INSTANTONS AND CURVE COUNTING

Richard Szabo

Heriot–Watt University, Edinburgh

Maxwell Institute for Mathematical Sciences

Noncommutative Algebraic Geometry and D-Branes

Simons Center for Geometry and Physics

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Outline

I. Generalized instantons
and curve counting on toric Calabi–Yau 3-folds

II. Instantons and curve counting on toric surfaces

III. Instanton counting on noncommutative toric varieties

with Michele Cirafici, Lucio Cirio, Amir Kashani-Poor, Giovanni Landi & Annamaria Sinkovics

Part I

Generalized instantons and curve counting on toric Calabi–Yau 3-folds

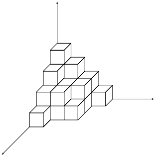
Curve counting on toric Calabi–Yau 3-folds X

- $I_k(X,\beta) \;=\;$ Hilbert scheme of curves $Y \subset X$ with no component of codim 1, $k \;=\; \chi(\mathcal{O}_Y)$, $\beta \;=\; [Y] \in H_2(X)$;
  parametrizes rank 1 torsion free sheaves $\mathcal{I}$ with $\det \mathcal{I}$ trivial

- Donaldson–Thomas partition function:

$$Z_{\mathrm{DT}}(X) \;=\; \sum_{\beta \in H_2(X)} Q^\beta \, \mathtt{DT}_\beta(X;q)\,, \quad \mathtt{DT}_\beta(X;q) \;=\; \sum_{k \in \mathbb{Z}} q^k \int_{[I_k(X,\beta)]^{\mathrm{vir}}} 1$$

- $\mathtt{DT}_0(X;q) \;=\; M(q)^{\chi(X)} \;=\; \prod_{n=1}^{\infty} \frac{1}{\big(1-q^n\big)^{n\,\chi(X)}}$



$M(q)$ enumerates plane partitions π
(3D Young diagrams)

Topological vertex formalism

- Trivalent planar toric graph Γ with:

  (1) $3D$ Young diagram π_v at each vertex v
  (2) $2D$ Young diagram λ_e at each edge e (asymptotics of π_v)

- "Topological string" partition function

  (Aganagic *et al.* '05; Okounkov, Reshetikhin & Vafa '06; Maulik *et al.* '06):

$$Z_{\mathrm{DT}}(X) = \sum_{\substack{\text{Young diagrams} \\ \lambda_e}} \prod_{\text{edges } e} Q_e^{|\lambda_e|} \prod_{\substack{\text{vertices} \\ v=(e_1,e_2,e_3)}} M_{\lambda_{e_1},\lambda_{e_2},\lambda_{e_3}}(q)$$

- $$M_{\lambda,\mu,\nu}(q) = \sum_{\pi\,:\,\partial\pi=(\lambda,\mu,\nu)} q^{|\pi|}$$

  Generating function for plane partitions π with boundaries λ,μ,ν

- GW/DT correspondence $\equiv$ gauge/string theory duality

6D cohomological gauge theory

(Iqbal *et al.* '06)

- $\mathcal{N} = 2$ topologically twisted $U(1)$ Yang–Mills on Kähler 3-fold (X, ω) localizes at BRST fixed points:

$$F_A^{2,0} = 0 = F_A^{0,2} , \qquad F_A^{1,1} \wedge \omega \wedge \omega = 0$$

- **Donaldson–Uhlenbeck–Yau equations:**
  BPS D6–D2–D0 states $\equiv$ (generalized) instantons

- Localization of path integral onto instanton moduli space M computes " $Z_X = \int_{\mathsf{M}} e(\mathcal{N})$ "
  $e(\mathcal{N})$ = Euler characteristic class of obstruction bundle $\mathcal{N}$

- Stability in $\mathbf{D}(X)$? B-field/noncommutative deformation, non-linear/higher-derivative corrections, worldsheet instantons, ...

Singular instanton solutions

- Instanton equations on noncommutative deformation $\mathbb{C}^3_\theta$ have "ADHM form" $\left[Z^i\,,\,Z^j\right] \;=\; 0,\;\left[Z^i\,,\,Z_i^\dagger\right] \;=\; 3$ on Fock module $\mathcal{H} \;=\; \mathbb{C}\big[\bar z^1,\bar z^2,\bar z^3\big]|0\rangle$

- Solutions parametrized by monomial ideals $I\subset\mathbb{C}[z^1,z^2,z^3]$, $\mathcal{H}_I \;=\; I(\bar z^1,\bar z^2,\bar z^3)|0\rangle$; correspond to plane partitions π with $k \;:=\; \mathrm{ch}_3(E) \;=\; |\pi|$

- In "Coulomb branch" $U(1)^r$ noncommutative instantons correspond to coloured partitions $\vec\pi \;=\; (\pi_1,\dots,\pi_r)$; after toric localization:

$$Z^r_{\rm gauge}\big(\mathbb{C}^3\big) \;=\; \sum_{\vec\pi}(-1)^{(r+1)\,|\vec\pi|}\,q^{|\vec\pi|} \;=\; M\big((-1)^{r+1}\,q\big)^r$$

  Degenerate central charge limit of higher-rank invariants (Stoppa '09); not dual to topological string theory

Stacky gauge theories

- G-equivariant instantons on $\mathbb{C}^3$ for finite $G \subset (\mathbb{C}^\times)^3 \subset SL(3,\mathbb{C})$ with weights ρ_i, natural rep $Q = \mathbb{C}^3$; count G-equivariant closed subschemes of $\mathbb{C}^3$ (substacks of $[\mathbb{C}^3/G]$)
- Instanton equations $Z_i^{(\rho+\rho_j)}\, Z_j^{(\rho)} = Z_j^{(\rho+\rho_i)}\, Z_i^{(\rho)}$ on $\mathcal{H} = \bigoplus_{\rho\in\widehat{G}} \mathcal{H}_\rho$, $Z_i = \bigoplus_{\rho\in\widehat{G}} Z_i^{(\rho)}$, $Z_i^{(\rho)} \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_\rho, \mathcal{H}_{\rho+\rho_i})$; solutions parametrized by $\widehat{G}$-coloured plane partitions $\pi = (\pi_\rho)_{\rho\in\widehat{G}}$
- Framed moduli space of torsion free sheaves $\mathcal{E}$ on $\mathbb{P}^3/G$, $\mathrm{ch}_0(\mathcal{E}) = r$, $\mathrm{ch}_3(\mathcal{E}) = k \equiv$ reps $(V = \mathbb{C}^k, W = \mathbb{C}^r; B, I)$, $B \in \mathrm{Hom}_G(V, Q\otimes V)$, $I \in \mathrm{Hom}_G(W, V)$ of framed McKay quiver
- McKay correspondence: $\mathrm{ch}(\mathcal{E})$ determined by exceptional curves on crepant resolution $X = \mathrm{Hilb}^G(\mathbb{C}^3)$ via Beilinson's theorem

Instanton quantum mechanics

- Topological matrix model with stability condition and "orbifold ADHM equations" $B_i^{(\rho+\rho_j)}\, B_j^{(\rho)} \;=\; B_j^{(\rho+\rho_i)}\, B_i^{(\rho)}$

- In "Coulomb branch" BRST fixed points correspond to coloured plane partitions $\vec{\pi} \;=\; (\pi_1,\dots,\pi_r)$ with $|\vec{\pi}| \;=\; k$ and $\pi_l = (\pi_{l,\rho})_{\rho\in\widehat{G}}$, $\sum_l |\pi_{l,\rho}| \;=\; \dim_{\mathbb{C}}(V_\rho)$

- Local model for instanton moduli space near fixed point of $\widetilde{T} \;=\; (\mathbb{C}^\times)^3\times(\mathbb{C}^\times)^r$:

$$\mathrm{Hom}_G(V_{\vec{\pi}},V_{\vec{\pi}}) \longrightarrow \begin{array}{c}\mathrm{Hom}_G(V_{\vec{\pi}},V_{\vec{\pi}}\otimes Q)\\ \oplus \\ \mathrm{Hom}_G(W_{\vec{\pi}},V_{\vec{\pi}})\\ \oplus \\ \mathrm{Hom}_G(V_{\vec{\pi}},V_{\vec{\pi}}\otimes \bigwedge^3 Q)\end{array} \longrightarrow \begin{array}{c}\mathrm{Hom}_G(V_{\vec{\pi}},V_{\vec{\pi}}\otimes \bigwedge^2 Q)\\ \oplus \\ \mathrm{Hom}_G(V_{\vec{\pi}},W_{\vec{\pi}}\otimes \bigwedge^3 Q)\end{array}$$

G-equivariant version of instanton deformation complex

Orbifold invariants

- Partition function:

$$Z^{\mathbf{r}}_{\rm gauge}\big([\mathbb{C}^3/G]\big) \ = \ \sum_{\vec\pi}\,(-1)^{\mathcal{K}(\vec\pi;\mathbf{r})}\,q^{{\rm ch}_3(\mathcal{E}_{\vec\pi})}\,Q^{{\rm ch}_2(\mathcal{E}_{\vec\pi})}$$

  $\mathbf{r} \ = \ (\dim_{\mathbb{C}}(W_1),\dots,\dim_{\mathbb{C}}(W_r))$
  Expressed in terms of intersection theory on $X \ = \ {\rm Hilb}^G(\mathbb{C}^3)$

- Simple change of variables $(q,Q) \ \longmapsto \ (p_\rho)_{\rho\in\widehat{G}}$ with $\prod_{\rho\in\widehat{G}} p_\rho \ = \ q$:

$$Z^{\mathbf{r}}_{\rm gauge}\big([\mathbb{C}^3/G]\big) \ = \ \sum_{\vec\pi}\,(-1)^{\mathcal{K}(\vec\pi;\mathbf{r})}\,\prod_{\rho\in\widehat{G}} p_\rho^{\sum_{l=1}^N\,|\pi_{l,\rho}|}$$

  G-equivariant instanton charges are relevant variables in noncommutative crepant resolution chamber

  (Bryan & Young '10; Joyce & Song '11)

Part II

Instantons and curve counting on toric surfaces

Curve counting on toric surfaces X

- Hilbert scheme of curves $Y \subset X$, $\beta = [Y] \in H_2(X)$, $k = \chi(\mathcal{O}_Y)$:

$$I_k(X,\beta) \;\cong\; I_{k_\beta}(X,\beta) \;\times\; X^{[k-k_\beta]}$$

$k_\beta \;=\; -\frac{1}{2}\,\beta\cdot(\beta+K_X)$ (divisorial part)
$\dim_{\mathbb{C}}\left(X^{[m]}\right) \;=\; 2m$ (punctual part)

- Partition function:

$$Z_{\rm curve}(X) \;=\; \sum_{k\in\mathbb{Z}}\;\sum_{\beta\in H_2(X)} q^k\,Q^\beta\;\int_{I_k(X,\beta)} e\big(TI_k(X,\beta)\big)$$

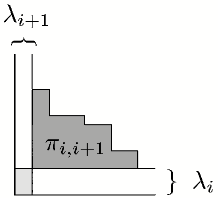
- Göttsche's formula:

$$\sum_{n\geq 0} q^n\,\chi\big(X^{[n]}\big) \;=\; \hat\eta(q)^{-\chi(X)} \;=\; \prod_{n=1}^\infty \frac{1}{\big(1-q^n\big)^{\chi(X)}}$$

$\hat\eta(q)^{-1}$ enumerates Young diagrams $\lambda = (\lambda_1,\lambda_2,\dots)$

Curve counting — Torus fixed points

- Localization theorem in equivariant Chow theory (Edidin & Graham '98):



- $\{\infty \text{ Young diagrams}\} \;\cong\; \mathbb{Z}^2_{\geq 0} \;\times\; \{\text{finite Young diagrams}\}$

- For compact toric invariant divisor $D \;=\; \sum_i \lambda_i\, D_i,\;\; \lambda_i \in \mathbb{Z}_{\geq 0}$ with $a_i \;=\; -D_i^2$:

$$\chi(\mathcal{O}_D) \;=\; -\tfrac{1}{2}\, D\cdot(D+K_X) \;=\; \sum_i \left(a_i\, \frac{\lambda_i\,(\lambda_i - 1)}{2} + \lambda_i - \lambda_i\,\lambda_{i+1} \right)$$

Vertex formalism for toric surfaces

- Partition function on bivalent planar toric graph Γ:

$$Z_{\rm curve}(X) \;=\; \sum_{\lambda_e\in\mathbb{Z}_{\geq 0}} \;\prod_{\text{edges } e} G_{\lambda_e}(q,Q_e) \;\prod_{\substack{\text{vertices}\\ v=(e_1,e_2)}} V_{\lambda_{e_1},\lambda_{e_2}}(q)$$

$$V_{\lambda_{e_1},\lambda_{e_2}}(q) \;=\; \hat\eta(q)^{-1}\, q^{-\lambda_{e_1}\,\lambda_{e_2}} \quad , \quad G_{\lambda_e}(q,Q_e) \;=\; q^{a_e\,\frac{\lambda_e\,(\lambda_e-1)}{2}+\lambda_e}\, Q_e^{\lambda_e}$$

- Question: Is there a 4D "topological string theory" that reproduces this counting?

Vafa–Witten theory

(Vafa & Witten '94)

- $\mathcal{N} = 4$ topologically twisted $U(1)$ Yang–Mills on Kähler surface X, with instanton and monopole charges

$$k = \mathrm{ch}_2(E) \in H^4(X, \mathbb{Z}) \ , \qquad u = c_1(E) \in H^2(X, \mathbb{Z})$$

- Path integral computes Euler character of moduli space of $U(1)$ instantons on X (anti-self-dual connections $\star F_A = -F_A$)

- Conjectural exact expression on Hirzebruch–Jung spaces
  (Fucito, Morales & Poghossian '06; Griguolo *et al.* '07)

- Conjectured factorization for rank $r > 1$:

$$Z^r_{\mathrm{gauge}}(X) = \big(Z_{\mathrm{gauge}}(X)\big)^r$$

Instanton moduli spaces $\mathsf{M}_X(\beta, n)$

- Moduli space of rank 1 torsion free sheaves $\mathcal{I}$ ("noncommutative instantons"), $k = \mathrm{ch}_2(\mathcal{I})$, $\beta = \mathrm{ch}_1(\mathcal{I}) \in H_2(X)$:

$$\mathsf{M}_X(\beta, k) \cong \mathrm{Pic}_\beta(X) \times X^{[k-k_\beta]}$$

$$\mathrm{ch}_2\big(\mathcal{O}_X(D) \otimes \mathcal{I}_Z\big) = \tfrac{1}{2}\, D \cdot D - \chi(\mathcal{O}_Z)$$

- Partition function:

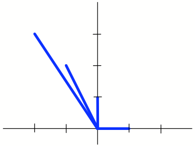
$$Z_{\mathrm{gauge}}(X) = \sum_{k \in \mathbb{Q}} \sum_{\beta \in H_2(X)} q^{-k}\, Q^\beta \int_{\mathsf{M}_X(\beta,k)} e\big(T\mathsf{M}_X(\beta, k)\big)$$

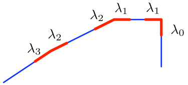
- Using linear equivalence, complete set of non-compact torically invariant divisors to integral generating set for Picard group (Kronheimer & Nakajima '90):

$$e^i = \sum_j \big(C^{-1}\big)^{ij} D_j \ , \qquad C_{ij} = D_i \cdot D_j$$

Example — ALE spaces

- Resolution of A_n singularity $\mathbb{C}^2/\mathbb{Z}_{n+1}$:





- Curve counting: $Z_{\rm curve}(A_1) \; = \; \dfrac{1}{\hat{\eta}(q)^2} \displaystyle\sum_{\lambda=0}^{\infty} q^{\lambda^2} \, Q^{\lambda}$

- Gauge theory: $Z_{\rm gauge}(A_1) \; = \; \dfrac{1}{\hat{\eta}(q)^2} \displaystyle\sum_{u=-\infty}^{\infty} q^{-\frac{1}{4}\,u^2} \, Q^{u}$

- Problems related but not identical in 4D!

Part III

Instanton counting on noncommutative toric varieties

Cocycle twist quantization

(Majid '95)

- H commutative Hopf algebra
  $F : H \otimes H \longrightarrow \mathbb{C}$ convolution-invertible unital two-cocycle on H

- H_F – new Hopf algebra, $H = H_F$ as coalgebra, but with:

$$h \times_F g := F(h_{(1)}, g_{(1)}) \, (h_{(2)} \, g_{(2)}) \, F^{-1}(h_{(3)}, g_{(3)})$$

- Simultaneously deforms all H-covariant constructions as functorial isomorphism of categories of left comodules:

$$\mathcal{Q}_F : {}^{H}\mathcal{M} \longrightarrow {}^{H_F}\mathcal{M}$$

Notation: $\Delta_L : A \longrightarrow H \otimes A$ left coaction of H on A,
$\Delta_L(a) := a^{(-1)} \otimes a^{(0)}$

Comodule twisting of algebras

- Trivial "flip" braiding on monoidal category ${}^H\mathcal{M}$:

$$\Psi\ :\ A\otimes B\ \longrightarrow\ B\otimes A\ , \qquad \Psi(a\otimes b)\ =\ b\otimes a$$

- Twist into new braiding on ${}^{H_F}\mathcal{M}$:

$$\Psi_F\ :\ A_F\otimes B_F\ \longrightarrow\ B_F\otimes A_F\ , \quad \Psi_F(a\otimes b)\ =\ F^{-2}\big(b^{(-1)},a^{(-1)}\big)\ \big(b^{(0)}\otimes a^{(0)}\big)$$

- A — H-comodule algebra $\Longrightarrow$ $A_F\ =\ \mathcal{Q}_F(A)$ — H_F-comodule algebra with new product:

$$a\cdot b\ :=\ F\big(a^{(-1)}\,,\ b^{(-1)}\big)\ \big(a^{(0)}\,b^{(0)}\big)$$

Noncommutative algebraic torus $T_\theta = (\mathbb{C}_\theta^\times)^n$

- $H := \mathbb{C}(t_1, \dots, t_n) = A(T)$ generated by $t^p := t_1^{p_1} \cdots t_n^{p_n}$, $p \in \mathbb{Z}^n$ with:

$$\Delta(t^p) = t^p \otimes t^p \ , \qquad \epsilon(t^p) = 1 \ , \qquad S(t^p) = t^{-p}$$

- Cocycle: $F(t_i, t_j) = \exp\left(\frac{\mathrm{i}}{2}\,\theta_{ij}\right) =: q_{ij}, \ \theta_{ij} = -\theta_{ji} \in \mathbb{C}$
  $H = H_F$ as Hopf algebras, but category of H-comodules twisted

- $\Delta : H \longrightarrow H \otimes H$ makes H into comodule algebra in ${}^H\mathcal{M}$, so cotwisted torus has:

$$t_i \cdot t_j = F(t_i, t_j)\ t_i\, t_j = F^2(t_i, t_j)\ t_j \cdot t_i = q_{ij}^2\ t_j \cdot t_i$$

  Noncommutative torus $A(T_\theta)$ as object of ${}^{H_F}\mathcal{M}$

Quantization of toric varieties $X \longrightarrow X_\theta$

(Ingalls)

- Noncommutative affine toric varieties $\sigma \longmapsto A\big(U_\theta[\sigma]\big)$
  finitely-generated H_F-comodule subalgebras of $A(T_\theta)$

- **Example:** $A(\mathbb{C}^n_\theta) = \mathbb{C}_\theta[z_1, \ldots, z_n]$, $z_i\, z_j = q_{ij}^2\, z_j\, z_i$
  "Algebraic Moyal plane"; Generally modulo relations

- Gluing rules $\Longrightarrow$ algebra automorphisms in category ${}^{H_F}\mathcal{M}$

- Uses **same fan**, deforms coordinate algebra of each cone σ

Noncommutative projective plane $\mathbb{P}^2_\theta$

- Maximal cones: $U_\theta[\sigma_i] \;\cong\; \mathbb{C}^2_\theta, \;\; i \;=\; 1,2,3$

- Edges: $U_\theta[\sigma_i \cap \sigma_{i+1}] \;\cong\;$ noncommutative projective line $\mathbb{P}^1_\theta$:

$$w_1\,w_2 \;=\; q^2\;w_2\,w_1\;,\;\; w_1\,w_2^{-1} \;=\; q^{-2}\;w_2^{-1}\,w_1\;, \qquad q \;:=\; q_{12}$$

- Homogeneous coordinate algebra: $A \;=\; \mathbb{C}_\theta[w_1,w_2,w_3]$
  graded algebra object in ${}^{H_F}\mathcal{M}$ (Auroux, Katzarkov & Orlov '08):

$$w_1\,w_2 \;=\; q^2\;w_2\,w_1\;, \qquad w_1\,w_3 \;=\; w_3\,w_1\;, \qquad w_2\,w_3 \;=\; w_3\,w_2$$

- Degree 0 left Ore localization $A\big[w_i^{-1}\big]_0 \;\cong\; A\big(U_\theta[\sigma_i]\big)$

Instanton moduli spaces $\mathrm{M}_\theta(r,k)$

- $\mathrm{M}_\theta(r,k) \;=\;$ isomorphism classes of framed torsion-free A-modules M with fixed trivialization $M_{\mathbb{P}^1_\theta} \;:=\; M/M\cdot w_3 \;\cong\; W\otimes A_{\mathbb{P}^1_\theta}$, $W \;=\; \mathbb{C}^r$, $A_{\mathbb{P}^1_\theta} \;:=\; A/A\cdot w_3$, and $\dim_{\mathbb{C}} \mathrm{Ext}^1(A, M(-1)) \;=\; k$

- Invariants: $\mathrm{rank}(M) \;=\; r$,
$$\chi(M) \;=\; \sum_{p\geq 0} (-1)^p \dim_{\mathbb{C}} \mathrm{Ext}^p(A,M) \;=\; r-k$$

- Noncommutative ADHM construction: $V \;=\; \mathbb{C}^k$, $W \;=\; \mathbb{C}^r$
$\mathrm{M}_\theta(r,k) \;=\; \big\{ B_1\,,\, B_2\,,\, I\,,\, J \big\}$ where $B_1, B_2 \;\in\; \mathrm{End}_{\mathbb{C}}(V)$, $I \;\in\; \mathrm{Hom}_{\mathbb{C}}(W,V)$, $J \;\in\; \mathrm{Hom}_{\mathbb{C}}(V,W)$ satisfy:
$$[B_1,B_2]_\theta + I\,J \;=\; 0$$
$[B_1,B_2]_\theta \;:=\; B_1\,B_2 - q^{-2}\;B_2\,B_1$ braided commutator
modulo stability and free proper action of $GL(k,\mathbb{C})$

Instanton moduli spaces — Properties

- $\mathsf{M}_\theta(r,k) \;=\;$ fine moduli space (Nevins & Stafford '07)
- Smooth of dimension $2\,r\,k$, $T_{[M]}\mathsf{M}_\theta(r,k) \;=\; \mathrm{Ext}^1\big(M\,,\,M(-1)\big)$
- At $[M] \;=\; [(B_1,B_2,I,J)]$, $T_{[M]}\mathsf{M}_\theta(r,k) \;=\;$ cohomology H^1 of instanton deformation complex:

$$0 \longrightarrow \mathrm{End}_{\mathbb{C}}(V) \longrightarrow \begin{array}{c}\mathrm{End}_{\mathbb{C}}(V)^{\oplus 2}\\ \oplus \\ \mathrm{Hom}_{\mathbb{C}}(W,V)\\ \oplus \\ \mathrm{Hom}_{\mathbb{C}}(V,W)\end{array} \longrightarrow \mathrm{End}_{\mathbb{C}}(V) \longrightarrow 0$$

Instanton moduli spaces — Torus fixed points

- $\widetilde{T} = (\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^r$, coaction of $\widetilde{H}_F = H_F \otimes \mathbb{C}(\rho_1, \dots, \rho_r)$:

  $\Delta_L(B_1, B_2, I, J) = (t_1 \otimes 1 \otimes B_1, t_2 \otimes 1 \otimes B_2, t_1\, t_2 \otimes \rho^{-1} \otimes I, 1 \otimes \rho \otimes J)$

  Makes V, W objects, (B_1, B_2, I, J) morphisms in ${}^{\widetilde{H}_F}\mathcal{M}$

- Coequivariant modules $[M] \in \mathrm{M}_\theta(r,k)^{\widetilde{T}} \cong$ finite set of length r sequences $\vec{\lambda} = (\lambda^1, \dots, \lambda^r)$ of Young diagrams of size $|\vec{\lambda}| = k$

- Restriction of instanton deformation complex to fixed point $\vec{\lambda}$ complex in ${}^{\widetilde{H}_F}\mathcal{M}$ with (Nakajima & Yoshioka '05)

$$\mathrm{ch}_{\widetilde{T}}(V_{\vec{\lambda}}) = \sum_{l=1}^{r} \sum_{p \in \lambda^l} \rho_l\ t_1^{1-p_1}\ t_2^{1-p_2}\ , \qquad \mathrm{ch}_{\widetilde{T}}(W_{\vec{\lambda}}) = \sum_{l=1}^{r} \rho_l$$

  Hence equivariant instanton counting and (pure) gauge theory partition functions same as in classical case $\theta = 0$