AN INTRODUCTION TO FLOER HOMOLOGY

DANIEL RUBERMAN

Floer homology is a beautiful theory introduced in 1985 by Andreas Floer [8]. It combined new ideas about Morse theory, gauge theory, and Casson's approach [1, 14] to homology 3-spheres and the representations of their fundamental groups into Lie groups such as SU(2) and SO(3). From its inception, it was related to the study of the anti-self-dual Yang-Mills equations on 4-manifolds, and is the receptacle for the relative Donaldson invariants of 4-manifolds with boundary [5].

Floer introduced two versions, one for Lagrangian submanifolds of a symplectic manifold, and another (Instanton Homology) for homology spheres. These threads were reunited with the introduction of Heegaard Floer homology some years later by Ozsváth and Szabó, and monopole homology by Kronheimer and Mrowka [10]. Even with these great advances, the instanton theory retains great interest due to its close connection with the fundamental group–the most basic invariant of a 3-manifold. Remarkable results in knot theory were proved by Kronheimer and Mrowka by developing versions of instanton homology for knots and links in a 3-manifold. There is still much to be learned from Floer's original ideas!

The plan for these two lectures is to briefly review the (ordinary) Morse background and to introduce basic notions about connections. Then we will see the definition and basic properties of the Chern-Simons invariant, and a sketch of the construction of the instanton homology of a homology 3-sphere. A second pair of lectures by Nikolai Saveliev will develop the basics of the instanton knot homology. The prerequisites are a general understanding of

These are notes for the FRG Graduate Workshop on 4-manifolds, Simons Center, August 2014. The author was supported by NSF FRG grant 1065827.

Morse theory (for finite dimensional manifolds) plus a basic familiarity with connections and curvature on principal bundles.

1. Morse theory

I will briefly review how a Morse function f on a finite-dimensional smooth manifold M gives rise to a chain complex that computes the ordinary homology of M. For simplicity, we assume that M is compact and without boundary. The theory of Morse theory on manifolds with boundary plays a key role in the construction of the monopole Floer theory by Kronheimer and Mrowka.

Definition 1.1. A Morse function is a smooth function $f : M \to \mathbb{R}$ whose critical points are all non-degenerate.

Recall that non-degeneracy means that for each $p \in \mathrm{Crit}(f)$, the Hessian is invertible. The Hessian, being self-adjoint, has real spectrum, and we define $\mathrm{ind}_p(f)$ to be the dimension of the sum of the negative eigenspaces. We assume that f is self-indexing, in the sense that for all $p, q \in \mathrm{Crit}(f)$, we have $\mathrm{ind}_p(f) > \mathrm{ind}_q(f) \Rightarrow f(p) > f(q)$.

Choose a Riemannian metric g on M; then the 1-form df gets converted into the gradient vector field ∇f, and we can study its flow equation

$$\dot{\gamma}(t) = -\nabla_{\gamma(t)} f \tag{*}$$

For $p, q \in \mathrm{Crit}(f)$ we have the moduli space of flow lines $\mathcal{M}(p, q)$ of solutions to (*) such that

$$\lim_{t\to-\infty} \gamma(t) = p \text{ and } \lim_{t\to\infty} \gamma(t) = q.$$

Since f decreases along flow lines, this is empty unless $f(p) > f(q)$. By a generic choice of metric, we can ensure that the following *Morse-Smale* transversality property holds for all $p \neq q \in \mathrm{Crit}(f)$:

$$\mathcal{M}(p, q) \text{ is a manifold of dimension } \mathrm{ind}_p(f) - \mathrm{ind}_q(f).$$

An alternate phrasing is that $\mathcal{M}(p,q)$ is the intersection of the descending manifold of p with the ascending manifold of q; the Morse-Smale condition is more correctly stated as saying that this intersection is transverse.

There is an $\mathbb{R}$-action on $\mathcal{M}(p,q)$ (free and proper if $p \neq q$) given by reparameterization, $\gamma(t) \to \gamma(t-T)$ for $T \in \mathbb{R}$, and we denote the quotient $\mathcal{M}(p,q)/\mathbb{R}$ by $\bar{\mathcal{M}}(p,q)$. For each $p \in \mathrm{Crit}(f)$, choose (arbitrarily) an orientation of the negative eigenspace of the Hessian at p; this orients both $\mathcal{M}(p,q)$ and $\bar{\mathcal{M}}(p,q)$.

In particular, if $\mathrm{ind}_p(f) = \mathrm{ind}_q(f)+1$, then $\bar{\mathcal{M}}(p,q)$ is a (compact) oriented 0-manifold, and its points may be counted with signs, yielding a signed count $\#\bar{\mathcal{M}}(p,q) \in \mathbb{Z}$.

1.1. **Morse homology.** Continuing the assumptions from above, we construct a chain complex as follows. Generators of the chain group correspond to the critical points:

$$C_k(M,f) = \bigoplus_{p \in \mathrm{Crit}(f),\ \mathrm{ind}_p(f)=k} \mathbb{Z}\langle p\rangle$$

The boundary operator comes from *counting flow lines*: for $p \in \mathrm{Crit}(f)$ with $\mathrm{ind}_p(f) = k$,

$$\partial p = \sum_{q \in \mathrm{Crit}(f),\ \mathrm{ind}_p(f)=k-1} \#\bar{\mathcal{M}}(p,q)\cdot q.$$

This chain complex appears, with a somewhat different description, in Milnor's exposition of the h-cobordism theorem [11]. There it is shown that this is the same as the cellular chain complex coming from a CW decomposition of M associated with f. It is non-trivial [15] to prove, directly from the definition, that $\partial^2 = 0$. The key idea in calculating the p,r component of $\partial\partial(p)$ for $\mathrm{ind}_p(f) = \mathrm{ind}_r(f)+2$ is to consider the space $\bar{\mathcal{M}}(p,r)$. This has dimension 1, but is no longer compact. Its compactification consists of 'broken flow lines' from p to r, that are made of flow lines from p to q and q to r, where $\mathrm{ind}_p(f) = \mathrm{ind}_q(f)+1$. The quotient $\bar{\mathcal{M}}(p,r)$ is indicated by the horizontal curve; the hollow endpoints correspond to two ends of

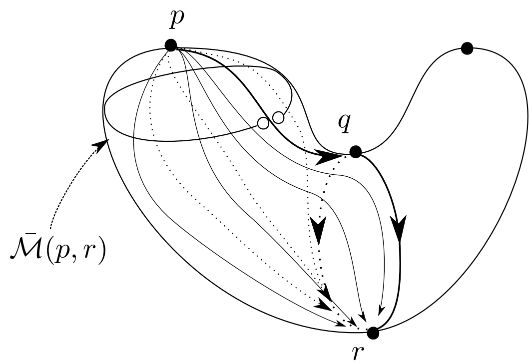

FIGURE 1. Sketch for $\partial^2 = 0$

$\bar{\mathcal{M}}(p,r)$. The compactification of $\bar{\mathcal{M}}(p,r)$ is formed by adding boundary points for each of these ends, resulting (in general) in a union of closed intervals. Those boundary points correspond to broken trajectories which are sketched in Figure **??**

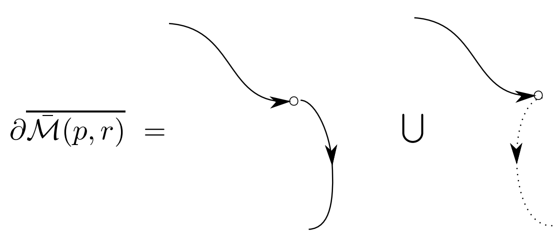

FIGURE 2. The boundary of the compactification

The fact that $\partial^2 = 0$ comes from the fact that the boundary of an oriented interval is 0. The proof that this argument works takes some careful analysis: it involves a study of sequences of solutions to (*) and a gluing theorem to show that broken trajectories can be combined into nearby trajectories from p to r. These analytical facts have (harder) cognates in the Floer homology setting.

1.2. **The idea of Floer homology.** Before we get to details, here is the executive summary of how Floer homology works. The terminology will be developed over the next few sections. Instead of a finite dimensional manifold, we study the space of connections on a principal SU(2) or SO(3) bundle over a 3-manifold Y, modulo the action of the gauge group. The role of the Morse function is played by the Chern-Simons function. Critical points for this function correspond to flat connections, or equivalently to representations of $\pi_1(Y)$.

At this point, the technical bogeymen come out from under the bed

- The space (connections mod gauge group) on which we want to do gauge theory is not a manifold.
- The Chern-Simons function is now circle-valued, not $\mathbb{R}$-valued.
- There's no reason that the Chern-Simons function should have non-degenerate critical points.
- The negative and positive eigenspaces for the Hessian are always infinite dimensional, as are descending and ascending manifolds. (This latter statement needs a big grain of salt.)
- Compactness theorems for the analog of $\bar{\mathcal{M}}(p,q)$ are considerable harder to prove, as are gluing theorems for 'broken flow lines'.

2. Connections

We briefly review the theory of connections on a principal G-bundle $\pi : P \to X$. The cases of interest will be $G = \mathrm{SU}(2)$ and $G = \mathrm{SO}(3)$, where the base space is a closed oriented 3-manifold. In the former setting, the bundle P will be trivial (but not trivialized!) and in the latter will primarily be of interest when $w_2(P) \neq 0$. For the present, we will let G be one of these groups, and $\mathfrak{g}$ its Lie algebra. As usual, the Lie bracket on $\mathfrak{g}$ is denoted $[\cdot,\cdot]$. For $G = \mathrm{SU}(2)$, there is an associated Hermitian $\mathbb{C}^2$ bundle $E = P \times_{\mathrm{SU}(2)} \mathbb{C}^2$, and for $G = \mathrm{SO}(3)$ there is the associated bundle $E = P \times_{\mathrm{SO}(3)} \mathbb{R}^3$ and in both cases we denote by $\operatorname{ad} P$ the bundle $P \times_G \mathfrak{g}$ where G acts via the adjoint representation.

A connection A on P can be viewed in many ways:

- A G-invariant 'horizontal' subbundle $H_A \subset TP$ transversal to the vertical tangent space $VTP = \ker(d\pi)$;

- a system of parallel transport in P (lifts of paths in X to horizontal paths in P);
- a $\mathfrak{g}$ valued, ad-equivariant 1 form equal to $i_p^{-1} : VTP_p \to \mathfrak{g}$ (where i_p is the differential of the action of G on P);
- a covariant derivative $\nabla_A : \mathcal{C}^\infty(X; E) \to \mathcal{C}^\infty(Y; T^*X \otimes E)$ satisfying a Leibniz rule $\nabla_A(f\sigma) = f\nabla_A(\sigma) + df \otimes \sigma$.

The covariant derivative extends to any associated bundle, and we write $d_A : \mathcal{C}^\infty(X; \operatorname{ad} P) \to \mathcal{C}^\infty(X; T^*X \otimes \operatorname{ad} P)$. The Leibniz rule gives an extension of d_A to higher forms $d_A : \Omega^k(X; \operatorname{ad} P) \to \Omega^{k+1}(X; \operatorname{ad} P)$.

The simplest example arises if P is trivialized, i.e. given an isomorphism with $X \times G$. The *trivial connection* A has $H_A = TX$ and connection 1-form induced by projection onto G. Sections of E are identified with functions $X \to \mathbb{C}^2$, and ∇_A becomes the ordinary directional derivative.

Although every bundle is locally trivial, not every connection is locally trivial, as measured by the curvature 2-form, denoted F_A. This has (not surprisingly) various definitions as well. If we denote by X^h the horizontal projection into $(H_A)_p$, then $F_A(V, W)_p = -A_p([V^h, W^h])$. As a 2-form on P, the curvature is given by the expression

$$F_A = dA + \frac{1}{2}[A \wedge A].$$

This last term combines the wedge product and Lie bracket (and in particular is not skew-symmetric on 1-forms!). From either expression, we see that $F_A \in \mathcal{C}^\infty(X; \Lambda^2 \otimes \operatorname{ad} P)$. The trivial connection has F identically 0.

The difference between two connections descends to a $\mathfrak{g}$-valued 1-form on X, so the set of connections on P forms an affine space over $\Omega^1(X; \operatorname{ad} P)$. We denote the set of connections by $\mathcal{A}(P)$ (or just $\mathcal{A}$ when P is understood), and write $A' = A + a$ for the action of $a \in \Omega^1(X; \operatorname{ad} P)$ on $A \in \mathcal{A}$. A key calculation is that $F_{A'} = F_A + d_A a + \frac{1}{2}[a \wedge a]$.

2.1. **Gauge transformations.** A gauge transformation is a bundle automorphism of P covering the identity map of X. Given $g : P \to P$, we get a new map $\hat{g} : P \to G$ by writing $g(p) = p\hat{g}(p)$. It has the property that

$$\hat{g}(ph) = h^{-1}\hat{g}(p)h$$

So the set of gauge transformations, called the gauge group $\mathcal{G}$, is identified with sections of $\operatorname{Ad} P = P \times_{\operatorname{Ad}} G$. There is a natural action of $\mathcal{G}$ on $\mathcal{A}$, given by pulling back a 1-form on P, in symbols $A \to g^*(A)$. The curvature transforms in a simple way

$$F_{g^*A} = \operatorname{Ad}_{\hat{g}} \circ F_A. \tag{1}$$

2.2. **Holonomy and flat connections.** Fix a connection A on P. For a path $\gamma(t)$ in X, parallel transport gives an isomorphism $T_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$. This isomorphism behaves naturally under concatenation of paths, and is conjugated in an obvious way if A is replaced by g^*A for $g \in \mathcal{G}$. If γ is a loop based at $x \in X$, then it gives an automorphism of the fiber P_x called the holonomy (of A) around γ, and is denoted $\operatorname{hol}_A(\gamma)$.

Suppose now that A has the property that its curvature F_A is identically 0; such a connection is called a *flat* connection. Flat connections are locally isomorphic to the trivial connection; this is proved by parallel transport in a coordinate neighborhood in X. On the other hand, the holonomy shows that a flat connection is not necessarily globally trivial. The key result is that if γ and γ' are homotopic loops (rel endpoints) based at x, then

$$\operatorname{hol}_A(\gamma) = \operatorname{hol}_A(\gamma').$$

Thus the holonomy map of a flat connection A gives a homomorphism $\operatorname{hol}_A : \pi_1(X, x) \to G$.

Let us denote by $\operatorname{Flat}(P)$ the set of flat connections on P. By (1), a connection that is gauge equivalent to a flat connection is itself flat, so that $\mathcal{G}$ acts on $\operatorname{Flat}(P)$. For $G = \mathrm{SU}(2)$, the holonomy gives a bijection

$$\operatorname{hol} : \operatorname{Flat}(P)/\mathcal{G} \to \operatorname{Hom}(\pi_1(X), G)/G \equiv \mathfrak{R}(X, G)$$

The inverse of hol is the following construction. Given $\alpha : \pi_1(X) \to G$, form the quotient $P_\alpha = \widetilde{X} \times_{\pi_1(X)} G$ where $\pi_1(X)$ acts on G via α. One has to choose an isomorphism with P, but this ambiguity created by this choice disappears in the quotient by the gauge group. For $G = \mathrm{SO}(3)$, the bijection is a little more subtle, as one has to choose the subset of $\mathfrak{R}(X, \mathrm{SO}(3))$ such that this construction yields a bundle isomorphic to P. For appropriate topologies on $\mathrm{Flat}(P)/\mathcal{G}$ and $\mathfrak{R}(X, G)$, the holonomy map is in fact a homeomorphism. Since $\mathfrak{R}(X, \mathrm{SO}(3))$ is a quotient of a closed subset of G^n (with $n =$ the number of generators in a presentation of $\pi_1(X)$) we deduce the important fact (that also has a strictly analytic proof) that $\mathrm{Flat}(P)/\mathcal{G}$ is compact.

3. The Chern-Simons function

Suppose for the moment that $G = \mathrm{SU}(2)$. Then the bundle P has one characteristic class, the second Chern class $c_2(P) \in H^2(X; \mathbb{Z})$. (This would more normally be written as $c_2(E)$.) The Chern-Weil formalism (see for instance [12, Appendix C]) gives a de Rham cohomology representative for the image of this class in $H^2(X; \mathbb{R})$ via the formula

$$c_2(A) = \frac{1}{8\pi^2} \operatorname{tr}(F_A \wedge F_A).$$

Chern-Weil theory ensures that this is a closed 2-form, and its cohomology class is independent of A. Since it agrees with the integral class $c_2(P)$, its integral over any closed 4-manifold is an integer.

For the case of $G = \mathrm{SO}(3)$ the corresponding class is the first Pontrjagin class of $P \times_{\mathrm{SO}(3)} \mathbb{R}^3$ and is given by

$$p_1(A) = -\frac{1}{2\pi^2} \operatorname{tr}(F_A \wedge F_A).$$

Note, for the moment, the missing factor of 4 in the denominator.

For dimensional reasons, these characteristic classes vanish on 3-manifolds. But they give rise to an interesting 'secondary' characteristic class that depends on A; this is called the Chern-Simons invariant. Here is one definition;

we will see equivalent versions shortly. Let X be a 4-manifold with oriented boundary Y. Then the quantity

$$\int_X \operatorname{tr}(F_A \wedge F_A) \pmod{8\pi^2\mathbb{Z}}$$

depends only on the gauge equivalence class of $A|_Y$, and not on X nor on A in the interior of X. For if $A' = g^*A$ extends to a connection A' on a bundle $P' \to X'$ then we can glue X to X' along Y, and patch the bundles together along Y to get a new bundle $P'' \to X''$. Then

$$\int_X \operatorname{tr}(F_A \wedge F_A) - \int_{X'} \operatorname{tr}(F_{A'} \wedge F_{A'}) = \int_{X''} \operatorname{tr}(F_{A''} \wedge F_{A''}) \in 8\pi^2\mathbb{Z}.$$

For a connection B on $P \to Y$, we define the *Chern-Simons* invariant $\mathrm{CS}(B)$ to be the residue (mod $8\pi^2\mathbb{Z}$) of $\int_X \operatorname{tr}(F_A \wedge F_A)$ where X is arbitrary, and A is any extension of B to a connection on some bundle over X. There is a little topological point here; we need to know that any $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$ bundle on a closed oriented 3-manifold extends over some compact oriented 4-manifold. This follows using the fact that for $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$, the cobordism groups $\Omega_3(BG) = H_3(BG)$; see [9] for a hands-on approach to this fact. (The high-tech version is to use the Atiyah-Hirzebruch spectral sequence.) Moreover, these groups vanish by a standard spectral sequence calculation. Once we have extended the bundle, a standard argument with partitions of unity extends the connection.

The freedom to choose the extension leads to an alternative formulation. Let B_0 be a fixed reference connection on P, and let B be any connection. There is a path of connections B_t, $t \in [0,1]$ with $B_1 = B$, which we may regard as a connection A on $I \times P \to I \times Y$. Then we define the relative Chern-Simons invariant

$$\mathrm{CS}_{B_0}(B) = \int_{I\times Y} \operatorname{tr}(F_A \wedge F_A).$$

If $G = \mathrm{SU}(2)$, then we choose a trivialization, giving a trivial connection that we take for B_0. Writing $B = B_0 + b$, there is an obvious path $B_t = B_0 + tb$.

Then we calculate

$$F_A = d(tb) + \frac{t^2}{2}[b \wedge b] = dt \wedge b + tdb + \frac{t^2}{2}[b \wedge b]. \tag{2}$$

This gives

$$\operatorname{tr}(F_A \wedge F_A) = dt \wedge \operatorname{tr}(b \wedge (2tdb + t^2[b \wedge b])).$$

Carry out the integral defining $\mathrm{CS}(B)$ over the t-variable to get

$$\begin{aligned} \mathrm{CS}(B) &= \int_{I \times Y} \operatorname{tr}(F_A \wedge F_A) \\ &= \int_{I \times Y} dt \wedge \operatorname{tr}(b \wedge (2tdb + t^2[b \wedge b])) \\ &= \int_Y \operatorname{tr}(b \wedge db + \frac{2}{3} b \wedge b \wedge b) \end{aligned}$$

It is worth reiterating that two choices of trivialization differ by a gauge transformation, and the resulting relative CS invariants differ by an integer multiple of $8\pi^2$. Similarly, we have the following fundamental fact about the behavior of CS under gauge transformations. If $G = \mathrm{SU}(2)$ then P is trivial and so is $\operatorname{Ad} P$, so a gauge transformation (aka section of $\operatorname{Ad} P$) can be identified with an ordinary map $g : Y \to \mathrm{SU}(2)$. As such, it has a degree.

Proposition 3.1. *If B is a connection on the* $\mathrm{SU}(2)$ *bundle P, and $g : Y \to$* $\mathrm{SU}(2)$ *a gauge transformation, then*

$$\mathrm{CS}(B) - \mathrm{CS}(g^* B) = 8\pi^2 \deg(g).$$

Proof. The integrand expressing the left-hand side is the integral of $\operatorname{tr}(F_A \wedge F_A)$, where A is a connection on the bundle arising from the mapping torus of g. Thus it computes $c_2(E_g)$, where E_g is the associated $\mathbb{C}^2$ bundle. This is the same as the Euler class, which is computed by counting zeroes of a section. It is a nice exercise to construct a section of E_g with $\deg(g)$ transverse zeroes. An alternate approach is to show that the bundle E_g is pulled back from the bundle over the suspension of Y with clutching function g via a degree-one map from $S^1 \times Y$. It is easy to see that the Euler class of the bundle over the suspension is given by the degree of g. The formula may also be proved via the Atiyah-Patodi-Singer index theorem [2, 3]. $\square$

Exercise 3.2. If B_0 does not arise from a trivialization show that

$$\mathrm{CS}_{B_0}(B_1) = \int_Y \mathrm{tr}(2b \wedge F_{B_0} + b \wedge db + \frac{2}{3} b \wedge b \wedge b).$$

This will be useful in the lectures on Floer homology for manifolds with non-trivial homology (using a non-trivial 'admissible' SO(3) bundle).

4. Floer homology for homology spheres

It is time to start chasing those bogeymen back under the bed where they belong. We will deal exclusively with SU(2) bundles, and assume that our manifold Y is an integral homology sphere (oriented, as always), equipped with a Riemannian metric. First, the space on which we want to do Morse theory is $\mathcal{C} = \mathcal{A}/\mathcal{G}$. The standard procedure is make this into an infinite dimensional manifold by completing both $\mathcal{A}$ and $\mathcal{G}$ with respect to topologies coming from Sobolev norms. The topology on the quotient comes from a local slice theorem for the action of $\mathcal{G}$ on $\mathcal{A}$. We will slide over this (and many other analytic) details, but even so there is a big issue in that $\mathcal{G}$ does not act freely.

The non-freeness comes in two forms, one benign and the other trickier to deal with. $\mathcal{G}$ contains a central $\mathbb{Z}_2$ subgroup consisting of constant gauge transformations with value $\pm I \in \mathrm{SU}(2)$, and this $\mathbb{Z}_2$ acts trivially on $\mathcal{A}$. This is dealt with by working with $\mathcal{G}/\mathbb{Z}_2$ instead, and is not worth much further discussion. On the other hand, there are points in $\mathcal{A}$ with much bigger stabilizers. If Γ is the trivial connection (with respect to some trivialization) then Γ is fixed by any gauge transformation corresponding to a constant function $g : Y \to \mathrm{SU}(2)$, so the $\mathcal{G}$ orbit of Γ has an SU(2) (well, $\mathrm{SU}(2)/\mathbb{Z}_2 = \mathrm{SO}(3)$) stabilizer. In the case when Y is a homology sphere, this fixed point is dealt with in the crudest way possible, by simply deleting it, and setting $\mathcal{A}^* = \mathcal{A}-$ (the $\mathcal{G}$ orbit of Γ). If Y is not a homology sphere, then this procedure causes no end of trouble, and no completely satisfactory workaround has been found despite plenty of effort.

The tangent space to $\mathcal{A}^*$ at a connection B is the linear space $\Omega^1(Y; \operatorname{ad} P)$. To understand the tangent space at $[B] \in \mathcal{C}$, we make use of the Hodge star operator defined by the metric and orientation on Y. As a reminder, an orientation and Riemannian metric on a manifold X^n defines the Hodge $*$-operator $* : \Omega^k(X) \to \Omega^{n-k}(X)$, characterized by

$$\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle, \text{ for } \alpha, \beta \in \Omega^k(X)$$

with $\langle\ ,\ \rangle$ the inner product on forms and μ the volume form. It satisfies $*^2 = (-1)^{k(n-k)}$ on k-forms. The (formal) adjoint of d_B may be written as $d_B^* = - * d_B *$, and its kernel (assuming that B is an irreducible connection) is the tangent space at $[B]$.

We consider the Chern-Simons function $CS : \mathcal{A}^*/\mathcal{G} \to \mathbb{R}/(8\pi^2\mathbb{Z})$ and discuss what we can do to make it act like a Morse function. Actually, it will be convenient to replace CS by $\mathcal{L} = -\frac{1}{2}\,\mathrm{CS}$. Acting like good calculus students, we start by computing the directional derivative of $\mathcal{L}$ at a connection B, in the direction of a 1-form $c \in \Omega^1(Y; \operatorname{ad} P)$. This is computed in $\mathcal{A}^*$ by taking $-\frac{1}{2}\frac{d}{ds}\,\mathrm{CS}(B + sc)|_{s=0}$, and we use the formulation as in (2). (To compute in $\mathcal{A}^*/\mathcal{G}$, we should restrict to directions orthogonal to the orbits of $\mathcal{G}$; this important point will largely be suppressed.) On $I \times Y$, using coordinate t in the I direction, we introduce a connection $A_s = B + (st)c$, and compute

$$F_{A_s} = F_B + d_B^{(4)}(stc) + \frac{s^2t^2}{2}[c \wedge c]$$

so that

$$\frac{d}{ds} F_{A_s}|_{s=0} = dt \wedge c + t d_B c.$$

The derivative of $\mathcal{L}$ is then

$$\begin{aligned}
-\frac{1}{2}\frac{d}{ds}\operatorname{CS}(B+sc)|_{s=0} &= -\frac{1}{2}\frac{d}{ds}\int_{I\times Y}\operatorname{tr}(F_{A_s}\wedge F_{A_s})|_{s=0} \\
&= -\int_0^1\int_Y \operatorname{tr}((dt\wedge c + t d_B c)\wedge F_B) \\
&= -\int_0^1\int_Y dt\wedge \operatorname{tr}(c\wedge F_B) \\
&= -\int_Y \operatorname{tr}(c\wedge F_B).
\end{aligned}$$

It follows that the critical points of $\mathcal{L}$ (and CS as well) on $\mathcal{A}^*$ are *flat* connections; on $\mathcal{C}^*$ the critical points are gauge-equivalence classes of flat connections. The gradient of $\mathcal{L}$ on $\mathcal{A}^*$ is given by $*_Y F_B$ where $*_Y$ is the Hodge $*$-operator associated to the metric on Y. The gradient on $\mathcal{C}^*$ is a little trickier, as we should project into the slice $\ker(d_B^*)$. See [13, Chapter 2] for details.

Mimicking the finite-dimensional theory, let's say that a critical point B is *non-degenerate* if the Hessian of CS at $[B]$ is invertible. If all critical points are non-degenerate, then $\mathcal{L}$ behaves in some sense like a Morse function. By a computation similar to our computation of the gradient, the Hessian is

$$H_B = \frac{1}{2}\operatorname{proj}_{\ker(d_B^*)} * d_B : \ker(d_B^*) \to \ker(d_B^*).$$

If B is flat, the kernel of this operator is (by Hodge theory) the twisted cohomology group $H^1(Y;\operatorname{ad}(B))$ where $\operatorname{ad}(B)$ is the adjoint of the $\mathrm{SU}(2)$ representation of $\pi_1(Y)$ coming from the holonomy of B. A big difference with ordinary Morse theory is that the spectrum of H_B is unbounded in both positive and negative directions. (We are being deliberately vague about exactly what function spaces such operators live on.)

Let us pretend (in the way of children everywhere) that the bogeyman of degenerate flat connections does not exist, summarized crudely by saying that $\mathcal{L}$ is a Morse function. That is of course not necessarily true, and it is dealt with by choosing a gauge-invariant perturbation of $\mathcal{L}$. Carrying this out is somewhat technical (and indeed caused some issues in the early days).

The issue is that one needs to specify some class of perturbations of $\mathcal{L}$ that will contain Morse functions, and yet still have the property that their zero loci are still compact.

4.1. **The Floer complex: chains and grading.** Let Y be an oriented homology sphere, with a Riemannian metric. Assuming that $\mathcal{L}$ is a Morse function, we define the Floer chains to be the $\mathbb{Z}$-module with generators $[B] \in \mathrm{Flat}(Y \times \mathrm{SU}(2))$. Let us sort out the grading, which in finite dimensions is given by the indices of the critical points, a notion that no longer makes sense.

For B_0, B_1 flat connections, we choose a path B_t connecting them, and study the corresponding path of operators H_{B_t}. Such a path has a spectral flow [4], defined by tracking the net number of eigenvalues crossing 0. In other words (and somewhat informally) we define $\mathrm{sf}(B_0, B_1)$ to be the number of eigenvalues that pass (as we move along the path H_{B_t}) from negative to positive minus the number that go the other way. In finite dimensions, this would be precisely the difference in the index of the critical points, and so we think of it as the *relative index* of (B_0, B_1).

The spectral flow between flat *connections* is an integer, but we are dealing with equivalence classes, and so must understand the effect of gauge transformations on the relative index.

Proposition 4.1. *If* $g \in \mathcal{G}$, *then* $\mathrm{sf}(g^*B_0, B_1) - \mathrm{sf}(B_0, B_1) = 8\deg(g)$

The result should remind you of Proposition 3.1, but the proof is more strenuous, and leans on the work of Atiyah-Patodi-Singer [2, 3, 4]. The upshot is that the relative grading is not a $\mathbb{Z}$-grading, but rather a $\mathbb{Z}_8$-grading. An absolute $\mathbb{Z}_8$-grading can be fixed by comparing flat connections to the trivial connection, which is declared to have grading 0. There are some subtleties here, so we skip the details. With additional work [7] the $\mathbb{Z}_8$ grading can be lifted to a $\mathbb{Z}$-grading.

4.2. **The boundary operator.** The definition of the boundary operator is similar to that in the finite-dimensional case, in that we want to count gradient flow lines between (gauge equivalence classes) of flat connections that have relative index one. The best way to say this carefully involves a remarkable observation relating the flow of the Chern-Simons function to the Yang-Mills equation.

The downward flow equation for $\mathcal{L}$ for a family $B(t)$ of connections is $\frac{dB}{dt} = -2(*_Y F_B)$. Let us view $B(t)$ as defining a connection A on $\mathbb{R} \times P \to X = \mathbb{R} \times Y$, where the $\mathbb{R}$ direction in $\mathbb{R} \times P$ is horizontal. The curvature of A on the 4-manifold X is given by

$$F_A = dt \wedge \frac{dB}{dt} + F_B.$$

Note that

$$*F_A = dt \wedge *_Y F_B + *_Y \frac{dB}{dt} = - * F_A.$$

This is the big miracle–the *anti-self-dual Yang Mills equation* $*F_A = - * F_A$ makes sense on any (oriented, Riemannian) 4-manifold, not just the product $\mathbb{R} \times Y$. Moreover, its solutions are invariant under a bigger gauge group $\mathcal{G}_X = Aut(\mathbb{R} \times P) = \{h : \mathbb{R} \times Y \to \mathrm{SU}(2)\}$. The gauge group for P sits inside $\mathcal{G}_X$ as the automorphisms that are constant in t.

The boundary operator is defined by counting points in a certain moduli space. Let α and β be flat connections; again, we are assuming that they are non-degenerate and non-trivial. We define $\mathcal{M}(\alpha, \beta)$ to be the quotient by $\mathcal{G}_X$ of the set of connections A on $\mathbb{R} \times P$ satisfying

- $F_A = - * F_A$ (A is anti-self-dual)
- $\lim_{t \to -\infty} [A|_{t \times Y}] = \alpha$
- $\lim_{t \to \infty} [A|_{t \times Y}] = \beta$
- $- \int_X \mathrm{tr}(F_A \wedge *F_A) < \infty$ (A has finite energy).

Once we have done the analysis to put a topology on $\mathcal{A}^*/\mathcal{G}_X$, we can give $\mathcal{M}(\alpha, \beta)$ the subspace topology. The first three conditions are exactly what we required in ordinary Morse theory; the last one would automatically be satisfied in that setting but not in the gauge-theory context. Note that shifting the t variable again defines an $\mathbb{R}$ action on $\mathcal{M}(\alpha,\beta)$, and we define $\bar{\mathcal{M}}(\alpha,\beta) = \mathcal{M}(\alpha,\beta)/\mathbb{R}$.

The analog of the Morse-Smale condition is that $\bar{\mathcal{M}}(\alpha,\beta)$ be a manifold. It's a long story as to why one would even expect it to have finite dimension, let alone such a nice structure. This is achieved by choosing an appropriate perturbation, say ϵ of $\mathcal{L}$, resulting in moduli spaces $\mathcal{M}_\epsilon(\alpha,\beta)$ and $\bar{\mathcal{M}}_\epsilon(\alpha,\beta)$.

Theorem 4.2. *There is a perturbation ϵ so that $\bar{\mathcal{M}}_\epsilon(\alpha,\beta)$ is a smooth oriented manifold of dimension* $\mathrm{sf}(\alpha,\beta)-1$. *If this dimension is* 0, *then $\bar{\mathcal{M}}_\epsilon(\alpha,\beta)$ is compact.*

The choice of orientation is another subtle point, as the usual Morse-theory procedure (orient the sum of the negative eigenspaces of the Hessian) breaks down due to the infinite-dimensionality of that sum.

Definition 4.3. The instanton Floer chain complex $CF_*(Y)$ has chains generated by the gauge-equivalence classes of flat connections on the trivial $\mathrm{SU}(2)$ bundle over Y. The boundary operator is given by

$$\partial\alpha = \sum_{\beta\in\mathrm{Flat}(P),\ \mathrm{sf}(\alpha,\beta)=1} \#\bar{\mathcal{M}}(\alpha,\beta)\cdot\beta.$$

The proof that this is a chain complex follows the same basic scheme as in finite dimensions; the requisite compactness and gluing theorems are (not surprisingly) a lot more strenuous. Once that's done, one needs to go back and prove the existence of good perturbations, and then show the independence of the resulting homology groups from the choice of perturbation and Riemannian metric. The resulting homology is called instanton Floer homology, $I_*(Y)$.

Example 4.4. The Poincaré homology sphere Q is defined as the quotient of $\mathrm{SU}(2)$ by the binary icosahedral group I^*. Up to conjugacy, there are two irreducible $\mathrm{SU}(2)$ representations of $\pi_1(Q)$: the inclusion $\rho : I^* \to$

SU(2), and its complex conjugate $\bar{\rho}$. It is easy to check that these are non-degenerate. With a little more work one computes that their gradings are 5 and 1 mod 8 (ask Nikolai which one is which!). In particular, the boundary operator is 0, so that $I_*(Q) = \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(5)}$. This calculation can be carried out for any Seifert-fibered homology sphere with 3 exceptional fibers [6]. When there are more exceptional fibers, the space of flat connections is a manifold, but not of the correct dimension, so that perturbations are required.

References

[1] S. Akbulut and J. McCarthy, *Casson's invariant for oriented homology 3-spheres – an exposition*, Mathematical Notes, vol. 36, Princeton University Press, Princeton, 1990.

[2] M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry: I*, Math. Proc. Camb. Phil. Soc. **77** (1975), 43–69.

[3] ———, *Spectral asymmetry and Riemannian geometry: II*, Math. Proc. Camb. Phil. Soc. **78** (1975), 405–432.

[4] ———, *Spectral asymmetry and Riemannian geometry: III*, Math. Proc. Camb. Phil. Soc. **79** (1976), 71–99.

[5] S. K. Donaldson, *Floer Homology Groups in Yang–Mills Theory*, Cambridge University Press, 2002.

[6] Ronald Fintushel and Ronald J. Stern, *Instanton homology of Seifert–fibered homology 3–spheres*, Proc. Lond. Math. Soc. **61** (1990), 109–137.

[7] Ronald Fintushel and Ronald J. Stern, *Integer graded instanton homology groups for homology three-spheres*, Topology **31** (1992), 589–604.

[8] A. Floer, *An instanton invariant for 3-manifolds*, Comm. Math. Phys. **118** (1989), 215–240.

[9] C. McA. Gordon, *On the G-signature theorem in dimension four*, À la recherche de la topologie perdue, Progr. Math., vol. 62, Birkhäuser Boston, Boston, MA, 1986, pp. 159–180.

[10] P. B. Kronheimer and T. S. Mrowka, *Monopoles and Three-Manifolds*, Cambridge University Press, Cambridge, UK, 2008.

[11] J. Milnor, *Lectures on the h-cobordism theorem*, Mathematical Notes, vol. 1, Princeton University Press, Princeton, N.J., 1965.

[12] J. Milnor and J. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N.J., 1974.

[13] J. Morgan, T. Mrowka, and D. Ruberman, *The L^2-moduli space and a vanishing theorem for Donaldson invariants*, Monographs in Geometry and Topology, vol. 2, International Press, 1994.

[14] Nikolai Saveliev, *Lectures on the topology of 3-manifolds*, revised ed., de Gruyter Textbook, Walter de Gruyter & Co., Berlin, 2012, An introduction to the Casson invariant.

[15] Matthias Schwarz, *Morse homology*, Progress in Mathematics, vol. 111, Birkhäuser Verlag, Basel, 1993.