Quantization of 3-Calabi–Yau moduli spaces

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These slides available at
http://people.maths.ox.ac.uk/∼joyce/

Plan of talk:

1. PTVV's shifted symplectic geometry

Work in the context of Toën and Vezzosi's theory of *derived algebraic geometry*, for simplicity over the field $\mathbb{C}$. This gives ∞-categories of *derived* $\mathbb{C}$*-schemes* $\mathbf{dSch}_{\mathbb{C}}$ and *derived stacks* $\mathbf{dSt}_{\mathbb{C}}$. For this talk we are interested in derived schemes, though we are working on extensions to derived Artin stacks. Think of a derived $\mathbb{C}$-scheme $\mathbf{X}$ as a geometric space which can be covered by Zariski open sets $\mathbf{Y} \subseteq \mathbf{X}$ with $\mathbf{Y} \simeq \operatorname{Spec} A$ for $A = (A, \mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{C}$.

Cotangent complexes of derived schemes and stacks

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a notion of *k-shifted symplectic structure* on a derived scheme or derived stack $\mathbf{X}$, for $k \in \mathbb{Z}$. This is complicated, but here is the basic idea. The *cotangent complex* $\mathbb{L}_{\mathbf{X}}$ of $\mathbf{X}$ is an element of a derived category $L_{\mathrm{qcoh}}(\mathbf{X})$ of quasicoherent sheaves on $\mathbf{X}$. It has exterior powers $\Lambda^p \mathbb{L}_{\mathbf{X}}$ for $p = 0, 1, \ldots$. The *de Rham differential* $\mathrm{d}_{dR} : \Lambda^p \mathbb{L}_{\mathbf{X}} \to \Lambda^{p+1} \mathbb{L}_{\mathbf{X}}$ is a morphism of complexes, though not of $\mathcal{O}_{\mathbf{X}}$-modules. Each $\Lambda^p \mathbb{L}_{\mathbf{X}}$ is a complex, so has an internal differential $\mathrm{d} : (\Lambda^p \mathbb{L}_{\mathbf{X}})^k \to (\Lambda^p \mathbb{L}_{\mathbf{X}})^{k+1}$. We have $\mathrm{d}^2 = \mathrm{d}_{dR}^2 = \mathrm{d} \circ \mathrm{d}_{dR} + \mathrm{d}_{dR} \circ \mathrm{d} = 0$.

p-forms and closed p-forms

A *p-form of degree k* on $\mathbf{X}$ for $k \in \mathbb{Z}$ is an element $[\omega^0]$ of $H^k(\Lambda^p \mathbb{L}_{\mathbf{X}}, \mathrm{d})$. A *closed p-form of degree k* on $\mathbf{X}$ is an element

$$[(\omega^0, \omega^1, \ldots)] \in H^k\bigl(\textstyle\bigoplus_{i=0}^{\infty} \Lambda^{p+i} \mathbb{L}_{\mathbf{X}}[i], \mathrm{d} + \mathrm{d}_{dR}\bigr).$$

There is a projection $\pi : [(\omega^0, \omega^1, \ldots)] \mapsto [\omega^0]$ from closed p-forms $[(\omega^0, \omega^1, \ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.
Note that a closed p-form *is not a special example of a p-form*, but a p-form with an extra structure. The map π from closed p-forms to p-forms can be neither injective nor surjective.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\mathbf{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$, where $\mathbb{T}_{\mathbf{X}} = \mathbb{L}_{\mathbf{X}}^{\vee}$ is the tangent complex of $\mathbf{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ is a quasi-isomorphism.
If $\mathbf{X}$ is a derived scheme then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\mathbf{X}$ is a smooth classical scheme, and if $k = -1$ then $\mathbf{X}$ is quasi-smooth.
A closed 2-form $\omega = [(\omega^0, \omega^1, \ldots)]$ of degree k on $\mathbf{X}$ is called a *k-shifted symplectic structure* if $[\omega^0] = \pi(\omega)$ is nondegenerate.

Calabi–Yau moduli schemes and moduli stacks

Pantev et al. prove that if Y is a Calabi–Yau m-fold and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a natural $(2-m)$-shifted symplectic structure ω. So Calabi–Yau 3-folds give -1-shifted derived schemes or stacks.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E]\in\boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{1-i}(E,E)^*$.

The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E)\cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0:\mathbb{T}_{\boldsymbol{\mathcal{M}}}\to\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

Lagrangians and Lagrangian intersections

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\mathbf{L}$ in $(\mathbf{X},\omega)$, which is a morphism $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ of derived schemes or stacks together with a homotopy $i^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\mathbf{L}}\simeq\mathbb{L}_{\mathbf{L}/\mathbf{X}}[k-1]$.

If $\mathbf{L},\mathbf{M}$ are Lagrangians in $(\mathbf{X},\omega)$, then the fibre product $\mathbf{L}\times_{\mathbf{X}}\mathbf{M}$ has a natural $(k-1)$-shifted symplectic structure.

If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

2. A Darboux theorem for shifted symplectic schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived scheme for $k<0$. If $k \not\equiv 2 \mod 4$, then each $x\in\mathbf{X}$ admits a Zariski open neighbourhood $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq \mathrm{Spec}\, A$ for (A,d) an explicit cdga generated by graded variables x_j^{-i}, y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\mathbf{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l, y_j^l have degree l, and

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in (A,d) is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$.

If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 in the degree $k/2$ variables depending on some invertible functions.

The case of -1-shifted symplectic derived schemes

When $k=-1$ the Hamiltonian H in the theorem has degree 0. Then the theorem reduces to:

Corollary

Suppose $(\mathbf{X},\omega)$ is a -1-shifted symplectic derived scheme. Then $(\mathbf{X},\omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H:U\to\mathbb{A}^1)$, for U a smooth classical scheme and $H:U\to\mathbb{A}^1$ a regular function. Hence, the underlying classical scheme $X=t_0(\mathbf{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$.

Combining this with results of Pantev et al. from §1 gives:

Corollary

Let Y be a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth scheme.

N.B. Heuristically, $\mathcal{M}$ is the critical locus (in infinite-dimensions) of the holomorphic Chern–Simons functional.

Corollary

Let (S, ω) be a classical smooth symplectic scheme, and $L, M \subseteq S$ be smooth algebraic Lagrangians. Then the intersection $L \cap M$, as a subscheme of S, is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth scheme.

The case of -2-shifted symplectic derived schemes

When $k = -2$ the theorem implies:

Corollary

Suppose $(\mathbf{X}, \omega)$ is a -2-shifted symplectic derived scheme. Then for each x in the classical scheme $X = t_0(\mathbf{X})$, there exist a smooth scheme U, a vector bundle $E \to U$, a nondegenerate quadratic form q on E, and a section $s \in H^0(E)$ with $q(s, s) = 0$, such that a Zariski open neighbourhood of x in X is isomorphic to the closed subscheme $s^{-1}(0)$ in U.

If $\mathbf{X}$ is a derived moduli scheme of simple coherent sheaves S on a Calabi–Yau 4-fold and $x = [S]$, we may take $\dim U = \dim \mathrm{Ext}^1(S, S)$ and $\mathrm{rank}\, E = \dim \mathrm{Ext}^2(S, S)$. This gives new local models for 4-Calabi–Yau moduli schemes.

D–T style counting invariants for Calabi–Yau 4-folds?

In work in progress with Dennis Borisov, I hope to prove:

- Given a -2-shifted symplectic derived $\mathbb{C}$-scheme $(\mathbf{X},\omega)$, define the structure $\mathbf{X}_{\rm dm}$ of a derived smooth manifold (d-manifold) on the underlying topological space X.
- 'Orientations' on $(\mathbf{X},\omega)$ correspond to orientations on $\mathbf{X}_{\rm dm}$.
- If $(\mathbf{X},\omega)$ is compact (proper) and oriented, we get a *virtual cycle* on $\mathbf{X}_{\rm dm}$ (not necessarily of dimension zero).
- Hence, define new invariants 'counting' stable coherent sheaves on Calabi–Yau 4-folds with 'orientation data', like D–T invariants, or (a better analogy?) complexified Donaldson invariants.

Question: do these invariants have an interpretation in String Theory? Maybe to do with reducing from holonomy SU(4) ($N=2$ supersymmetries) to holonomy Spin(7) ($N=1$ supersymmetry)?

3. D-critical loci

Theorem (Joyce arXiv:1304.4508)

Let X be a classical $\mathbb{C}$-scheme. Then there exists a canonical sheaf $\mathcal{S}_X$ of $\mathbb{C}$-vector spaces on X, such that if $R\subseteq X$ is Zariski open and $i:R\hookrightarrow U$ is a closed embedding of R into a smooth $\mathbb{C}$-scheme U, and $I_{R,U}\subseteq\mathcal{O}_U$ is the ideal vanishing on $i(R)$, then

$$\mathcal{S}_X|_R\cong\operatorname{Ker}\left(\frac{\mathcal{O}_U}{I_{R,U}^2}\xrightarrow{\rm d}\frac{T^*U}{I_{R,U}\cdot T^*U}\right).$$

Also $\mathcal{S}_X$ splits naturally as $\mathcal{S}_X=\mathcal{S}_X^0\oplus\mathbb{C}_X$, where $\mathbb{C}_X$ is the sheaf of locally constant functions $X\to\mathbb{C}$.

The meaning of the sheaves $\mathcal{S}_X, \mathcal{S}_X^0$

If $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then taking $R = X$, $i =$inclusion, we see that $f + I_{X,U}^2$ is a section of $\mathcal{S}_X$. Also $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{C}$ is locally constant, and if $f|_{X^{\mathrm{red}}} = 0$ then $f + I_{X,U}^2$ is a section of $\mathcal{S}_X^0$. Note that $f + I_{X,U} = f|_X$ in $\mathcal{O}_X = \mathcal{O}_U / I_{X,U}$. The theorem means that $f + I_{X,U}^2$ makes sense *intrinsically on* X, without reference to the embedding of X into U.

That is, if $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then we can remember f up to second order in the ideal $I_{X,U}$ as a piece of data on X, not on U.

Suppose $X = \mathrm{Crit}(f : U \to \mathbb{A}^1) = \mathrm{Crit}(g : V \to \mathbb{A}^1)$ is written as a critical locus in two different ways. Then $f + I_{X,U}^2$, $g + I_{X,V}^2$ are sections of $\mathcal{S}_X$, so we can ask whether $f + I_{X,U}^2 = g + I_{X,V}^2$. This gives a way to compare isomorphic critical loci in different smooth classical schemes.

The definition of d-critical loci

Definition (Joyce arXiv:1304.4508)

An (*algebraic*) *d-critical locus* (X, s) is a classical $\mathbb{C}$-scheme X and a global section $s \in H^0(\mathcal{S}_X^0)$ such that X may be covered by Zariski open $R \subseteq X$ with an isomorphism $i : R \to \mathrm{Crit}(f : U \to \mathbb{A}^1)$ identifying $s|_R$ with $f + I_{R,U}^2$, for f a regular function on a smooth $\mathbb{C}$-scheme U.

That is, a d-critical locus (X, s) is a $\mathbb{C}$-scheme X which may Zariski locally be written as a critical locus $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, and the section s remembers f up to second order in the ideal $I_{X,U}$.

We also define *complex analytic d-critical loci*, with X a complex analytic space locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ for U a complex manifold and f holomorphic.

Orientations on d-critical loci

Theorem (Joyce arXiv:1304.4508)

Let (X, s) be an algebraic d-critical locus and $X^{\rm red}$ the reduced $\mathbb{C}$-subscheme of X. Then there is a natural line bundle $K_{X,s}$ on $X^{\rm red}$ called the **canonical bundle**, *such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$ then $K_{X,s}$ is locally modelled on $K_U^{\otimes^2}|_{\mathrm{Crit}(f)^{\rm red}}$, for K_U the usual canonical bundle of U.*

Definition

Let (X, s) be a d-critical locus. An *orientation* on (X, s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on $X^{\rm red}$.

This is related to *orientation data* in Kontsevich–Soibelman 2008.

PTVV's shifted symplectic geometry
A Darboux theorem for shifted symplectic schemes
D-critical loci
Categorification using perverse sheaves
Motivic Milnor fibres

Orientations, spin structures, and String Theory

Orientations (or *orientation data*) are an extra structure on 3-Calabi–Yau moduli spaces $\mathcal{M}$. The obstruction to existence of an orientation lies in $H^2(\mathcal{M}, \mathbb{Z}_2)$, and if they exist, the family of orientations is parametrized by $H^1(\mathcal{M}, \mathbb{Z}_2)$. Orientations are essential for categorified and motivic Donaldson–Thomas theory.
There is a version of orientations for 4-Calabi–Yau moduli spaces $\mathcal{M}$, obstruction in $H^1(\mathcal{M}, \mathbb{Z}_2)$, family parametrized by $H^0(\mathcal{M}, \mathbb{Z}_2)$, needed for 4-C–Y counting invariants, as in §2.
There is also a notion of *spin structure* on 3-C–Y moduli spaces $\mathcal{M}$, with obstruction in $H^3(\mathcal{M}, \mathbb{Z}_2)$, and family of spin structures parametrized by $H^2(\mathcal{M}, \mathbb{Z}_2)$. They appear to be essential in double categorification using matrix factorization categories.
Question: what is the meaning of orientations and spin structures in String Theory? I think they should be important.

Dominic Joyce, Oxford University Quantization of 3-Calabi–Yau moduli spaces

A truncation functor from −1-symplectic derived schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\boldsymbol{X},\omega)$ be a -1-shifted symplectic derived scheme. Then the classical scheme $X=t_0(\boldsymbol{X})$ extends naturally to an algebraic d-critical locus (X,s). The canonical bundle of (X,s) satisfies $K_{X,s}\cong\det\mathbb{L}_{\boldsymbol{X}}|_{X^{\rm red}}$.

That is, we define a *truncation functor* from −1-shifted symplectic derived schemes to algebraic d-critical loci. Examples show this functor is not full. Think of d-critical loci as *classical truncations* of −1-shifted symplectic derived schemes.
An alternative semi-classical truncation, used in D–T theory, is *schemes with symmetric obstruction theory*. D-critical loci appear to be better, for both categorified and motivic D–T theory.

The corollaries in §2 imply:

Corollary

Let Y be a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M},s)$. The canonical bundle satisfies $K_{\mathcal{M},s}\cong\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$, where $\phi:\mathcal{E}^\bullet\rightarrow\mathbb{L}_{\mathcal{M}}$ is the natural (symmetric) obstruction theory on $\mathcal{M}$.

Corollary

Let (S,ω) be a classical smooth symplectic scheme, and $L,M\subseteq S$ be smooth algebraic Lagrangians. Then $X=L\cap M$ extends naturally to a d-critical locus (X,s). The canonical bundle satisfies $K_{X,s}\cong K_L|_{X^{\rm red}}\otimes K_M|_{X^{\rm red}}$. Hence, choices of square roots $K_L^{1/2},K_M^{1/2}$ give an orientation for (X,s).

4. Categorification using perverse sheaves

Theorem (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)

Let (X,s) be an algebraic d-critical locus, with an orientation $K_{X,s}^{1/2}$. Then we can construct a canonical perverse sheaf $P_{X,s}^{\bullet}$ on X, such that if (X,s) is locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$, then $P_{X,s}^{\bullet}$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}_{U,f}^{\bullet}$ of (U,f).
Similarly, we can construct a natural $\mathscr{D}$-module $D_{X,s}^{\bullet}$ on X, and a natural mixed Hodge module $M_{X,s}^{\bullet}$ on X.

The first corollary in §2 implies:

Corollary

Let Y be a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi:\mathcal{E}^{\bullet}\to\mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^{\bullet})^{1/2}$ for $\det(\mathcal{E}^{\bullet})$ (i.e. **orientation data**, *K–S). Then we have a natural perverse sheaf $P_{\mathcal{M},s}^{\bullet}$ on $\mathcal{M}$.*

The *hypercohomology* $\mathbb{H}^*(P_{\mathcal{M},s}^{\bullet})$ is a finite-dimensional graded vector space. The pointwise Euler characteristic $\chi(P_{\mathcal{M},s}^{\bullet})$ is the *Behrend function* $\nu_{\mathcal{M}}$ of $\mathcal{M}$. Thus

$$\sum_{i\in\mathbb{Z}}(-1)^i\dim\mathbb{H}^i(P_{\mathcal{M},s}^{\bullet})=\chi(\mathcal{M},\nu_{\mathcal{M}})=DT(\mathcal{M}).$$

That is, $\mathbb{H}^*(P_{\mathcal{M},s}^{\bullet})$ is a *categorification* of the Donaldson–Thomas invariant $DT(\mathcal{M})$.

Question: is $\mathbb{H}^*(P_{\mathcal{M},s}^{\bullet})$ a mathematical definition of a space of BPS states in String Theory? Relevance of orientations?

Categorifying Lagrangian intersections

The second corollary in §2 implies:

Corollary

Let (S,ω) be a classical smooth symplectic scheme of dimension $2n$, and $L, M \subseteq S$ be smooth algebraic Lagrangians, with square roots $K_L^{1/2}, K_M^{1/2}$ of their canonical bundles. Then we have a natural perverse sheaf $P^\bullet_{L,M}$ on $X = L \cap M$.

We think of the hypercohomology $\mathbb{H}^*(P^\bullet_{L,M})$ as being morally related to the *Lagrangian Floer cohomology* $HF^*(L,M)$ by

$$\mathbb{H}^i(P^\bullet_{L,M}) \approx HF^{i+n}(L,M).$$

We are working on defining 'Fukaya categories' for algebraic/complex symplectic manifolds using these ideas.
Relation of these ideas to Kapustin and Rozansky 2-category $\ddot{L}(S)$ of complex symplectic manifold in String Theory, arXiv:0909.3642?

5. Motivic Milnor fibres

By similar arguments to those used to construct the perverse sheaves $P^\bullet_{X,s}$ in §4, we prove:

Theorem (Bussi, Joyce and Meinhardt arXiv:1305.6428)

Let (X,s) be an algebraic d-critical locus with an orientation $K_{X,s}^{1/2}$. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat\mu$-equivariant motives $\overline{\mathcal{M}}^{\hat\mu}_X$ on X, such that if (X,s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, then $MF_{X,s}$ is locally modelled on $\mathbb{L}^{-\dim U/2}([X] - MF^{\mathrm{mot}}_{U,f})$, where $MF^{\mathrm{mot}}_{U,f}$ is the **motivic Milnor fibre** *of f.*

Relation to motivic D–T invariants

The first corollary in §2 implies:

Corollary

Let Y be a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural motive $MF^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

This motive $MF^\bullet_{\mathcal{M},s}$ is essentially the motivic Donaldson–Thomas invariant of $\mathcal{M}$ defined (partially conjecturally) by Kontsevich and Soibelman 2008. K–S work with motivic Milnor fibres of formal power series at each point of $\mathcal{M}$. Our results show the formal power series can be taken to be a regular function, and clarify how the motivic Milnor fibres vary in families over $\mathcal{M}$.