Holographic fluids and superfluids.

Literature: New J. Phys. 14 (2012) 115009
hep-th/0201253 ("traditional" one)
my PhD thesis. (T. KALAYDZHYAN)

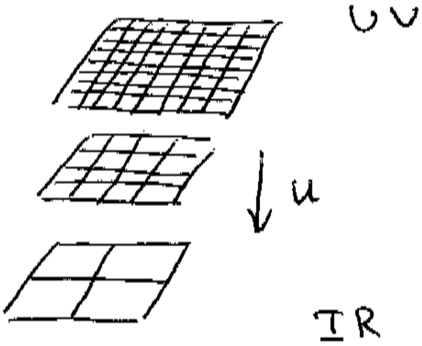
We will focus on the "weak" version of the correspondence:

Classical gravity in (D+1) $\rightsquigarrow$ Strongly coupled CFT on D-dim boundary.

Intuitive arguments:

Consider a field theory on a lattice with a Hamiltonian

$$H = \sum_{x,i} \underset{\substack{\uparrow \\ \text{source (i.e. coupling)}}}{J_i(x)}\ \mathcal{O}^i(x) \leftarrow \text{operator}$$



On a coarse-grained lattice 2a, 4a, ... one can average the multiple sites and tune sources $\{J_i\}$ preserving the ground state and the physics of low-energy excitations, i.e.

$$\text{RG:} \quad u \frac{\partial}{\partial u} J_i(x,u) = \underset{\nwarrow \text{ beta-function}}{\beta_i}\left(J_i(x,u), u\right), \qquad u = a, 2a, 4a \ldots$$

If we make a stack of lattices, then J_i become fields with an additional RG-coordinate u and UV-asymptotics: $J_i(x,a) = J_i(x)$. What kind of theory this can be?

there should be $\mathcal{O} = T_{\mu\nu}$, then $J = g^{\mu\nu}$. Also the physics on the layer $u' > u$ is defined by layer u only (because of R.G.-flow). Therefore we deal with something line a gravitational holography, i.e. we can restore information in some region by the information on the boundary of that region.

So, we have a hint of gravity being the theory on the stack of lattices. (there are, of course, more elaborated tests!)

Now, if we take a continuum limit and consider a conformal theory on the boundary of resulting space, then we can deduce the metric of this space. the most general metric consistent with D-dim Poincaré transformations:

$$ds^2 = \Omega^2(z)\,(-dx_0^2 + dx_i^2 + dz^2), \qquad i = \overline{1, D-1}.$$

$\Omega = \Omega(z)$, not (x, z) because of translational sym. in x^μ.

Conformal invariance gives us the symmetry $x^\mu \to \Lambda x^\mu$. z also transforms $z \to \Lambda z$, because it's a scale.

Therefore, $\Omega \to \Lambda^{-1}\Omega$ with $z \to \Lambda z$,

i.e. $\Omega = \frac{\text{const}}{z}$. Finally,

$$ds^2 = \frac{R^2}{z^2}\left(-dx_0^2 + dx_i^2 + dz^2\right), \qquad i = \overline{1, D-1}.$$

we see, that CFT_D is "dual" to AdS_{D+1} with curvature radius R, yet unfixed (explain here, what is AdS); it will depend later on the degrees of freedom in CFT.

Formal definition and example (the most elabor. one)

$$\mathcal{N}=4 \ SU(N) \ SYM \ \leftrightsquigarrow \ \text{Type IIB St. Th. on } AdS_5 \times S^5$$

(I) (II)

(I): 1 vector, 4 fermions, 6 scalars (adjoint)

$$S_{\mathcal{N}=4} = -\frac{1}{g_{YM}^2}\int d^4x \ \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(D_\mu \varphi^i\right)^2 + \left[\Phi^i, \Phi^j\right]^2\right) +$$
$$+ \text{fermions}.$$

Here $\beta = 0$ and at $N \to \infty$ the perturbative expansion is controlled by the 't Hooft coupling $\lambda = g_{YM}^2 N$.

(II): Type IIB String theory with coupling g_s on $AdS_5 \times S^5$ with $R_{AdS_5} = R_{S^5} = R$.

Correspondence between parameters:

$g_{YM}^2 = 4\pi g_s$, $\quad g_{YM}^2 N = \frac{R^4}{\ell_s^4}$, $\quad N = \int_{S^5} F_5^+$

(RR 4-form flux, from D_3 br.)

the "weak" form of the duality in this case corresponds to the limit: $\lambda \to \infty$, $N \to \infty$, $g_s \to 0$, and we have SUGRA in the bulk.

$$\left\langle e^{-i \sum_i \int d^4x\, J_i(x)\, \Theta^i(x)} \right\rangle_{CFT} = \exp\left\{-i\, S_{min}[AdS_5 \times S^5]\right\}$$

$$\left(J_i(x,z)\big|_{z=0} = J_i(x) \right)$$

$J_i(x,z)$ are classical solutions of Type IIB SUGRA.

Main idea: Bulk fields are the couplings promoted to dynamical fields on the RG-extended spacetime. The partition function of QFT is equal to the exponent of classical GR action defined on that fields.

Next step: Suppose we have to study a real QFT (with less symmetry or additional properties), then we should modify CFT and, hence, deform the gravity dual. There are two ways: top-down approach (starting from existing string/SUGRA backgrounds, when the FT dual is known), bottom-up (don't care about being precise about string constructions, start from necessary properties of FT and find a gravity dual.).

the main tool we use to link the quantities from the both sides is the <u>holographic renormalization</u>: the boundary counterterms cancel the UV-divergencies of the bulk theory.

<u>Example</u>: consider an asymptotically AdS_{D+1} space with cosmological constant $\Lambda = -\frac{D(D-1)}{2}$ in Fefferman-Graham coordinates:

$$ds^2 = g_{MN}(x,z)\, dx^M dx^N = \frac{g_{\mu\nu}(x,z) dx^\mu dx^\nu + dz^2}{z^2},$$

where $M,N = 0 \dots D+1$, $\mu,\nu = 0 \dots D$.

Near-boundary expansion:

$$g(x,z) = g^{(0)}(x) + g^{(2)}(x) z^2 + \dots + g^{(D)}(x) z^D + h^{(D)} z^D \log z^2 + O(z^{D+1}).$$

For a 4D CFT:

$$g_{\mu\nu}(x,z) = \eta_{\mu\nu} + 4\pi G_N \langle T_{\mu\nu}(x) \rangle z^4 + \dots$$

for arbitrary dimensions, $g^{(D)}_{\mu\nu} \sim \langle T_{\mu\nu} \rangle$,

the structure $\Phi =$ "Source" + "VEV" $z^{\#} + \dots$

is general for the bound. expansion of the bulk fields.

From this point we start adding items to the "holographic dictionary", see page A.

Black holes:

for the thermal theories we consider an AdS-BH.

$$ds^2 = \frac{1}{z^2}\left(-f(z)\,dt^2 + d\vec{x}^{\,2} + \frac{1}{f(z)}\,dz^2\right),$$

where $f(z) = +1 - \left(\frac{z}{z_H}\right)^D.$

Thermodynamic parameters are given by

$$T = -\frac{f'(z_H)}{4\pi} = \frac{D}{4\pi z_H}, \qquad \epsilon = \frac{D-1}{16\pi G_N z_H^D}, \qquad s = \frac{1}{4 G_N z_H^{D-1}}$$

Charged BH

$$S_{EM} = \frac{1}{16\pi G_N}\int d^{D+1}x\,\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F^{MN}F_{MN}\right).$$

ds^2 is the same, but

$$f(z) = 1 - \left(1 + \frac{z_H^2\mu^2}{\zeta^2}\right)\left(\frac{z}{z_H}\right)^D + \left(\frac{z_H^2\mu^2}{\zeta^2}\right)\left(\frac{z}{z_H}\right)^{2(D-1)},$$

$$A = A_t(z)\,dt, \qquad A_t(z) = \mu\left(1 - \left(\frac{z}{z_H}\right)^{D-2}\right).$$

↑ chem. potential. ↴

$\zeta^2 \equiv \frac{2(D-1)}{D-2}$ and ch. dens. $\rho = \frac{D-1}{\zeta^2 8\pi G_N}\cdot\frac{\mu}{z_H^D}$

Fluid-gravity

we can change variables in the BH metric:

$$z \to \frac{\tilde{z}}{\sqrt{1+\tilde{z}^4/\tilde{z}_H^4}} \, , \qquad z_H \to \tilde{z}_H/\sqrt{2} \, ,$$

So it becomes

$$ds^2 = -\frac{(1-\tilde{z}^4/\tilde{z}_H^4)^2}{(1+\tilde{z}^4/\tilde{z}_H^4)\,\tilde{z}^2}\,dt^2 + \left(1+\frac{\tilde{z}^4}{\tilde{z}_H^4}\right)\frac{d\vec{x}^2}{\tilde{z}^2} + \frac{d\tilde{z}^2}{\tilde{z}^2},$$

This form is suitable for the hol. renorm. and gives us

$$\langle T_{\mu\nu}\rangle = \frac{1}{4\pi G_N}\, g^{(4)}_{\mu\nu} = \frac{1}{16\pi G_N}\,\mathrm{diag}\left(\frac{3}{z_H^4}, \frac{1}{z_H^4}, \frac{1}{z_H^4}, \frac{1}{z_H^4}\right) =$$

$$= \mathrm{diag}\left(\epsilon, \frac{\epsilon}{3}, \frac{\epsilon}{3}, \frac{\epsilon}{3}\right).$$ (see eqs. above)

this is a conformal fluid at rest ($\epsilon = 3P$).

if we boost the BH solution along u_μ, then

$$T_{\mu\nu} = (\epsilon + P)\, u_\mu u_\nu + P\, g_{\mu\nu}.$$

One can systematically correct it by including higher-order (in ∂) terms (see other talks!)

the main algorithm:

1) Fluid on the boundary, gravity in the bulk.
Input = zero-order parameters: energy density, anomalies, bacuground fields, e.t.c.

2) Fix the metric components (and gauge field components), Chern-Simons parameters, e.t.c. in the bulk.

3) Solve equations of motion for the bulk fields (Einstein-Maxwell eqns., for instance).

4) Read off a nontrivial result from the near-boundary expansion of the bulk fields (e.g. transport coefficients).

Example:
$$\begin{cases} \partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda \\ \partial_\mu j^\mu = 0 \\ \partial_\mu j_5^\mu = C E^\lambda B_\lambda \end{cases}$$
$$u_\mu u^\mu = -1.$$

$$\begin{cases} j^\mu = \rho u^\mu + \kappa_\omega \omega^\mu + \kappa_B B^\mu + \dots \\ j_5^\mu = \rho_5 u^\mu + \xi_\omega \omega^\mu + \xi_B B^\mu + \dots \end{cases}$$
$$B^\mu \equiv \varepsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$$
$$\omega^\mu \equiv \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta \swarrow \text{vorticity.}$$

Gr. dual: AdS-BH with two U(1) charges + CS term.
Result:

$$\boxed{\text{CVE}} \quad \kappa_\omega = 2C\mu\mu_5\left(1-\frac{\mu\rho}{\epsilon+P}\right), \qquad \kappa_B = C\mu_5\left(1-\frac{\mu\rho}{\epsilon+P}\right) \quad \boxed{\text{CME}}$$

$$\boxed{\text{AVE}} \quad \xi_\omega = C\mu^2\left(1-2\,\frac{\mu_5\rho_5}{\epsilon+P}\right), \qquad \xi_B = C\mu\left(1-\frac{\mu_5\rho_5}{\epsilon+P}\right) \quad \boxed{\text{CSE}}$$

Superfluid / superconductor

First, we describe the field theory in hydro language:

Suppose, we have a scalar Φ (complex scalar) with a Mexican hat potential, then, when we break the global U(1) spontaneously,

$$\langle \Phi \rangle = \frac{f}{\sqrt{2}} e^{i\varphi} \quad \text{Goldstone mode}$$

In the absence of external fields:

$$\begin{cases} \partial_\mu T^{\mu\nu} = 0 \\ \partial_\mu j^\mu = 0 \\ u^\mu \partial_\mu \varphi + \mu = 0 \quad \leftarrow \text{"Josephson equation".} \end{cases}$$

$$\mathcal{L}_\varphi \sim \frac{f^2}{2}(\partial_\mu \varphi)^2 \sim \rho_s \mu_s + \dots$$

$$\Downarrow$$

$$\dot{\varphi} \sim \mu + \text{boost along } u^\mu$$

$$\Downarrow$$

these eqns. can be solved in the derivative expansion:

$$\begin{cases} T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P \eta^{\mu\nu} + f^2 \partial^\mu \varphi \partial^\nu \varphi + \dots \\ j^\mu = \underbrace{n u^\mu}_{\text{normal component}} + \underbrace{f^2 \partial^\mu \varphi}_{\text{curl-free superfluid component.}} + \dots \end{cases}$$

note: ϵ is defined by $\epsilon + P = Ts + n\mu$,

also $dP = s\,dT + n\,d\mu - f^2 d\left(\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi\right)$.

and $n_s = f^2 \mu$ – superfluid charge density, also the superfluid component doesn't contribute to entropy (a field).

Holographic case (Gubser, Hartnoll, Son, Herzog)

1) J^μ on the boudary $\leftrightsquigarrow$ A^μ in the bulk

2) $\langle \mathcal{O} \rangle$ on the bound. $\leftrightsquigarrow$ non-trivial profile of a bulk scalar Φ charged under $U(1)$
(order parameter of condensation)

3) Spont. breaking of global $U(1)$ $\leftrightsquigarrow$ spont. breaking of the bulk gauge $U(1)$ → Higgs mech.

$$S_{\text{bulk}} \propto \int d^{d+1}x \sqrt{-g} \left(R + \frac{d(d-1)}{L^2} - \frac{1}{e^2}\left[\frac{1}{4}F^2 + |D\Phi|^2 + m^2|\Phi|^2 \right] \right).$$

we can take a probe limit. EOM: (BH background).

$$z^2 f(z) A_t'' - \underbrace{\underbrace{2|\Phi|^2}_{\text{mass of the gauge field (Higgs mech.)}} A_t}_{\text{charge density (r.h.s. of the Maxwell's eqn.)}} = 0$$

$$z^2 f(z) \Phi'' + (z f'(z) - 2f(z)) z \Phi' - \underbrace{\left(m^2 - \frac{z^2 A_t^2}{f(z)} \right)}_{m^2_{\text{eff}} \ (!)} \Phi = 0$$

Breitenlohner-Freedman bound: $m^2 = -D^2/4L^2$.

At large charge density or close to the BH horison the effective mass of the scalar can become tachyonic, which is a hint of the condensation.

Near-boundary asymptotics: (taken the Ansatz $A = A_t(z)$, $\Phi = \Phi(z)$).

$A_t \sim \mu + \rho\, z^{D-2} + \dots,$ (USUALLY, AdS$_4$.)

$\Phi \sim \phi_{D-\Delta}\, r^{D-\Delta} + \phi_D\, r^{\Delta} + \dots$ (see page A.)

here $m^2 L^2 = \Delta(\Delta - D)$, Convenient choice: $D=3$, $\Delta = 2$.

Δ is the scaling dimension of the operator $\mathcal{O}$.

For a spontaneous condensation: $\phi_{D-\Delta} = 0$

$\phi_\Delta = \langle \mathcal{O} \rangle.$

One can study $\langle \mathcal{O} \rangle$ as a function of temperature and restore the phase diagram. One can also study transport coefficients and critical exponents...

Scalar field in AdS

$$S_\Phi \propto \int d^{d+1}x \sqrt{-g}\left(-\frac{1}{2}(\partial\Phi)^2 - \frac{m^2}{2}\Phi^2\right).$$

$\Phi(x,z) = \Phi(z)\, e^{ikx}$, the wave eqn. for the scalar:

$$z^2 f(z)\,\Phi''(z) - z\left[z f'(z) - (d-1) f(z)\right]\Phi'(z) - \left[k^2 z^2 + m^2 L^2\right]\Phi(z) = 0$$

in the background of a charged BH.

Additional Ansatz: $A = A_t(z)\, dt$.

Near the boundary ($z \to 0$):

$$\Phi \sim \phi_{d-\Delta}(k)\, z^{d-\Delta} + \phi_\Delta(k)\, z^{\Delta} + \dots$$

$\Delta(\Delta - d) = m^2 L^2$, Δ is the scaling dim. of the dual op. $\mathcal{O}$.

Note: 1) $\phi_{d-\Delta}(k)$ is non-normalizable and reqires a counter-term on the boundary.

2) $$\phi_\Delta = \frac{\Gamma(\frac{d}{2} - \Delta)}{2^{\Delta - d/2}\,\Gamma(\Delta - \frac{d}{2})}\,(\omega^2 + k^2)^{\Delta - \frac{d}{2}}\,\phi_{d-\Delta}$$

Source: $J(k) = \phi_{d-\Delta}(k) = \lim_{z \to 0} z^{\Delta - d}\, \Phi(k, z)$

VEV: $\langle \mathcal{O}(k) \rangle = \frac{2\Delta - d}{L}\, \phi_\Delta(k)$, where

$\mathcal{O}(k)$ corresponds to $J(k)$ and $\Delta = \dim[\mathcal{O}]$, so near the boundary

$$\Phi \sim \underset{\nwarrow \text{source}}{J(k)}\, z^{d-\Delta} + \# \underset{\nearrow \atop \text{VEV.}}{\langle \mathcal{O} \rangle}\, z^{\Delta} + \dots$$

Holographic dictionary

LB

BOUNDARY		BULK $(\sim AdS_5)$
$T_{\mu\nu}$	$\leftrightsquigarrow$	$g_{\mu\nu}^{(4)}$
T, ϵ, s, P	$\leftrightsquigarrow$	z_H
S	$\leftrightsquigarrow$	A_H
μ	$\leftrightsquigarrow$	$A_0(z=0) - A_0(z_H)$
J_μ	$\leftrightsquigarrow$	$A_\mu^{(2)}$
C^{abc}	$\leftrightsquigarrow$	S_{cs}^{abc}
Mag. field	$\leftrightsquigarrow$	Mag. field
free energy	$\leftrightsquigarrow$	on-shell bulk action
Global symmetry	$\leftrightsquigarrow$	Gauge symmetry
$\mathcal{O}(x)$	$\leftrightsquigarrow$	$\Phi(x,z)$
$\Delta_{\mathcal{O}}$	$\leftrightsquigarrow$	m_Φ
Strength of interactions, λ	$\leftrightsquigarrow$	Curvature radius in String units, $\left(\frac{D}{\ell_s}\right)^4$.