module. Let $\mathfrak{M}_r^{0\,\overset{Az^f}{p}}(Y)$ be the *stack of punctual D0-branes of rank r on a variety Y*. It is an Artin stack with atlas constructed from Quot-schemes. There is a morphism $\pi_Y : \mathfrak{M}_r^{0\,\overset{Az^f}{p}}(Y) \to Y$ that takes φ to $Supp\,(\varphi_* E)$ with the reduced scheme structure. π_Y is essentially the Hilbert-Chow/Quot-Chow morphism.

- In term of this, note that:

  · *Looking only at the internal part, then each element in $\mathcal{M}_{\vec{\zeta}_\bullet}$ corresponds to a punctual D0-brane on $[\mathbb{C}^2/\Gamma]$.*

It follows that the result of Douglas and Moore [D-M] of D-brane resolution of ADE surface singularities reviewed above can be rephrased as: (resuming the notation $\mathbb{A}^2$ for the affine variety behind $\mathbb{C}^2$.)

Proposition 3.2. [Douglas-Moore: D-brane resolution of ADE singularities]. *There is an embedding $\widetilde{\mathbb{A}^2/\Gamma} \to \mathfrak{M}_1^{0\,\overset{Az^f}{p}}([\mathbb{A}^2/\Gamma])$ that descends to a resolution $\widetilde{\mathbb{A}^2/\Gamma} \to \mathbb{A}^2/\Gamma$ of singularities of $\mathbb{A}^2/\Gamma$.*

- This, together with other existing examples of D-brane resolution of singularities – including the case of conifolds – and the richness and complexity of the stack $\mathfrak{M}_r^{0\,\overset{Az^f}{p}}(Y)$, motivates the following abundance conjecture:

Conjecture 3.3. [abundance]. *Let Y be a reduced quasi-projective variety over $\mathbb{C}$. Then, any birational model $Y' \to Y$ of and over Y factors through an embedding of Y' into the moduli stack $\mathfrak{M}_r^{0\,\overset{Az^f}{p}}(Y)$ of punctual D0-branes of rank r on Y, for r sufficiently large.*

In particular,

Conjecture 3.4. [D0-brane resolution of singularity]. *Let Y be a reduced quasi-projective variety over $\mathbb{C}$. Then, any resolution $\rho : Y' \to Y$ of the singularities of Y factors through an embedding of Y' into $\mathfrak{M}_r^{0\,\overset{Az^f}{p}}(Y)$, for r sufficiently large.*

- As a simple test, one has the following proposition:

Proposition 3.5. [D0-brane resolution of curve singularity]. ([L-Y10 (L-(Baosen Wu)-Yau): D(9.1), Proposition 2.1].) *Conjecture 3.4 holds in the case of curves over $\mathbb{C}$. Namely, let C be a (proper, Noetherian) reduced singular curve over $\mathbb{C}$ and*

$$\rho \,:\, C' \,\longrightarrow\, C$$

be the resolution of singularities of C. Then, there exists an $r_0 \in \mathbb{N}$ depending only on the tuple $(n_{p'})_{\rho(p') \in C_{sing}}$ and a (possibly empty) set $\{\,b.i.i.(p) : p \in C_{sing},\, C$ has multiple branches at $p\,\}$, both associated to the germ of C_{sing} in C, such that, for any $r \geq r_0$, there exists an embedding $\tilde{\rho} : C' \hookrightarrow \mathfrak{M}_r^{0\,\overset{Az^f}{p}}(C)$ that makes the following diagram commute:

Here,

· $n_{p'} \in \mathbb{N}$, for $\rho(p') \in$ the singular locus $C_{sing} \subset C$, is a *multiplicity* related to how the graph Γ_ρ of ρ intersects $C' \times \{\rho(p')\}$ (scheme-theoretically) in the product $C' \times C$;

· $b.i.i.(p) \in \mathbb{N}$ is the *branch intersection index* of $p \in C_{sing}$; it is the least upper bound of the length of the 0-dimensional schemes from the (scheme-theoretical) intersections of pairs of distinct branches of C at p .

• Two remarks I should mention:

Remark 3.6. [*another aspect*]. (Cf. [L-Y10: Remark 0.1] (D(9.1)).) It should be noted that there is another direction of D-brane resolutions of singularities (e.g. [As1], [Br], [Ch]), from the point of view of (hard/massive/solitonic) D-branes (or more precisely B-branes) as objects in the bounded derived category of coherent sheaves. Conceptually that aspect and ours (for which D-branes are soft in terms of string tension) are in different regimes of a refined Wilson's theory-space of $d = 2$ supersymmetric field theory-with-boundary on the open-string world-sheet. Being so, there should be an interpolation between these two aspects. It would be very interesting to understand such details.

Remark 3.7. [*string-theoretical remark*]. (Cf. [L-Y10: Remark 1.7] (D(9.1)).) A standard setting (cf. [D-M]) in D-brane resolution of singularities of a (complex) variety Y (which is a singular Calabi-Yau space in the context of string theory) is to consider a super-string target-space-time of the form $\mathbb{R}^{(9-2d)+1} \times Y$ and an (effective-space-time-filling) D$(9-2d)$-brane whose world-volume sits in the target space-time as a submanifold of the form $\mathbb{R}^{(9-2d)+1} \times \{p\}$. Here, d is the complex dimension of the variety Y and $p \in Y$ is an isolated singularity of Y. When considering only the geometry of the internal part of this setting, one sees only a D0-brane on Y. This explains the role of D0-branes in the statement of Conjecture 1.5 and Conjecture 1.6. On the physics side, the exact dimension of the D-brane (rather than just the internal part) matters since supersymmetries and their superfield representations in different dimensions are not the same and, hence, dimension does play a role in writing down a supersymmetric quantum-field-theory action for the world-volume of the D$(9-2d)$-brane probe. In the above mathematical abstraction, these data are now reflected into the richness, complexity, and a master nature of the stack $\mathfrak{M}_r^{0_p^{Az^f}}(Y)$ that is intrinsically associated to the internal geometry. The precise dimension of the D-brane as an object sitting in or mapped to the whole space-time becomes irrelevant.

Epilogue.

In view of the fundamental role of Azumaya geometry for D-branes and the fact that Azumaya noncommutativity is lost under Morita equivalence and for that reason, most standard noncommutative algebraic geometers current days who follow the categorical language don't treat it as a significant noncommutative geometry, one cannot help making the following moral, derived from Lao-Tzu (600 B.C.), *Tao-te Ching* (*The Scripture on the Way and its Virtue*), Chapter 11:

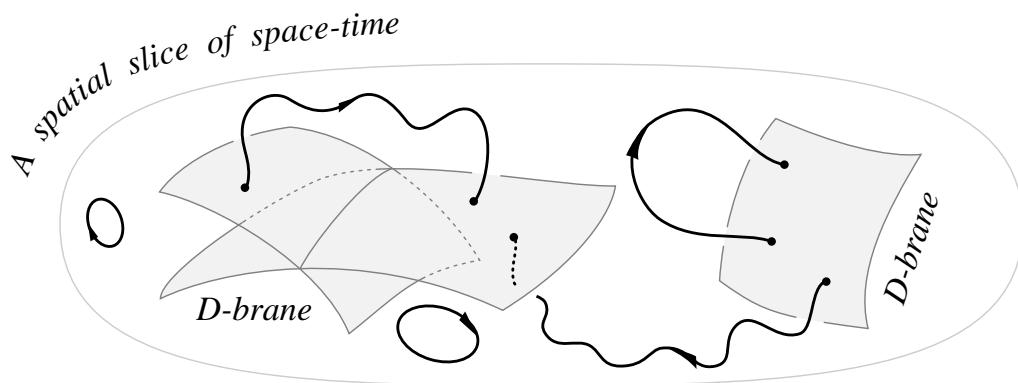*What's naught could be the most useful!*

FIGURE 1-1. D-branes as boundary conditions for open strings in space-time. This gives rise to interactions of D-brane world-volumes with both open strings and closed strings. Properties of D-branes, including the quantum field theory on their world-volume and deformations of such, are governed by open and closed strings via this interaction. Both oriented open (resp. closed) strings and a D-brane configuration are shown.
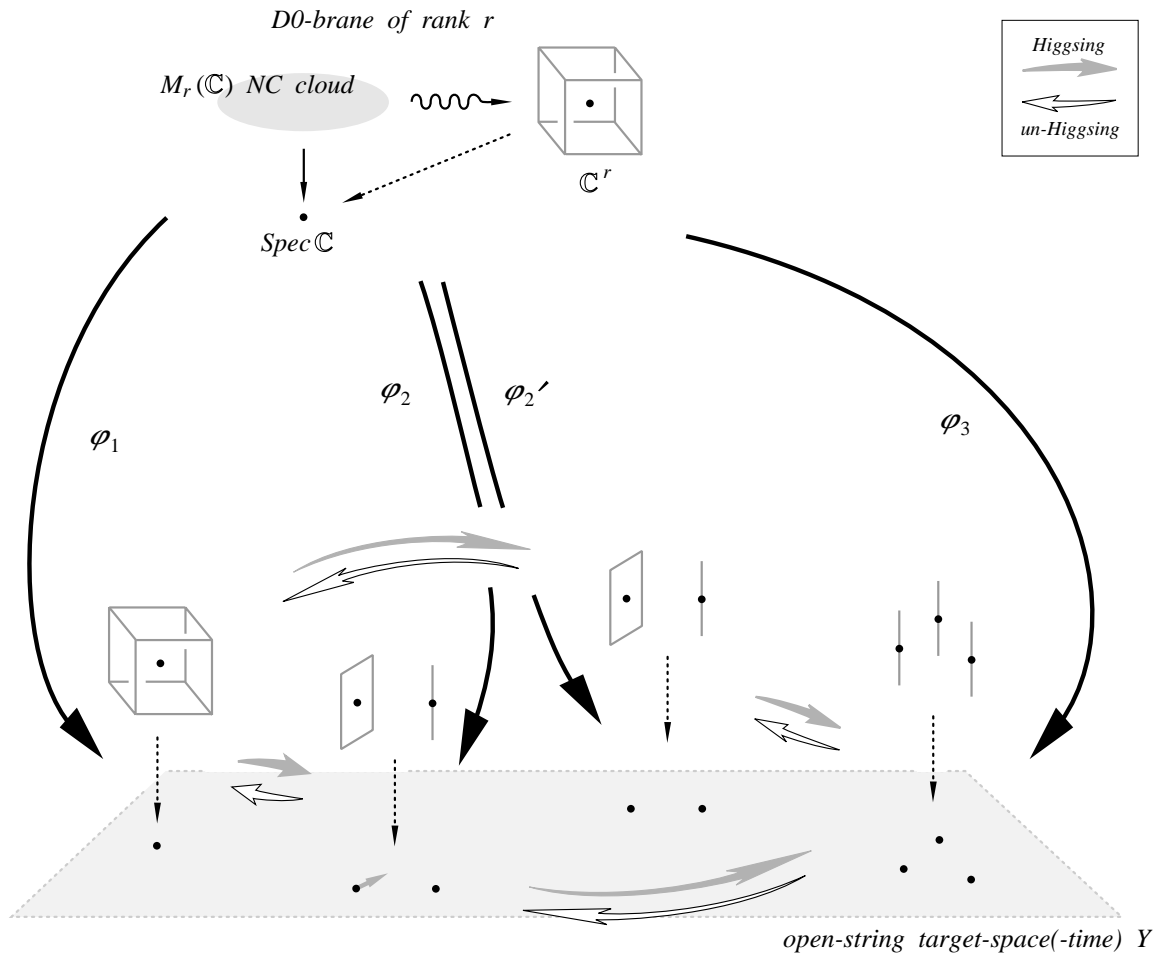
FIGURE 1-2. (Cf. [L-Y7: FIGURE 2-1-1] (D(6)).) Despite that $Space\,M_r(\mathbb{C})$ may look only one-point-like, under morphisms the Azumaya "noncommutative cloud" $M_r(\mathbb{C})$ over $Space\,M_r(\mathbb{C})$ can "split and condense" to various image schemes with a rich geometry. The latter image schemes can even have more than one component. The Higgsing/un-Higgsing behavior of the Chan-Paton module of D0-branes on Y $(=\mathbb{A}^1$ in Example) occurs due to the fact that when a morphism $\varphi : Space\,M_r(\mathbb{C}) \to Y$ deforms, the corresponding push-forward $\varphi_* E$ of the fundamental module $E = \mathbb{C}^r$ on $Space\,M_r(\mathbb{C})$ can also change/deform. These features generalize to morphisms from Azumaya schemes with a fundamental module to a scheme Y. Despite its simplicity, this example already hints at a richness of Azumaya-type noncommutative geometry. In the figure, a module over a scheme is indicated by a dotted arrow ⋯⋯▷ .
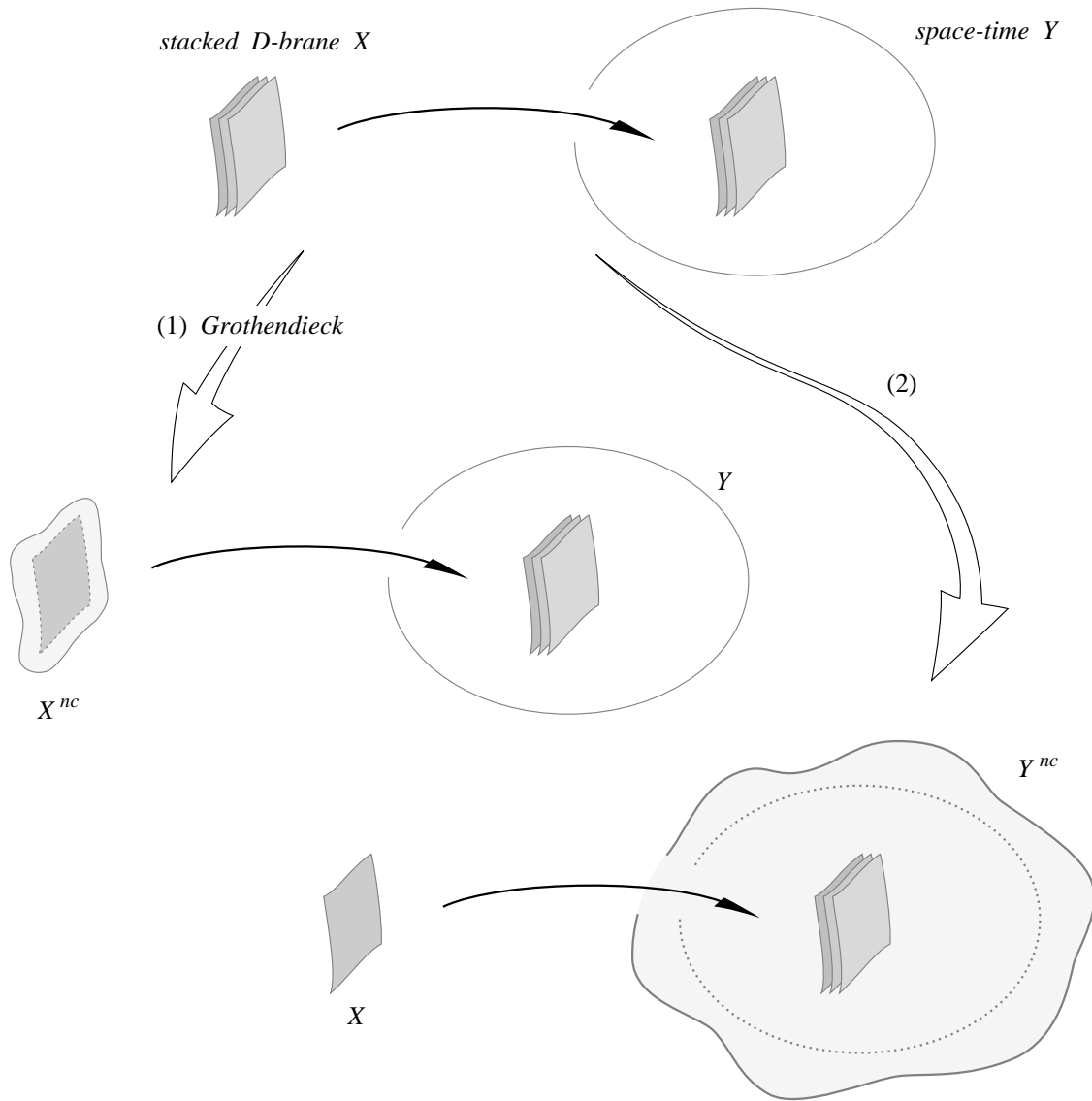
FIGURE 1-3. (Cf. [L-Y7: FIGURE 1-1-2] (D(6)).) Two counter (seemingly dual but not quite) aspects on noncommutativity related to coincident/stacked D-branes: (1) noncommutativity of D-brane world-volume as its fundamental/intrinsic nature versus (2) noncommutativity of space-time as probed by stacked D-branes. (1) leads to the *Polchinski-Grothendieck Ansatz* and is more fundamental from Grothendieck's viewpoint of contravariant equivalence of the category of local geometries and the category of function rings. The matrix/Azumaya structure on coincident D-brane world-volume was also found in the work of Pei-Ming Ho and Yong-Shi Wu [P-W] (1996) in their own path. Their significant observation was unfortunately ignored by the majority of string-theory community. The latter pursued Path (2), following a few equally pival works including [Do] (1997) of Michael Douglas.
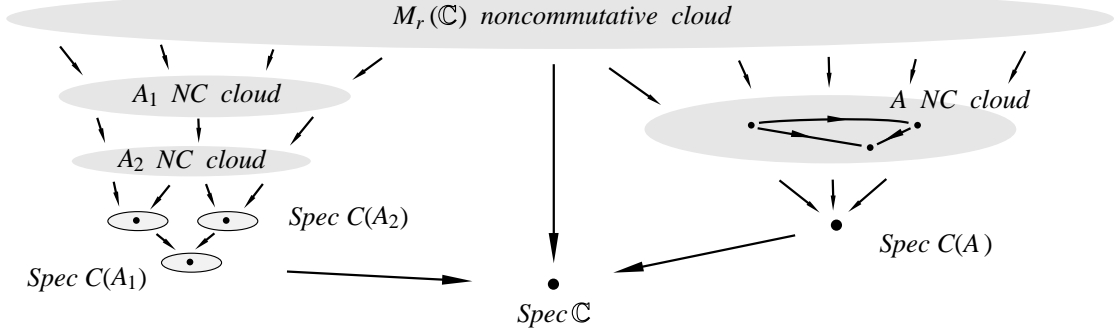
FIGURE 3-1. (Cf. [L-Y4: FIGURE 0-1] (D(3)).) An Azumaya scheme contains a very rich amount of geometry, revealed via its surrogates; cf. [L-L-S-Y: FIGURE 1-3]. Indicated here is the geometry of an Azumaya point $pt^{Az} := (Spec\,\mathbb{C}, M_r(\mathbb{C}))$. Here, A_i are $\mathbb{C}$-subalgebras of $M_r(\mathbb{C})$ and $C(A_i)$ is the center of A_i with

$$
\begin{array}{ccccccc}
M_r(\mathbb{C}) & \supset & A_1 & \supset & A_2 & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\mathbb{C}\cdot\mathbf{1} & \subset & C(A_1) & \subset & C(A_2) & \subset & \cdots \quad .
\end{array}
$$

According to the Polchinski-Grothendieck Ansatz, a D0-brane can be modelled prototypically by an Azumaya point with a fundamental module of type r, $(Spec\,\mathbb{C}, End\,(\mathbb{C}^r), \mathbb{C}^r)$. When the target space Y is commutative, the surrogates involved are commutative $\mathbb{C}$-sub-algebras of the matrix algebra $M_r(\mathbb{C}) = End\,(\mathbb{C}^r)$. This part already contains an equal amount of information/richness/complexity as the moduli space of 0-dimensional coherent sheaves of length r. When the target space is noncommutative, more surrogates to the Azumaya point will be involved. Allowing r to go to ∞ enables Azumaya points to probe "infinitesimally nearby points" to points on a scheme to arbitrary level/order/depth. In (commutative) algebraic geometry, a resolution of a scheme Y comes from a blow-up. In other words, a resolution of a singularity p of Y is achieved by adding an appropriate family of infinitesimally nearby points to p. Since D-branes with an Azumaya-type structure are able to "see" these infinitesimally nearby points via morphisms therefrom to Y, they can be used to resolve singularities of Y. Thus, from the viewpoint of Polchinski-Grothendieck Ansatz, the Azumaya-type structure on D-branes is why D-branes have the power to "see" a singularity of a scheme not just as a point, but rather as a partial or complete resolution of it. Such effect should be regarded as a generalization of the standard technique in algebraic geometry of probing a singularity of a scheme by arcs of the form $Spec\,(\mathbb{C}[\varepsilon]/(\varepsilon^r))$, which leads to the notion of jet-schemes in the study of singularity and birational geometry.
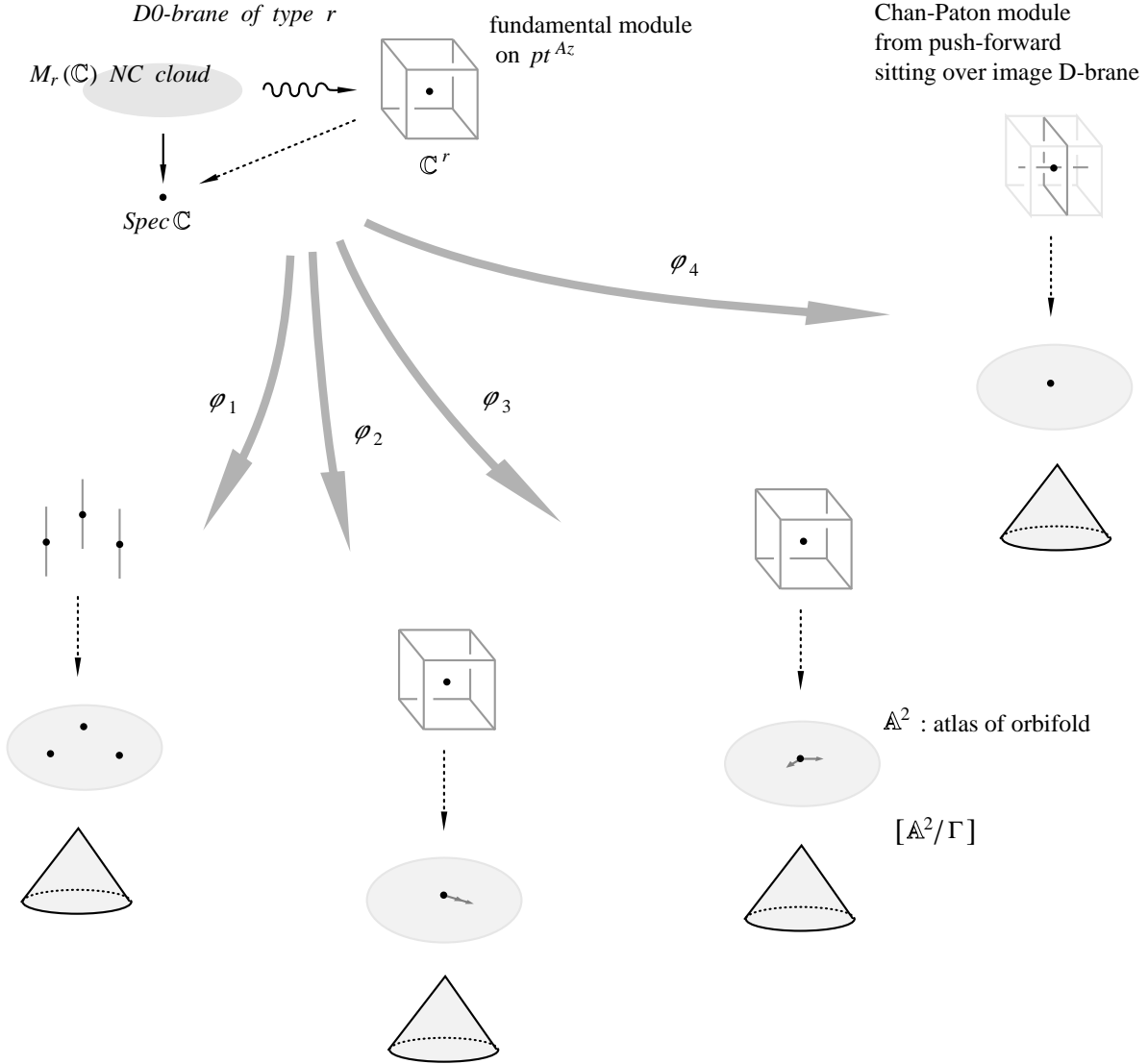
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D0-brane of type r

$M_r(\mathbb{C})$ *NC cloud*

$Spec\,\mathbb{C}$

fundamental module on pt^{Az}

$\mathbb{C}^r$

Chan-Paton module from push-forward sitting over image D-brane

φ_1

φ_2

φ_3

φ_4

$\mathbb{A}^2$: atlas of orbifold

$[\mathbb{A}^2/\Gamma]$

FIGURE 3-2. (Cf. [L-Y4: FIGURE 2-1] (D(3)).) Examples of morphisms from an Azumaya point with a fundamental module $(Spec\,\mathbb{C}, End\,(\mathbb{C}^r), \mathbb{C}^r)$, which models an intrinsic D0-brane according to the Polchinski-Grothendieck Ansatz, to the orbifold $[\mathbb{A}^2/\Gamma]$ are shown. Morphism φ_1 is in Case (a) while morphisms φ_2, φ_3, φ_4 are in Case (b). The image D0-brane under φ_i on the orbifold $[\mathbb{A}^2/\Gamma]$ is represented by a 0-dimensional Γ-subscheme of length $\leq r$ on the atlas $\mathbb{A}^2$ of $[\mathbb{A}^2/\Gamma]$.

Notes and acknowledgements added after the workshop.

This note was prepared before the lecture with only mild revision and addition after coming back to Boston. For that reason, it is intentionally kept lecture-like so that the readers can get to the key points and the key words immediately without being distracted by formality. When writing this note three days before the workshop, I had in mind of it as part of notes for a minicourse. For this particular workshop, I selected the main part of Sec. 1 and quick highlight in Sec. 3 and presented them mainly on the blackboard so that the audience can think over and digest the concept in real time. A vote was cast after presenting very slowly Example 1.5 and Remark 1.6 to decide whether the audience, particularly string-theorists, agree that my notion of D-branes following the line of Grothendieck does correctly reflect string-theorists' D-branes (in the appropriate region of the related Wilson's theory-space, cf. beginning of Sec. 1). It turned out that there is no objection to the setting; yet it received only cautious acceptance: "... can accept it but have to think more". This is another time I put the notion under the scrutinization of experts outside Yau's group and Harvard string-theory community since the first paper D(1) in the series that appeared in 2007. No objections do not necessarily imply believing it; there are still numerous themes in the series yet to be understood and completed.

Comments/corrections/objections to this preliminary lecture note may be sent to the following as part of the basis for its future revision/improvement (after the project is pushed far enough):

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