

# AN INTRODUCTION TO INSTANTON KNOT FLOER HOMOLOGY

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These notes are intended to serve as an informal introduction to instanton Floer homology of knots. We begin by developing a version of instanton Floer homology for closed 3-manifolds [2], [5] associated with admissible  $SO(3)$  bundles, and then briefly review three instanton knot homology theories due to Floer [5] and to Kronheimer and Mrowka [6], [7].

## 1. STIEFEL–WHITNEY CLASSES

Let  $Y$  be a finite CW-complex of dimension at most three and  $P \rightarrow Y$  an  $SO(3)$  bundle. Associated with  $P$  are its Stiefel–Whitney classes  $w_i(P) \in H^i(Y; \mathbb{Z}/2)$ ,  $i = 1, 2, 3$ . Only one of them, namely,

$$w_2(P) \in H^2(Y; \mathbb{Z}/2),$$

will be of importance to us, for the reason that  $w_1(P) = 0$  since  $P$  is orientable, and  $w_3(P) = \text{Sq}^1(w_2(P))$  by the Wu formula. The class  $w_2(P)$  can be defined as follows. Let  $\{U_\alpha\}$  be a trivializing cover for the bundle  $P$  then the gluing functions

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SO(3) \tag{1}$$

define a class  $[P] \in H^1(Y; SO(3))$  in Čech cohomology. Conversely, a cohomology class in  $H^1(Y; SO(3))$  defines a unique  $SO(3)$  bundle up to an isomorphism. The short exact sequence of Lie groups

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow SU(2) \xrightarrow{\text{Ad}} SO(3) \longrightarrow 1,$$

where  $\mathbb{Z}/2 = \{\pm 1\}$  is the center of  $SU(2)$ , gives rise to the exact sequence in Čech cohomology,

$$\dots \rightarrow H^1(Y; SU(2)) \longrightarrow H^1(Y; SO(3)) \xrightarrow{\delta} H^2(Y; \mathbb{Z}/2) \rightarrow \dots$$

The class  $w_2(P)$  is the image of  $[P] \in H^1(Y; SO(3))$  under the connecting homomorphism  $\delta$ . More explicitly, given a set of gluing functions (1) corresponding to a nice cover (meaning that all  $U_\alpha$  and all their intersections are contractible), lift  $\varphi_{\alpha\beta}$  arbitrarily to

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SU(2).$$

One can then check that

$$\varepsilon_{\alpha\beta\gamma} = \psi_{\alpha\beta} \cdot \psi_{\beta\gamma} \cdot \psi_{\gamma\alpha} : U_\alpha \cap U_\beta \cap U_\gamma \rightarrow \mathbb{Z}/2$$

is a Čech 2-cocycle. Its cohomology class is the Stiefel–Whitney class  $w_2(P) \in H^2(Y; \mathbb{Z}/2)$ . By its very definition,  $w_2(P)$  is the obstruction to lifting  $P$  to an  $SU(2)$  bundle over  $Y$ . With a bit more work, one can show that the class  $w_2(P)$  completely classifies  $SO(3)$  bundles  $P \rightarrow Y$ .

## 2. ADMISSIBLE BUNDLES

Let  $Y$  be a closed oriented 3-manifold, and note that the universal coefficient theorem supplies us with the short exact sequence

$$0 \rightarrow \text{Ext}(H_1(Y), \mathbb{Z}/2) \longrightarrow H^2(Y; \mathbb{Z}/2) \xrightarrow{h} \text{Hom}(H_2(Y), \mathbb{Z}/2) \rightarrow 0$$

An  $SO(3)$  bundle  $P \rightarrow Y$  is called *admissible* if either (1)  $Y$  is an integral homology sphere, or else (2)  $h(w_2(P)) \neq 0$ .

**Proposition 2.1.** *Admissible bundles of type (2) do not carry reducible flat connections.*

*Proof.* Let  $P \rightarrow Y$  be an admissible bundle of type (2) carrying a reducible flat connection and let  $\alpha : \pi_1(Y) \rightarrow SO(3)$  be its holonomy. Each homology class in  $H_2(Y)$  is represented by an embedded surface  $i : F \rightarrow Y$ , therefore, by the admissibility of  $P$ , one can find a surface  $F$  such that

$$0 \neq i^*w_2(P) = w_2(i^*P) \in H^2(F; \mathbb{Z}/2) = \mathbb{Z}/2.$$

This implies that the  $SO(3)$  bundle  $i^*P$  does not lift to an  $SU(2)$  bundle, which in turn implies that the representation  $i^*\alpha : \pi_1(F) \rightarrow SO(3)$  does

not lift to an  $SU(2)$  representation. However, one can easily check that all reducible representations  $\pi_1(F) \rightarrow SO(3)$ , such as our representation  $i^*\alpha$ , do admit  $SU(2)$  lifts.  $\square$

**Remark 2.2.** Proposition 2.1 does not hold without the assumption of admissibility: if  $\gamma$  is the canonical line bundle over  $\mathbb{RP}^3$  then the  $SO(3)$  bundle  $P = 1 \oplus \gamma \oplus \gamma$  carries a reducible flat connection with the holonomy  $\text{diag}(1, -1, -1)$ . Of course, the bundle  $P$  is not admissible because  $H_2(\mathbb{RP}^3) = 0$ .

### 3. FLOER HOMOLOGY OF ADMISSIBLE BUNDLES

Proposition 2.1 allows to define Floer homology  $I_*(Y, P)$  for admissible bundles of type (2) by modifying the original construction of Floer [4] for integral homology spheres; see [2] and [5]. The original construction in fact simplifies since we no longer need to worry about the trivial connection; at the same time, the absence of the trivial connection makes it more difficult to define an absolute grading in  $I_*(Y, P)$ . The most important modification, however, comes from the change in the gauge group.

The group  $\mathcal{G}(P)$  of gauge transformations admits a homomorphism  $\eta : \mathcal{G}(P) \rightarrow H^1(Y; \mathbb{Z}/2)$  which gives an obstruction to lifting  $g \in \mathcal{G}(P)$  to an  $SU(2)$  gauge transformation. To give an accurate description of  $\eta$ , view automorphisms  $g \in \mathcal{G}(P)$  as sections of the bundle  $\text{Ad } P = P \times_{\text{Ad}} SO(3)$  and view cohomology classes in  $H^1(Y; \mathbb{Z}/2)$  as homomorphisms  $\pi_1(Y) \rightarrow \mathbb{Z}/2$ . For any loop  $\gamma : S^1 \rightarrow Y$ , the pull back bundle  $\gamma^* \text{Ad } P$  is trivial hence the section  $\gamma^*g$  can be viewed as a function  $S^1 \rightarrow SO(3)$  defined uniquely up to conjugation. Then the homomorphism  $\eta(g) : \pi_1(Y) \rightarrow \mathbb{Z}/2$  takes  $[\gamma] \in \pi_1(Y)$  to  $(\gamma^*g)(1) \in \pi_1(SO(3)) = \mathbb{Z}/2$ .

One can use obstruction theory to show that  $\eta : \mathcal{G}(P) \rightarrow H^1(Y; \mathbb{Z}/2)$  is surjective hence it gives rise to the short exact sequence

$$1 \longrightarrow \mathcal{G}_0(P) \longrightarrow \mathcal{G}(P) \xrightarrow{\eta} H^1(Y; \mathbb{Z}/2) \longrightarrow 1.$$

The Floer homology  $I_*(Y, P)$  is then defined as the Morse theory of the Chern–Simons function on the space  $\mathcal{A}(P)/\mathcal{G}_0(P)$ . Its points are usually interpreted as  $U(2)$  connections with a fixed trace modulo the determinant one gauge group [2] but we will not use this interpretation in these notes. The group  $H^1(Y; \mathbb{Z}/2)$  acts on  $\mathcal{A}(P)/\mathcal{G}_0(P)$  with the quotient  $\mathcal{A}(P)/\mathcal{G}(P)$  hence the critical points of the Chern–Simons function come in  $H^1(Y; \mathbb{Z}/2)$  orbits, one orbit for each representation  $\alpha : \pi_1(Y) \rightarrow SO(3)$  modulo conjugation; note that these orbits need not be free.

In what follows, we will get a feel for  $I_*(Y, P)$  by examining its generating set in several special cases arising in applications to instanton Floer homology of knots. To simplify our task, we will assume that we are in as generic a situation as possible, which will usually mean that no perturbation of the Chern–Simons function is needed.

#### 4. FLOER HOMOLOGY OF KNOTS: THE ORIGINAL VERSION

Let  $k$  be a knot in an integral homology sphere  $\Sigma$ , and  $Y$  the manifold obtained by 0–surgery along  $k$ . If  $K = \Sigma - \text{int } N(k)$  is the exterior of the knot  $k$  then

$$Y = K \cup (S^1 \times D^2),$$

where  $\partial K$  is identified with  $\partial(S^1 \times D^2)$  by matching the meridian  $m$  with the circle factor of  $S^1 \times D^2$  and the longitude  $\ell$  with  $\partial D^2$ . Note that  $H^2(Y; \mathbb{Z}/2) = \text{Hom}(H_2(Y), \mathbb{Z}/2) = \mathbb{Z}/2$ . Therefore, there is just one admissible bundle  $P \rightarrow Y$  and we define the Floer homology of  $k$  as  $I_*(Y, P)$ .

**4.1. The generators.** The bundle  $P \rightarrow Y$  is uniquely characterized by the property that  $w_2(P) \neq 0$  hence all we need to do is look for representations  $\pi_1(Y) \rightarrow SO(3)$  which do not lift to  $SU(2)$  representations. Since the group  $\pi_1(Y)$  is obtained from  $\pi_1(K)$  by imposing the relation  $\ell = 1$ , we can identify representations  $\pi_1(Y) \rightarrow SO(3)$  with representations  $\alpha : \pi_1(K) \rightarrow SO(3)$  such that  $\alpha(\ell) = 1$ . Since  $H^2(K; \mathbb{Z}/2) = 0$ , a representation  $\alpha$  lifts to a representation  $\tilde{\alpha} : \pi_1(K) \rightarrow SU(2)$  which sends  $\ell$  to either  $+1$  or  $-1$ . The former option would correspond to the original representation  $\pi_1(Y) \rightarrow$

$SO(3)$  lifting to an  $SU(2)$  representation, therefore, it is the representations  $\tilde{\alpha} : \pi_1(K) \rightarrow SU(2)$  with  $\tilde{\alpha}(\ell) = -1$  that we are after. The conjugacy classes of these representations generate the Floer chain complex of  $I_*(Y, P)$ .

A lift  $\tilde{\alpha} : \pi_1(K) \rightarrow SU(2)$  of  $\alpha$  is not unique, however, any other lift must be of the form  $\chi \cdot \tilde{\alpha}$ , where  $\chi : \pi_1(K) \rightarrow \mathbb{Z}/2$  is a function taking values in the center of  $SU(2)$ . The fact that  $\chi \cdot \tilde{\alpha}$  is a homomorphism implies that  $\chi$  must be a homomorphism as well,  $\chi \in \text{Hom}(\pi_1(K), \mathbb{Z}/2) = H^1(K; \mathbb{Z}/2)$ . Thus each  $\alpha$  admits exactly two lifts,  $\tilde{\alpha}$  and  $\chi \cdot \tilde{\alpha}$ , where  $\chi$  is a generator of  $H^1(K; \mathbb{Z}/2) = \mathbb{Z}/2$ .

**Lemma 4.1.** *The representations  $\tilde{\alpha}$  and  $\chi \cdot \tilde{\alpha} : \pi_1(K) \rightarrow SU(2)$  are not conjugate to each other.*

*Proof.* Suppose on the contrary that there is  $u \in SU(2)$  such that  $\chi \cdot \tilde{\alpha} = u \tilde{\alpha} u^{-1}$ . Apply  $\chi$  twice to obtain  $\tilde{\alpha} = u^2 \tilde{\alpha} u^{-2}$ . It follows from Proposition 2.1 that  $\tilde{\alpha}$  is irreducible hence  $u^2$  must lie in the center of  $SU(2)$ , meaning  $u^2 = \pm 1$ . The option  $u^2 = 1$  would mean that  $u = \pm 1$  which leads to a contradiction because the equation  $-\tilde{\alpha}(m) = \tilde{\alpha}(m)$  cannot hold in  $SU(2)$ . Therefore,  $u^2 = -1$  and one may assume after conjugation that  $u = i$ . But then  $\pm \tilde{\alpha}(g) = i \tilde{\alpha}(g) i^{-1}$  for all  $g \in \pi_1(K)$ , which means that the image of  $\tilde{\alpha} : \pi_1(K) \rightarrow SU(2)$  is contained in the binary dihedral group  $U(1) \cup j \cdot U(1)$ , where  $U(1)$  stands for the circle of unit complex numbers in  $SU(2)$ . Since  $\ell$  belongs to the second commutator subgroup of  $\pi_1(K)$  (proving this is an exercise in knot theory), its image  $\tilde{\alpha}(\ell)$  must belong to the second commutator subgroup of the binary dihedral group, which is easily seen to be trivial. This contradicts the fact that  $\tilde{\alpha}(\ell) = -1$ .  $\square$

**Corollary 4.2.** *The  $H^1(Y; \mathbb{Z}/2)$  action on the generating set of  $I_*(Y, P)$  is free hence the conjugacy class of every representation  $\pi_1(Y) \rightarrow SO(3)$  not lifting to  $SU(2)$  gives rise to exactly two generators.*

**Remark 4.3.** One can show [2, Section 1.3] that the involution sending  $\tilde{\alpha}$  to  $\chi \cdot \tilde{\alpha}$  defines a free involution on  $I_*(Y, P)$  of degree 4 (mod 8). For this

reason,  $I_*(Y, P)$  is often truncated to just four groups and viewed as a  $\mathbb{Z}/4$  graded theory whose rank is half that of  $I_*(Y, P)$ .

In what follows, we will view representations  $\alpha : \pi_1(K) \rightarrow SU(2)$  with  $\alpha(\ell) = \pm 1$  as *projective* representations  $\alpha : \pi_1(Y) \rightarrow SU(2)$ , that is, set-theoretic maps which are only required to be homomorphisms up to  $\pm 1$ . They can be viewed as the holonomies of projectively flat  $U(2)$  connections.

**4.2. The pillowcase.** The above calculation admits a rather nice pictorial description. The inclusion  $i : \partial K \rightarrow K$  induces a map

$$i^* : \mathcal{R}(K) \rightarrow \mathcal{R}(\partial K), \tag{2}$$

for  $\mathcal{R}(K) = \text{Hom}(\pi_1(K), SU(2))/SU(2)$  and the similarly defined  $\mathcal{R}(\partial K)$ . The space  $\mathcal{R}(\partial K)$ , known in the literature as the *pillowcase*, is described as follows. The choice of meridian  $m$  and longitude  $\ell$  on  $\partial K$  establishes a homeomorphism  $\partial K = T^2$ . Every homomorphism  $\alpha : \pi_1(\partial K) \rightarrow SU(2)$  is then given by two  $SU(2)$  matrices  $\alpha(m)$  and  $\alpha(\ell)$  which commute with each other. Therefore, after conjugation if necessary, one may assume that  $(\alpha(m), \alpha(\ell)) \in U(1) \times U(1)$ . The residual conjugation by  $j$  acts on this 2-torus by the rule  $(z, w) \rightarrow (\bar{z}, \bar{w})$ , and the quotient space of this action is the pillowcase  $\mathcal{R}(\partial K)$  shown in Figure 1.

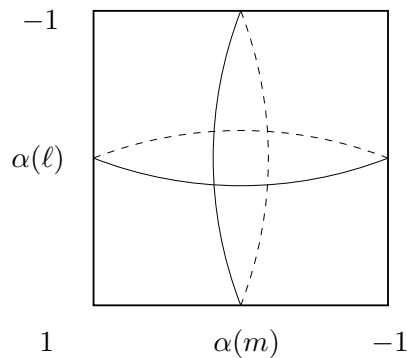


FIGURE 1. The pillowcase

The map (2) sends reducible representations  $H_1(K) \rightarrow SU(2)$  bijectively to the bottom edge of the pillowcase, and the irreducible ones (generically)

to a collection of immersed curves in  $\mathcal{R}(\partial K)$  consisting of circles and open intervals that are only allowed to limit to the bottom edge of the pillowcase. The representations sending the longitude to  $-1$  show up on the top edge of the pillowcase.

**Example.** Let  $k \subset S^3$  be the left-handed trefoil. The fundamental group of its exterior has presentation  $\pi_1(K) = \langle a, b \mid a^2 = b^3 \rangle$  with the meridian  $m = b^{-1}a$  and longitude  $\ell = a^{-2}m^6$ . After conjugation, one may assume that  $\alpha(m) = e^{i\varphi}$  with  $0 \leq \varphi \leq \pi$ . If  $\alpha : \pi_1(K) \rightarrow SU(2)$  is reducible then  $\alpha(\ell) = 1$  hence  $\alpha$  is sent bijectively to the bottom edge of the pillowcase by (2). Next, observe that  $a^2$  and  $b^3$  are in the center of  $\pi_1(K)$ . Therefore, if  $\alpha$  is irreducible, we must have  $\alpha(a)^2 = \alpha(b)^3 = -1$  which implies that  $\alpha(m) = i$  and  $\alpha(b)$  is conjugate to  $e^{\pi i/3}$ . Then  $\alpha(m) = \alpha(b)^{-1}\alpha(a) = e^{i\varphi}$  with  $\pi/6 < \varphi < 5\pi/6$ , and  $\alpha(\ell) = \alpha(a)^{-2}\alpha(m)^6 = -e^{6i\varphi} = e^{i(\pi+6\varphi)}$ . This gives the arc shown in Figure 2. The two representations sending the longitude to  $-1$  are shown as the black dots on the top edge of the pillowcase.

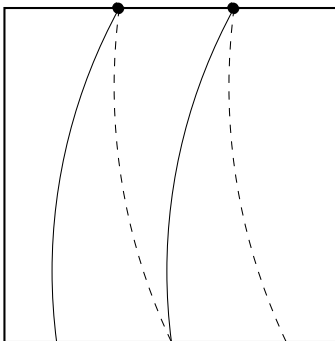


FIGURE 2.

**4.3. The Floer exact triangle.** Let  $k$  be a knot in an integral homology sphere  $\Sigma$ . Denote by  $Y$  the manifold obtained by 0-surgery along  $k$ , and by  $\Sigma'$  the manifold obtained by  $(-1)$ -surgery along  $k$ . The Floer homology groups of these three manifolds are related by the *Floer exact triangle*

$$\begin{array}{ccc}
& I_*(Y, P) & \\
-1 \swarrow & & \nwarrow \\
I_*(\Sigma) & \longrightarrow & I_*(\Sigma')
\end{array}$$

whose morphisms are given by the traces of the respective surgeries. The horizontal arrow has degree zero, and an absolute grading on  $I_*(Y, P)$  is chosen so that the other unmarked arrow has degree zero as well. The proof of this result is rather involved but here is a rough idea behind it.

Instead of working with the generating sets of the chain complexes of  $I_*(\Sigma)$ ,  $I_*(\Sigma')$  and  $I_*(Y, P)$ , look at their images in the pillowcase  $\mathcal{R}(\partial K)$ . These are cut out from the image of the map (2) by respectively the equations  $\alpha(m) = 1$ ,  $\alpha(m) = \alpha(\ell)$  and  $\alpha(\ell) = -1$ . One can isotope the diagonal  $\alpha(m) = \alpha(\ell)$  of the pillowcase into the union of the edges  $\alpha(m) = 1$  and  $\alpha(\ell) = -1$  as shown in Figure 3, thereby deforming the set of generators of the Floer chain complex of  $\Sigma'$  into a union of generators of the Floer chain complexes of  $\Sigma$  and  $Y$ . Some major effort is required [2] to make this into a short exact sequence of the three chain complexes, which will then lead as usual to the long exact sequence in homology.

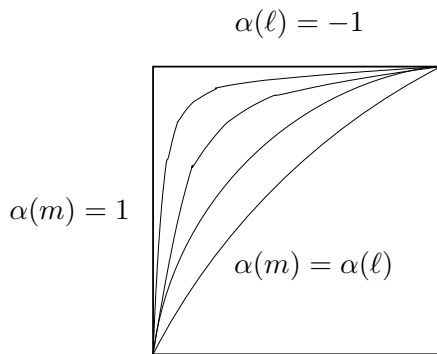


FIGURE 3. The isotopy

**Example.** It is an exercise in Kirby calculus to show that  $(-1)$ -surgery on the left-handed trefoil is the Poincaré homology sphere  $\Sigma(2, 3, 5)$ . Since  $I_*(S^3) = 0$ , the Floer exact triangle



$$\begin{array}{ccc}
& I_*(Y, P) & \\
\swarrow & & \searrow \\
0 & \longrightarrow & I_*(\Sigma(2, 3, 5))
\end{array}$$

establishes an isomorphism  $I_*(\Sigma(2, 3, 5)) \rightarrow I_*(Y, P)$ . More information can be gained from the pillowcase picture of the trefoil: for example, we see from Figure 4 that each of the chain complexes of  $I_*(\Sigma(2, 3, 5))$  and  $I_*(Y, P)$  has two generators, which show up in the figure below as the intersection points with the diagonal and the upper edge. The Floer indices of these generators are equal to 1 and 5 as computed by Fintushel and Stern [3] after one matches their orientation conventions to ours, see [9, Example 6.23].

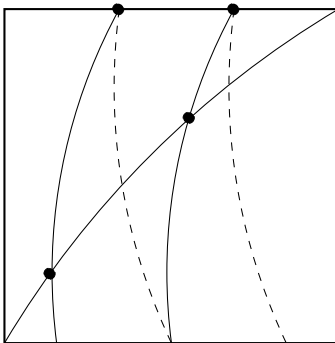


FIGURE 4.

**Remark 4.4.** The Floer exact triangle together with Taubes' work [10] and Casson's surgery formula [1] can be used to show that  $\chi(I_*(Y, P)) = \Delta''(1)$ , where  $\Delta(t)$  is the symmetrized Alexander polynomial of the knot  $k$ .

## 5. FLOER HOMOLOGY OF KNOTS: THE SUTURED VERSION

Let  $k$  be a knot in an integral homology sphere  $\Sigma$  and consider the punctured torus  $T = T^2 - \text{int}(D^2)$  with a basis  $a, b$  of simple closed curves in its first homology. Consider the closed 3-manifold

$$Z = K \cup (S^1 \times T)$$

obtained by matching the meridian  $m$  with the  $S^1$  factor and the longitude  $\ell$  with  $\partial T$ . Note that  $H_1(Z) = H_1(T^3)$  is generated by  $a, b$  and the circle factor  $c$ , and  $H_2(Z) = H_2(T^3)$  is generated by the tori  $a \times c$  and  $b \times c$  and a surface  $F$  obtained by capping off a Seifert surface of  $\ell$  by  $T$ . Let  $P$  be an admissible bundle whose  $w_2(P)$  evaluates non-trivially on the torus  $a \times c$  and trivially on the torus  $b \times c$  and the surface  $F$ . Define the Floer homology of  $k$  to be  $I_*(Z, P)$ . This is essentially the sutured Floer homology  $KHI(k)$  of Kronheimer and Mrowka [7]: the group  $I_*(Z, P)$  is in fact twice the rank of  $KHI(k)$ , see Remark 4.3.

**5.1. The trivial knot.** Let  $k$  be an unknot in  $S^3$  then  $Z = T^3$  and one can choose  $F = a \times b$ . The chain complex of  $I_*(T^3, P)$  is generated by the conjugacy classes of projective representations  $\beta : \pi_1(T^3) \rightarrow SU(2)$  such that

$$[\beta(a), \beta(b)] = 1, \quad [\beta(b), \beta(c)] = 1, \quad [\beta(a), \beta(c)] = -1. \quad (3)$$

**Lemma 5.1.** *Up to conjugation, the equation  $[A, B] = -1$  on the matrices  $A, B \in SU(2)$  has only one solution, namely,  $A = j$  and  $B = i$ .*

*Proof.* The equation  $[A, B] = -1$  can be written as  $ABA^{-1} = -B$  thus guaranteeing that  $\text{tr } B = 0$ . After conjugation, we may assume that  $B = i$ . Since  $A$  anti-commutes with  $i$ , some further conjugation by a unit complex number can be used to make  $A = j$  without changing  $B$ .  $\square$

It now follows that, up to conjugation, equations (3) have just two solutions,  $\beta(a) = j, \beta(b) = \pm 1$  and  $\beta(c) = i$ . These are obviously not conjugate to each other but they give rise to the same  $SO(3)$  representation. In particular, we see that the  $H^1(T^3; \mathbb{Z}/2) = (\mathbb{Z}/2)^3$  acts on the space of projective  $SU(2)$  representations on  $T^3$  with the stabilizer  $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ . Therefore,

$$I_*(T^3, P) = \mathbb{Z}^2 \quad \text{and} \quad KHI(k) = \mathbb{Z}.$$

**5.2. The general case.** In general, the fundamental group  $\pi_1(Z)$  is an amalgamated free product

$$\begin{array}{ccccc}
& & & \pi_1(K) & \\
& & i_* & \nearrow & \\
\pi_1(T^2) & & & & \searrow & \pi_1(Z) \\
& & j_* & \searrow & \nearrow & \\
& & & \pi_1(S^1 \times T) & & 
\end{array}$$

hence projective representations  $\pi_1(Z) \rightarrow SU(2)$  can be viewed as pairs  $\alpha * \beta$ , where  $\alpha : \pi_1(K) \rightarrow SU(2)$  is representation and  $\beta : \pi_1(S^1 \times T) \rightarrow SU(2)$  is a projective representation, such that  $i^*\alpha = j^*\beta$ . In terms of the meridian  $m$  and longitude  $\ell$ , the latter condition simply means that

$$\alpha(m) = \beta(c) \quad \text{and} \quad \alpha(\ell) = \beta(\partial T).$$

The space of the conjugacy classes of the projective representations  $\alpha * \beta$  as above will be denoted by  $\mathcal{R}_w(Z)$ .

The projective representations  $\beta : \pi_1(S^1 \times T) \rightarrow SU(2)$  are given by the equations  $[\beta(a), \beta(c)] = -1$  and  $[\beta(b), \beta(c)] = 1$ . One can easily see that, up to conjugation, there is a circle  $\beta(a) = j$ ,  $\beta(b) = e^{i\varphi}$  and  $\beta(c) = i$  of such representations parameterized by the angle  $\varphi$ . Since  $\beta(\partial T) = [\beta(a), \beta(b)] = e^{-2i\varphi}$ , the map  $j^*$  wraps this circle twice around the trace-zero circle in the pillowcase  $\mathcal{R}(T^2)$  shown in Figure 5.

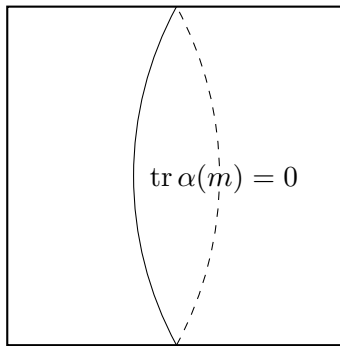


FIGURE 5.

The conditions on  $\alpha$  are then  $\alpha(m) = i$  and  $\alpha(\ell) = e^{-2i\varphi}$ , and there are two types of  $\alpha$  satisfying these conditions. The first type comes from the unique

reducible  $\alpha : \pi_1(K) \rightarrow SU(2)$  such that  $\alpha(m) = i$ . Since  $j^*$  is a two-to-one map, this gives two isolated points in  $\mathcal{R}_w(Z)$  characterized by

$$\alpha(m) = i, \quad \beta(a) = j, \quad \beta(b) = \pm 1 \quad \text{and} \quad \beta(c) = i.$$

The second type comes from irreducible  $\alpha : \pi_1(K) \rightarrow SU(2)$  such that  $\alpha(m) = i$ . There is a circle's worth of representations obtained by conjugating  $\alpha$  by a unit complex number (it has to be a complex number so that the conjugation preserves the condition  $\alpha(m) = i$ ). Since  $j^*$  is a two-to-one map, this gives rise to two circles in  $\mathcal{R}_w(Z)$  characterized by

$$\alpha(m) = i, \quad \alpha(\ell) = e^{-2i\varphi}, \quad \beta(a) = j, \quad \beta(b) = \pm e^{i\varphi} \quad \text{and} \quad \beta(c) = i.$$

To describe the outcome of this calculation, it is convenient to introduce the character variety  $\mathcal{R}_0(K)$  of trace-free representations, which consist of the conjugacy classes of representations  $\alpha : \pi_1(K) \rightarrow SU(2)$  with  $\text{tr } \alpha(m) = 0$ . The unique reducible point in  $\mathcal{R}_0(K)$  then gives rise to two isolated points in  $\mathcal{R}_w(Z)$ , and each irreducible point in  $\mathcal{R}_0(K)$  gives rise to two circles.

The presence of circles in  $\mathcal{R}_w(K)$  means that we find ourselves in a non-generic situation and hence the Chern–Simons function needs to be perturbed in order to define  $I_*(Z, P)$ . Since the Euler characteristic of a circle is zero, the Euler characteristic of  $KHI(k)$  equals one for all  $k$ .

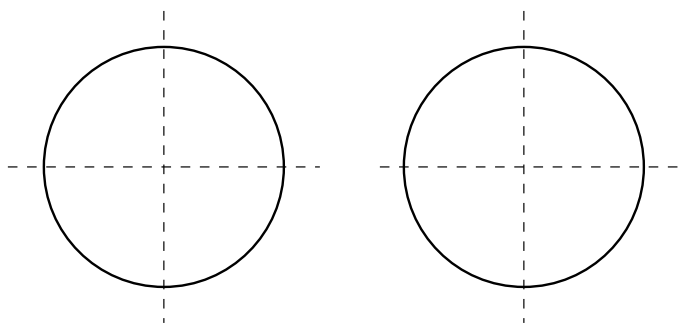


FIGURE 6.

**Example.** If  $k$  is a trefoil,  $\mathcal{R}_0(K)$  consists of one reducible and one irreducible representation, and  $\mathcal{R}_w(Z)$  consists of two isolated points and two

circles. With a bit of extra work one can calculate that  $H^1(Z; \mathbb{Z}/2) = (\mathbb{Z}/2)^3$  acts on the isolated points by permuting them, and on the two circles by permuting them and reflecting each of them with respect to the two sets of axes shown in Figure 6.

**5.3. A surgery description of  $Z$ .** It will be left as an exercise to show that the manifold  $Z$  is the result of the 0–framed surgery on the three–component link  $k \# B$  obtained by connect summing the knot  $k$  with the Borromean rings  $B$ . In particular, we see that  $Z$  is a homology 3–torus with the non-trivial triple cup product. Projective representations on homology 3-tori were studied in detail in [8].

## 6. FLOER HOMOLOGY OF KNOTS: THE SINGULAR VERSION

Let  $k$  be a knot in an integral homology sphere  $\Sigma$  and consider the two–component link  $k^\natural = k \# H$  obtained by connect summing  $k$  with the Hopf link  $H$ . The two components of  $k^\natural$  are the knot  $k$  itself and an ‘earring’  $\ell$ , see Figure 7.

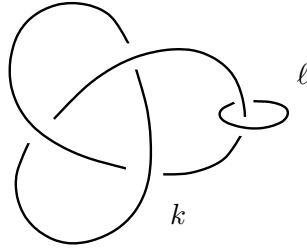


FIGURE 7.

The singular instanton knot homology  $I^\natural(k)$  of Kronheimer and Mrowka [6] is an orbifold version of the Floer theory for admissible bundles. Since  $k^\natural$  has two components,  $H^2(K; \mathbb{Z}/2) = \text{Hom}(H_2(K), \mathbb{Z}/2) = \mathbb{Z}/2$ . Therefore, there is a unique  $SO(3)$  bundle  $Q \rightarrow K$  with  $w_2(Q) \neq 0$ , and the Floer homology  $I^\natural(k)$  is defined as the Morse homology of the Chern–Simons function on  $Q$  restricted to the connections whose holonomy along the two meridians is (asymptotically) of order 2.

The generators of this Floer homology come from projective representations  $\alpha : \pi_1(K) \rightarrow SU(2)$ . Note that  $H_2(K)$  is generated by either of the two boundary components of  $K$  hence  $w_2(Q)$  evaluates non-trivially on both of them. The restriction of  $\alpha$  to each boundary component must then be conjugate to the unique projective representation  $\pi_1(T^2) \rightarrow SU(2)$  sending the generators to  $i, j \in SU(2)$ , see Lemma 5.1. In particular, both meridians of the link  $k^\natural$  are sent by  $\alpha$  to zero-trace matrices, whose adjoint  $SO(3)$  matrices have order two.

**Proposition 6.1.** *There exists a closed 3-manifold  $Y$  and an admissible bundle  $P \rightarrow Y$  such that  $I^\natural(k) = I_*(Y, P)$ .*

*Proof.* In fact, any manifold  $Y$  obtained by identifying the boundary components of  $K$  via an orientation reversing homeomorphism  $\varphi : T^2 \rightarrow T^2$  will do. At the level of generators, this follows from the fact that any projective representation  $\alpha : \pi_1(K) \rightarrow SU(2)$  defines a projective representation  $\pi_1(Y) \rightarrow SU(2)$  because the restrictions of  $\alpha$  to the two boundary components of  $K$  are conjugate to each other, see Lemma 5.1. The Floer excision principle, which essentially elevates this observation to the level of Floer homology, is used in [6] to get an accurate proof.  $\square$

We will now assume that  $k$  is a knot in  $S^3$  and give a surgery description of some of the manifolds  $Y$  obtained, as in the proof of Proposition 6.1, by identifying the boundary components of  $K$ . View  $S^3$  as the boundary of  $D^4$  then  $Y$  will be the boundary of the 4-manifold obtained by attaching the round handle  $D^2 \times S^1 \times I$  by matching the two connected components of  $D^2 \times S^1 \times \partial I$  with tubular neighborhoods of  $k$  and  $\ell$ . This attaching can be done in two stages. First choose an interval  $J \subset S^1$  and attach the 1-handle  $D^2 \times J \times I$  to  $D^4$  to obtain a copy of  $S^1 \times S^3$ . What's left of the round handle is a 2-handle, which is attached to the band sum of  $k$  and  $\ell$  with the band running once around the 1-handle.

The resulting manifold  $Y$  has surgery description as shown in Figure 8, where we exchanged the 1-handle for a 0-framed 2-handle. The framing  $p$

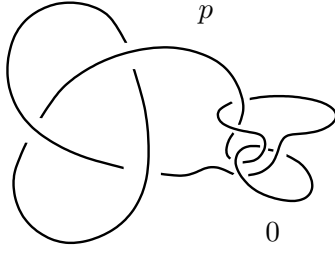


FIGURE 8.

on the other link component depends on the choice of the map  $\varphi : T^2 \rightarrow T^2$  and can be chosen arbitrarily by varying that map. Note that the link in Figure 8 is just a connected sum of  $k$  with the Whitehead link.

For the choice of  $p = 1$  the manifold  $Y$  has homology of  $S^1 \times S^2$  and we can identify  $I^\natural(k)$  with half the ‘original’ Floer homology  $I_*(Y, P)$  discussed in Section 4. One can use the general version of the Floer exact triangle [2] to show that  $I^\natural(k)$  is also isomorphic to half the Floer homology of the manifold  $Z$  obtained by 0–framed surgery on the link  $k \# B$ . Therefore, we have an identification

$$I^\natural(k) = KHI(k)$$

with the ‘sutured’ Floer homology theory discussed in Section 5. Either identification can be used to show that the Euler characteristic of  $I^\natural(k)$  is one for any knot  $k$ .

**Example.** Let  $k$  be the trivial knot then the manifold  $Y$  is obtained by the surgery on the Whitehead link whose components are framed by 0 and 1 or, after blowing down the 1–framed circle, by 0–surgery on a trefoil. The manifold  $Z$  is just the 3-torus. The Floer homology of both  $Y$  and  $Z$  is isomorphic  $\mathbb{Z} \oplus \mathbb{Z}$  hence  $I^\natural(k) = \mathbb{Z}$ .

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