Counting versus Integration

Anton Gerasimov (ITEP/TCD/HMI)

SCGP Workshop November 2012

Examples of counting:

1. Holomorphic maps of compact curves $\Sigma \to X$
2. Vortexes/Monopoles/Instantons
3. D-branes / sheaves of various types
4. Summing perturbative series in String Theory

In many examples

$$Z(t) = \sum_{\underline{k}} Z_{\underline{k}}(t) \sim \Psi(t)$$

is a wave function of another (dual) quantum system.
The wave function has an infinite-dimensional integral representation via the Hartle-Hawking representation in the dual system.
Sometimes it also has a nice finite-dimensional integral representation (or at least with a lower number of integration variables).

We discuss a possibility of the Hartle-Hawking type representation of the wave function in the original theory capturing counting sum $Z(t)$.

An old example of instanton counting for $\mathcal{N} = 4$ $d = 4$ SYM (Vafa-Witten 94')

$$Z(t) = \sum_{\underline{k}} Z_{\underline{k}}(t) \sim \Psi(t)$$

where $\Psi(t)$ is naturally a conformal block in some CFT.

Many examples for $\mathcal{N} = 2$ case.

Example: counting holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^\ell$

Counting holomorphic curves in homogeneous spaces such as projective spaces, flag spaces et cet after Givental. Recall the case of the target space $X = \mathbb{P}^\ell$.
We are interested in calculation of the sum

$$Z(x) \sim \sum_d Z_d(x)$$

of $G = S^1 \times U_{\ell+1}$-equivariant volumes of the spaces of of degree d holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^\ell$:

$$Z_d(x,\lambda) = \int_{(\mathbb{P}^1 \to \mathbb{P}^\ell)_d} e^{x\omega_G(\lambda)}$$

where S^1 acts on $\mathbb{P}^1$ by rotations, $U_{\ell+1}$ acts on target space $\mathbb{P}^\ell$ following the tautological representation $U_{\ell+1} \to \mathrm{End}(\mathbb{C}^{\ell+1})$.

The space of holomorphic maps shall be properly compactified. One way to do it is to use *the space of quasi-maps*. A quasi-map $\phi \in \mathcal{QM}_d(\mathbb{P}^\ell)$ of degree d is a collection

$$(a_0(y), a_1(y), \ \ldots \ a_\ell(y))$$

of homogeneous polynomials $a_i(y)$ in variables $y = (y_1, y_2)$ of degree d

$$a_k(y) \ = \ \sum_{j=0}^{d} a_{k,j} \, y_1^j y_2^{d-j}, \qquad\qquad k = 0, \ldots, \ell$$

considered up to the multiplication of all $a_i(y)$'s by a nonzero complex number.

Example: Rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$,

$$f(z) = \frac{p(z)}{q(z)}, \qquad \mathrm{deg} p(z) = \mathrm{deg} q(z) = d$$

When polynomials have common zero the degree of the map drops by one. Thus the space of degree d-maps is non-compact. One shall consider instead the space of pairs of polynomials modulo action of $\mathbb{C}^*$.

The space $\mathcal{QM}_d(\mathbb{P}^\ell)$ is a non-singular projective variety $\mathbb{P}^{(\ell+1)(d+1)-1}$ with the action of $(\lambda, g) \in \mathbb{C}^* \times GL_{\ell+1}$ on $\mathcal{QM}_d(\mathbb{P}^\ell)$ is induced by

$$\lambda \; : \qquad (y_1, y_2) \longrightarrow (\lambda y_1, y_2)$$

$$g \; : \quad (a_0, a_1, \, \ldots \, , a_\ell)) \longrightarrow \left(\sum_{k=1}^{\ell+1} g_{1,k} a_{k-1}, \, \ldots \, , \sum_{k=1}^{\ell+1} g_{\ell+1,k} a_{k-1} \right)$$

Thus we shall calculate the following integral

$$Z_d(x, \lambda, \hbar) = \int_{\mathbb{P}^{(\ell+1)(d+1)-1}} e^{x\omega_G(\lambda, \hbar)}$$

where ω_G is $G = S^1 \times U_{\ell+1}$-equivariant extension of the generator of $H^2(\mathbb{P}^\ell, \mathbb{Z})$. Here $\lambda = (\lambda_1, \dots, \lambda_{\ell+1})$ is an elements of the diagonal subalgebra of $\mathfrak{u}_{\ell+1}$ and $\hbar$ is a generator $\mathrm{Lie}(S^1)$ such that the $S^1 \times U_{\ell+1}$-equivariant cohomology ring of $\mathbb{P}^{(\ell+1)(d+1)-1}$ is given by

$$H^*_{S^1 \times U_{\ell+1}}(\mathbb{P}^{(\ell+1)(d+1)-1}, \mathbb{C}) = \mathbb{C}[\gamma, \hbar] \otimes \mathbb{C}[\lambda_1, \dots, \lambda_{\ell+1}]^{\mathfrak{S}_{\ell+1}} /$$

$$/ \prod_{j=1}^{\ell+1} \prod_{m=0}^{d} (\gamma - \lambda_j - \hbar m)\mathbb{C}[\gamma, \hbar] \otimes \mathbb{C}[\lambda_1, \dots, \lambda_{\ell+1}]^{\mathfrak{S}_{\ell+1}}$$

Recall that for $U_{\ell+1}$-equivariant cohomology of $\mathbb{P}^\ell$ realized as

$$H^*_{U_{\ell+1}}(\mathbb{P}^\ell, \mathbb{C}) = \mathbb{C}[\gamma] \otimes \mathbb{C}[\lambda_1, \dots, \lambda_{\ell+1}]^{\mathfrak{S}_{\ell+1}}$$

$$/ \prod_{j=1}^{\ell+1} (\gamma - \lambda_j) \mathbb{C}[\gamma] \otimes \mathbb{C}[\lambda_1, \dots, \lambda_{\ell+1}]^{\mathfrak{S}_{\ell+1}}$$

we have an integral representation for the pairing of cohomology classes with the $U_{\ell+1}$-equivariant fundamental cycle $[\mathbb{P}^\ell]$

$$\langle P, [\mathbb{P}^\ell] \rangle = \frac{1}{2\pi\imath} \oint_C \frac{P(\gamma, \lambda)\, d\gamma}{\prod_{j=1}^{\ell+1} (\gamma - \lambda_j)}, \qquad P \in H^*_{U_{\ell+1}}(\mathbb{P}^\ell, \mathbb{C})$$

where the integration contour C encircles the poles.

Taking $P = e^{x\omega_G}$ and generalizing to the case of the action of $S^1 \times U_{\ell+1}$ on $\mathcal{QM}_d = \mathbb{P}(\mathbb{C}^{(\ell+1)(d+1)})$ we obtain integral formula for equivariant volume of $\mathcal{QM}_d$

$$Z_d(x, \lambda, \hbar) \; = \; \frac{1}{2\pi\imath} \oint_C \frac{e^{\imath x\gamma} \; d\gamma}{\prod_{j=1}^{\ell+1} \prod_{m=0}^{d} (\gamma - \lambda_j - \hbar m)}$$

Taking the limit $d \to \infty$

Givental proposed to consider the limiting space $\mathcal{QM}_d(\mathbb{P}^\ell)$, $d \to +\infty$ as a substitute of the universal cover of the space $\widetilde{L\mathbb{P}^\ell}_+$ of holomorphic disks in $\mathbb{P}^\ell$. The algebraic version $\mathcal{L}\mathbb{P}^\ell_+$ of $\widetilde{L\mathbb{P}^\ell}_+$ is defined as a set of collections of regular series

$$a_i(z) = a_{i,0} + a_{i,1}z + a_{i,2}z^2 + \cdots, \qquad 0 \le i \le \ell$$

modulo the action of $\mathbb{C}^*$. This space inherits the action of $G = S^1 \times U(\ell + 1)$ defined previously on $\mathcal{QM}_d(\mathbb{P}^\ell)$.

Let us take the limit $d \to +\infty$ on the level of cohomology groups $H^*(\mathcal{QM}_d(\mathbb{P}^\ell))$. In the limit $d \to \infty$ we obtain

$$Z_*(x,\lambda,\hbar) \sim \int d\gamma\, e^{x\gamma/\hbar} \prod_{j=1}^{\ell+1} \prod_{n=0}^{\infty} \frac{1}{\gamma - \lambda_j - \hbar n}.$$

and we shall replace arising infinite products by Γ-functions

$$Z_*(x,\lambda,\hbar) = \int d\gamma \;\; e^{\frac{\gamma x}{\hbar}} \prod_{k=1}^{\ell+1} \hbar^{\frac{\lambda_k - \gamma}{\hbar}}\, \Gamma\Big(\frac{\lambda_k - \gamma}{\hbar}\Big),$$

This **finite-dimensional** integral is equal to the **infinite-dimensional** one

$$Z_*(x,\lambda,\hbar) = \int_{\mathbf{L}\mathbb{P}^\ell_+} e^{x\,\omega_G/\hbar}, \qquad \omega \in H^2_{S^1\times U_{\ell+1}}(\mathbf{L}\mathbb{P}^\ell_+, \mathbb{C})$$

The resulting function is a solution of the parabolic version of the Toda open chain

$$\Big\{ \prod_{j=1}^{\ell+1} \Big(\lambda_j - \hbar \frac{\partial}{\partial x}\Big) \,-\, e^x \Big\} Z_*(z,\lambda,\hbar) \;=\; 0$$

Note that solutions of this equation can be also written as matrix elements in infinite-dimensional representations of $GL_{\ell+1}(\mathbb{R})$.
It is known that counting function of Gromov-Witten invariants of $\mathbb{P}^\ell$ satisfies this equation and various solutions are distinguished by a choice of the particular two-point function (so we actually work on moduli space $\mathcal{M}_{0,2}$).

Direct derivation of an integral representation

It is instructive to directly calculate the infinite-dimensional integral. The integral is an integral over a toric manifold (limit of a projective spaces) i.e. modulo some divisors it is a product of a torus on a polyhedron. This allows to define an analog of angle-action variables. Integrand does not depend on the angle variables and integrating over angles one obtains the integral over a projection of the toric variety under the momentum map. For finite d the resulting integral can be written in the following form

$$Z^{(d)}(x,\lambda,\hbar) \sim \int \prod_{i=1}^{\ell+1}\prod_{n=0}^{d} dt_{i,n}\delta\Big(\sum_{i=1}^{\ell+1}\sum_{n=0}^{d} t_{i,n}-x\Big)\prod_{i=1}^{d} e^{\sum\limits_{n=0}^{d}(\lambda_i+n)t_{i,n}}$$

$$=\int dT_1\dots dT_{\ell+1}\delta\big(T_1+\dots+T_{\ell+1}-x\big)\prod_{i=1}^{d} e^{\lambda_i T_i}\,\Xi_d(T_i)$$

where $\Xi_d(T)$ is S^1-equivariant volume of $\mathbb{P}(\mathbb{C}[z]/z^{d+1}\mathbb{C}[z])$

$$\Xi_d(T)=\int\prod_{j=0}^{n} dt_n\ e^{\sum_{n=0}^{d} nt_n}\delta\big(\sum t_n-x\big)=\Big(1-e^{T}\Big)^{d}$$

Using renromalization $x \to x - (\ell + 1) \ln d$ and taking the limit $d \to \infty$ we obtain

$$Z(x, \lambda, \hbar) = \int\limits_{\mathbb{R}_+^{\ell+1}} dT_1 \dots dT_{\ell+1} \, e^{\sum\limits_{i=1}^{\ell+1} \lambda_i T_i} \times$$

$$\times \delta(x - \sum_{i=1}^{\ell+1} T_i) \prod_{i=1}^{\ell+1} \Xi_\infty(T_i) \,,$$

where

$$\Xi_\infty(T) = \lim_{d \to \infty} \left(1 - e^T / d\right)^d \sim e^{-e^T}$$

is an equivariant volume of $\mathbb{P}(\mathbb{C}[z])$.

Thus we arrive at the following Givental/Hori-Vafa integral representation of $\mathbb{P}^{\ell}$-parabolic Whittaker function:

$$Z_*(x,\lambda) \,=\, \int\limits_{T\in\mathbb{R}^{\ell+1}|\sum_j T_j=x} e^{\lambda_1 T_1 - e^{T_1}+\ldots+\lambda_{\ell+1} T_{\ell+1} - e^{T_{\ell+1}}}$$

QFT realization of the limit $d \to \infty$

One can show that the equivariant volume of the space of holomorphic maps of the disk D into $\mathbb{P}^{\ell}$ can be identified with a correlation function in type A topologically twisted linear gauged sigma model on a disk. This interpretation allows to make the previous considerations more natural and in particular to use mirror symmetry to obtain a finite-dimensional integral representation from the infinite-dimensional one. In the dual type B topologically twisted Landau-Ginzburg theory on a disk the corresponding correlation function is given by a finite-dimensional integral derived before

$$Z_*(x,\lambda) = \int\limits_{T\in\mathbb{R}^{\ell+1}|\sum_j T_j = x} e^{\lambda_1 T_1 - e^{T_1} + \ldots + \lambda_{\ell+1} T_{\ell+1} - e^{T_{\ell+1}}}$$

Note that we have derived **mirror symmetric** description A-model on $\mathbb{P}^\ell$ via the Landau-Ginzburg model with superpotential

$$W_0(T) = \sum_{j=1}^{\ell+1} e^{T_j}\Big|_{\sum_{j=1}^{\ell+1} T_j = x}$$

Lessons to learn from counting of holomorphic maps:

1. There is a way to replace the sum of the integrals over finite-dimensional moduli spaces of *compact* holomorphic curves by an integral over an infinite-dimensional space (*universal moduli space of curves*).

2. This universal moduli space of curves obtained by taking the degree of the map $d \to \infty$ can be interpreted as a space of maps of non-compact curves (disks).

3. This approach allows straightforward derivation of mirror symmetry map.

Vortex counting

Vortexes are close cousins to holomorphic maps (described via linear gauged sigma models) and defined as solutions of the following system of equations

$$\imath F(\nabla) + e^2\Big(\sum_{j=1}^{N_f} \varphi_j \varphi_j^\dagger - \xi \cdot \mathrm{id}_{N\times N}\Big) = 0$$

$$\nabla_{\bar{z}}\varphi_j = 0, \qquad \nabla_z \varphi_j^\dagger = 0$$

satisfying assymptotic conditions for $z \to \infty$

$$F(\nabla) \to 0, \qquad \nabla\varphi \to 0, \qquad \varphi \to const$$

Here $F(\nabla)$ is the curvature form of the connection ∇ in a principle $U(N)$ bundle and the Higgs field $\varphi \in \mathrm{Hom}(\mathbb{C}^{N_f}, \mathbb{C}^N)$ is a section of the associated vector bundle. The vortex charge is $k = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr} F$. We consider framed vortexes so that there is an action of $S^1 \times U(N)$ on the corresponding moduli space.

Vortex counting for rank one

Vortexes are characterized by zeros of φ and thus k-vortex moduli space is $S^k\mathbb{C}$. The corresponding S^1-equivariant volume is

$$\mathcal{Z}_k(\hbar) = \frac{1}{k!\hbar^k}$$

For the generating function of S^1-equivariant volumes of moduli spaces we obtain

$$\mathcal{Z}(x,\hbar) = \sum_{k=0}^{\infty} e^{kx}\mathcal{Z}_k = e^{\frac{1}{\hbar}e^x}$$

Infinite-dimensional integral representation:

We would like to construct an infinite-dimensional space with a natural action of the Lie group S^1 so that its S^1-equivariant volume would be equal to vortex counting function.
The vortex counting function given by the sum of integrals over k-vortex moduli spaces for the gauge group $U(1)$ can be represented as follows:

$$\mathcal{Z} = \mathrm{Vol}^{S^1}(\mathbb{P}(\mathbb{C}[z])).$$

Equivariant volume of $\mathrm{Vol}^{S^1}(\mathbb{P}(\mathbb{C}[z]))$:

$S^1 \times U(1)$-equivariant volume

$$Z(x, \hbar) = \int_{\mathbb{P}(\mathbb{C}[z])} e^{\hbar \tilde{H}_{S^1} + \tilde{\Omega}(x)}$$

can be computed by localization to make a connection with a sum over finite vortex contribution.

Duistermaat-Heckman formula

The way to see "particle structure" in $\mathbb{P}(\mathbb{C}[z])$ is to apply equivariant localization.

Let (M, Ω) be $2N$-dimensional symplectic manifold supplied with the Hamiltonian action of S^1 having only isolated fixed points. Let H_{S^1} be the corresponding momentum. The tangent space $T_{p_k}M$ to a fixed point $p_k \in M^{S^1}$ has the natural action of S^1. Let v be a generator of $\mathrm{Lie}(S^1)$ and let $\hat{v}$ be its action on $T_{p_k}M$

$$\int_M e^{\hbar H_{S^1}+\Omega} = \sum_{p_k \in M^{S^1}} \frac{e^{\hbar H_{S^1}(p_k)}}{\det_{T_{p_k}M} \hbar\hat{v}/2\pi}$$

Fixed points of S^1 acting on $\mathbb{P}\mathcal{M}(D, \mathbb{C})$ are given (in homogeneous coordinates) by

$$\varphi^{(n)}(z) = \varphi_n z^n, \qquad \varphi_n \in \mathbb{C}^* \quad n \in \mathbb{Z}_{\geq 0}.$$

The tangent space to $\mathcal{M}(D, \mathbb{C})$ at an S^1-fixed point $\varphi^{(n)}$ has natural linear coordinates φ_m/φ_n, $m \in \mathbb{Z}_{\geq 0}, m \neq n$ where $\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^k$.
Action of $\mathrm{Lie}(S^1)$ on the tangent space at the fixed point is given by a multiplication of each φ_m/φ_n on $(m-n)$. The regularized denominator in the right hand side of the Duistermaat-Heckman formula is given by

$$\frac{1}{\left[\prod_{m \in \mathbb{Z}_{\geq 0}, m \neq n} (m-n)\right]} \sim \frac{(-1)^n}{n!}$$

Difference of H_{S^1} at two fixed points is given by

$$H_{S^1}(\varphi^{(n)}) - H_{S^1}(\varphi^{(0)}) = \, nt$$

Now formal application of the Duistermaat-Heckman approach gives

$$Z(x, \hbar) \, \sim \, \sum_{n=0}^{\infty} (-1)^n \, \frac{e^{nx\hbar}}{n! \hbar^n} \, = \, e^{-\frac{1}{\hbar} e^x}$$

The resulting expression for equivariant volume is

$$\mathrm{Vol}^{S^1}(\mathbb{P}(\mathbb{C}[z])) = e^{-\frac{1}{\hbar} e^x}$$

By analogy with the case of holomorphic maps we would like to have an interpretation of the counting function as a matrix element of some kind. For $N = 1$ consider the following oscillator algebra

$$[H, a] = -a, \qquad [H, a^\dagger] = a^\dagger, \qquad [a^\dagger, a] = 1$$

Consider the following representation of this algebra

$$a^\dagger = e^{\partial_\gamma}, \qquad H = \gamma, \qquad a = (\gamma - \lambda)e^{-\partial_\gamma}$$

Now the analog of the Whittaker function in this case is given by

$$\Psi_\lambda(x) = \langle \psi_L | e^{xH} | \psi_R \rangle$$

where

$$a|\psi_L\rangle = |\psi_L\rangle, \qquad a^\dagger|\psi_R\rangle = |\psi_R\rangle$$

Explicitly we have

$$\psi_L(\gamma) = 1, \qquad \psi_R(\gamma) = \Gamma(\gamma - \lambda)$$

Corresponding analog of the Whittaker function has the integral representation

$$\Psi_\lambda(x) = \int d\gamma e^{\imath\gamma x}\Gamma(\gamma-\lambda) \sim e^{\imath\lambda x} e^{-e^x}$$

and satisfies the differential equation

$$(\partial_x - \imath\lambda - e^x)\; Z^{vortex}(x, \hbar) = 0$$

This equation is similar to quantum Toda chains.

This differential equation can be derived directly using $d \to \infty$. For finite d we can derive

$$\left(\partial_x - \imath\lambda - e^x(1 - d^{-1}e^x)\right)\; Z^{vortex}_d(x, \hbar) = 0$$

Interpretation of the limit $d \to \infty$

The moduli space of $N = 1$ vortexes is the configuration space $S^n\mathbb{C}$ of n indistinguishable points of $\mathbb{C}$. This space is non-singular and can be described as a set of zeroes of monic polynomials of degree n

$$f(z) = z^n + a_1 z^{n-1} + \cdots a_n$$

which leads to an obvious isomorphism

$$S^n\mathbb{C} = \mathbb{C}^n$$

Its compactification is done by adding strata corresponding to smaller number of points

$$\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \mathbb{C} \cup \mathrm{pt}$$

Formal compactification of the moduli space of infinite number of points gives

$$\mathcal{M}^{(1)} = \cup_n S^n\mathbb{C} = \mathbb{P}^\infty(\mathbb{C})$$

The compactification of the configuration space of n points can be represented as a space of polynomials of degree $\leq n$ up to multiplication on non-zero complex number

$$\mathcal{M}^{(1)} = \mathbb{P}(\mathbb{C}[z])$$

This is the universal moduli space we have arrived before.

The compactification process can be visualized as follows. One has $\mathbb{P}^1 = \mathbb{C} \cup \mathrm{pt}$ and all configurations are distinguished by the number of points sitting at the point pt. The configuration of the rest of the points is parametrized by $S^{n-k}\mathbb{C}$.

∞ number of points sits at the north pole and finite number of points is walking around.

There is another way to introduce the compactification of $S^n\mathbb{C}$

$$\mathbb{P}^n = S^n(\mathbb{P}^1)$$

Indeed the space $S^n(\mathbb{P}^1)$ is the space of effective divisors of degree n on $\mathbb{P}^1$ and thus the space of zeros of holomorphic sections of $\mathcal{O}(n)$

$$S^n(\mathbb{P}^1) = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(n)) = \mathbb{P}^n$$

Thus we formally have

$$\mathcal{M}^{(1)} = S^\infty(\mathbb{P}^1)$$

There is an obvious analogy with Wilson's Adelic Grassmannian here.

Counting of vortexes for general N

For the case of an arbitrary rank N the vortex counting function given by the sum of the integrals over k-vortex moduli spaces for the gauge group $U(N)$ is close to

$$\mathcal{Z} = \mathrm{Vol}^{U_N \times S^1}(\mathrm{Mat}_N(\mathbb{C}[z]/GL_N))$$

U_{n+m}-Equivariant volume of $Gr(n,n+m)$:

The bases of cohomology of $Gr(n,n+m)$ can be enumerated by Young diagrams emebedded in the $n\times m$-rectangle. Points of $Gr(n,n+m)$ can be describe by $n\times(n+m)$-matrices up to an action of GL_n from the left. Each element of $Mat_{n\times(n+m)}$ defines an embedding of the n-dimensional plane $\mathbb{C}^n$ into $\mathbb{C}^{n+m}$. The fixed points of the $(\mathbb{C}^*)^{n+m}$ are such configurations that the action of $(\mathbb{C}^*)^{n+m}$ from the right can be compensated by the action of GL_n from the left. Such n-planes are given by spans of the collections of vector $\{v_1,\cdots,v_n\}$ such that, in the standard bases $\{e_1,\cdots,e_{n+m}\}$ in $\mathbb{C}^{n+m}$, the coordinates of v_i are either 0 or 1 and the matrix $\|(v_i)j\|:=\|v_{ij}\|$ have in each coulomb only one 1. Using the action of GL_n from the left one can arrange vectors in such a way that

$$v_{ij}=\delta_{j,i+k_i},\qquad i+k_i<i+1+k_{i+1},\qquad k_i>0$$

Thus we can enumerate fixed points by partitions $\underline{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$ and

$$v_i = e_{i+k_i}, \qquad i = 1, \ldots, n,$$

The corresponding determinant in the denominator of localization formula for $U_{\ell+1}$-equivariant volume is given by

$$D_{\underline{k}} \det_{T_{\underline{k}} Gr(n,n+m)} \mathrm{diag}(\lambda_1, \ldots, \lambda_{\ell+1}) =$$

$$= \left(\prod_{i=1}^{n+m} \prod_{j=1}^{n} \right)' (\lambda_i - \lambda_{j+k_j})$$

$S^1 \times U_{\ell+1}$-Equivariant volume of $\mathrm{Mat}_N(\mathbb{C}[z]/GL_N)$:

Fixed points of $(\mathbb{C}^*)^{\ell+1} \times S^1$ are given by $(\ell+1) \times (\ell+1)$-matrices with only one non-zero entries in each coulomb and each row. The each non-zero entry is of the form z^n for some n. Using the left action of $GL_{\ell+1}$ one can rearrange the matrices in such a way that non-zero elements are only on diagonal. Thus we have fixed points of the form

$$M_{k_1,\cdots,k_{\ell+1}} = \mathrm{diag}(z^{k_1}, \ldots, z^{k_{\ell+1}})$$

The tangent space is generated by elements $E_{ij}z^{n_{ij}}$ where E_{ij}, $i, j = 1, \ldots, \ell+1$ are elementary matrices and $n_{ij} \in \mathbb{Z}_{\geq 0}$. Note that we omit elements of the form $E_{*i}z^{k_i}$, $i = 1, \cdots, \ell+1$. The action of $(\mathbb{C}^*)^{\ell+1} \times S^1$ is as follows

$$E_{ij}z^n \longrightarrow e^{a_j+n\hbar}\, E_{ij}z^n$$

Note that we work in the chart such that coefficients before $E_{ii}z^{k_i}$ are 1. This condition is not compatible with the right action of $(\mathbb{C}^*)^{\ell+1} \times S^1$. This shall be compensated by the left action of diagonal subgroup $(\mathbb{C}^*)^{\ell+1}$ of $GL_{\ell+1}(\mathbb{C})$. The combined action of $(\mathbb{C}^*)^{\ell+1} \times (\mathbb{C}^*)^{\ell+1} \times S^1$ is given by

$$E_{ij}z^n \longrightarrow e^{a_j + n\hbar + \alpha_i}\, E_{ij}z^n$$

and thus we find $\alpha_i = -(a_i + k_i\hbar)$. Thus the twisted action is given by

$$E_{ij}z^n \longrightarrow e^{a_j + n\hbar - (a_i + k_i\hbar)}\, E_{ij}z^n$$

The resulting sum is over partitions of inverse $D_{\underline{k}}$

$$D_{\underline{k}} = \left(\prod_{j=1}^{\ell+1}\prod_{i=1}^{n}\prod_{n=0}^{\infty}\right)' (a_j + n\hbar - (a_i + k_i\hbar))$$

The Mellin-Barnes type finite-dimensional integral representation for arbitrary (N, N_f) equivariant vortex counting function was constructed in [Gerasimov-Lebedev, arXiv:1011.0403]. The corresponding Givental/Hori-Vafa integral representation follows from [Oblezin, arXiv: 1011.4250, 1107.2998].

Vortex counting problem on $\mathbb{R}^2 \sim \mathbb{C}$ can be considered as a limit of counting instanton counting problem on $\mathbb{R}^4 \sim \mathbb{C}^2$. In the instanton case it is natural to consider $S^1 \times S^1$ equivariance with the corresponding parameters $\hbar_1$ and $\hbar_2$. Taking one of them to ∞ one recover vortex calculations. Thus one shall expect that the instanton partition function allows both as a finite and an infinite-dimensional integral representations providing integral representations of eigenfunction of the corresponding integrable system. *In particular these integral representations shall lead to direct reconstruction of Seiberg-Witten solution of $\mathcal{N} = 2$ theories (joint project with D. Lebedev and A. Sverdlikov).*

Return of the old idea of summing perturbation series via an integral over universal moduli space?! It was proposed long ago that the universal moduli space can be modeled on the moduli space $\mathcal{M}_\infty$ of curves of infinite genus. Kodaira-Spencer theory provides a realization of the sum of perturbative string theory theory as a wave function in some (integrable ?) system associated with extended moduli space of complex structures. Can we reconcile these pictures?