Dominic Joyce, String math conference 2013.

Title:
Quantization of 3-Calabi-Yau moduli spaces.

Abstract:
This is a report on a collection of projects in progress, joint with O. Ben-Bassat, C. Brav, V. Bussi, D. Dupont, S. Meinhardt, and B. Szendröi.

Let X be a Calabi-Yau 3-fold (over $\mathbb{C}$ or some other field), and M a proper moduli scheme of stable coherent sheaves on X. The *Donaldson-Thomas invariant* DT(M) (due to Thomas, Joyce-Song, and Kontsevich-Soibelman) is an integer which "counts" points in M. In String Theory it should be interpreted as a "number of BPS states". Behrend showed that DT(M) may be written as a weighted Euler characteristic DT(M) = $\chi(M,\nu)$, where $\nu : M \to \mathbb{Z}$ is a multiplicity function known as the "Behrend function".

One can also study the *derived* moduli space $\boldsymbol{M}$, as a derived scheme. Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) showed that derived moduli spaces of coherent sheaves $\boldsymbol{M}$ on a Calabi-Yau 3-fold X have a geometric structure called a –1-*shifted symplectic structure*. A typical example of a –1-shifted symplectic derived scheme is the derived critical locus **Crit(**f**)** of a function $f : U \to \mathbb{C}$ of a regular function on a smooth scheme U. The author (arXiv:1304.4508) defined *d-critical loci*, a geometric structure s on classical schemes M which is a classical truncation of PTVV's –1-shifted symplectic structures. Thus, classical moduli schemes of coherent sheaves on Calabi-Yau 3-folds are d-critical loci. We define notions of *orientation* and *spin structure* on d-critical loci (M,s). Orientations on Calabi-Yau 3-fold moduli spaces correspond roughly to the "orientation data" of Kontsevich-Soibelman. String Theorists should pay attention to such orientations and spin structures.

We prove that if (M,s) is a d-critical locus with an orientation, then one can define a natural perverse sheaf $P_{M,s}$ on M, such that if (M,s) is locally modelled on Crit(f) then $P_{M,s}$ is locally modelled on the perverse sheaf of vanishing cycles of f. The pointwise Euler characteristic $\chi(P_{M,s})$ is the Behrend function ν. The hypercohomology $\mathbb{H}^*(P_{M,s})$ is a graded vector space, a kind of generalized cohomology group of M, with $\Sigma_i(-1)^i \dim \mathbb{H}^i(P_{M,s}) = -$ DT(M).

Thus, the hypercohomology $\mathbb{H}^*(P_{M,s})$ is a *categorification* of the Donaldson-Thomas invariant DT(M). We claim that $\mathbb{H}^*(P_{M,s})$ should be interpreted in String Theory as mathematically rigorous definition of a *vector space of BPS states*.

This work leads off in many interesting directions that I probably will not have time to discuss, including Cohomological Hall Algebras (= algebras of BPS states?), motivic Donaldson-Thomas invariants, double categorification of Calabi-Yau 3-fold moduli schemes using matrix factorization categories, perverse sheaves on intersections of (derived) Lagrangians in symplectic manifolds, and defining a "Fukaya category" of (derived) Lagrangians in a complex symplectic manifold using perverse sheaves.