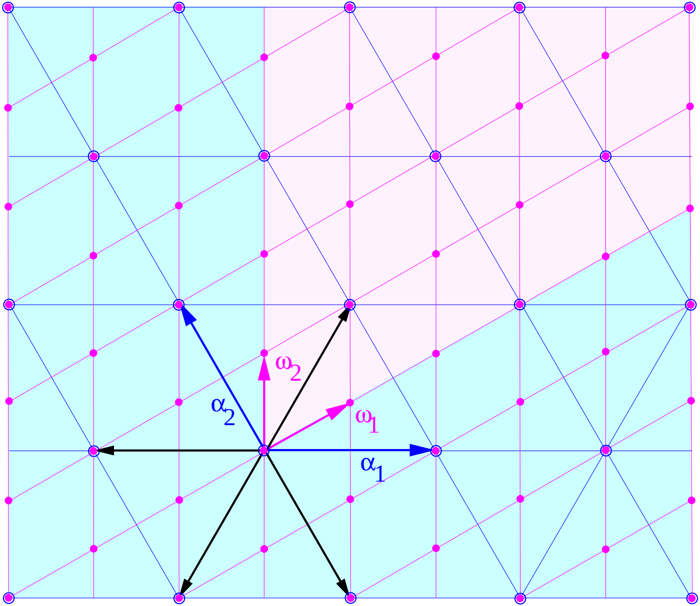
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Classical Anomalies

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Talk based on:

- MS, V. Dwivedi, *A Classical Version of the Non-Abelian Gauge Anomaly*, Phys. Rev. **D88** 045012 (2013).
- V. Dwivedi, MS, *Classical chiral kinetic theory and anomalies in even space-time dimensions*, J. Phys. **A47** 025401 (2103).

Outline

1. Classical Anomalies
   - Basic idea
2. Classical mechanics and Lie groups
   - Form language and Liouville's theorem
   - Lie groups as dynamical systems: Co-adjoint orbits
3. Wong equations
   - Liouville Measure
   - Classical and Quantum traces
   - Conclusions

Classical Anomalies

- Although the chiral anomaly is usually described as a quantum effect, there are classical mechanical versions.
- For massless fermions, use the incompressibility of phase-space flow to track flux through diabolical point.
- Works for both Abelian and non-Abelian theories in any dimension.

A diabolo

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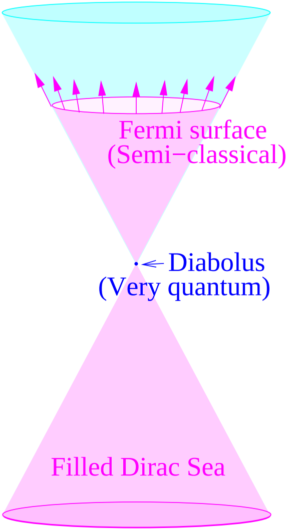

Stephanov & Yin

- Quantum Weyl Hamiltonian

$$\hat{H} = \hat{\boldsymbol{\sigma}} \cdot (\mathbf{p} - e\mathbf{A}) + e\phi$$

- Classical action with $\mathbf{k} = \mathbf{p} - e\mathbf{A}$

$$S[\mathbf{x}, \mathbf{k}] = \int dt \left\{ \mathbf{k} \cdot \dot{\mathbf{x}} - |\mathbf{k}| - e\phi(\mathbf{x}) + e\mathbf{A} \cdot \dot{\mathbf{x}} - \mathbf{a} \cdot \dot{\mathbf{k}} \right\}.$$

- Berry connection

$$\mathbf{b} = \nabla_{\mathbf{k}} \times \mathbf{a} = \frac{\hat{\mathbf{k}}}{2|\mathbf{k}|^2}, \quad \nabla \cdot \mathbf{b} = 2\pi\delta^3(\mathbf{k}).$$

Example

- Equation of motion

$$\begin{aligned}\dot{\mathbf{k}} &= \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}\\ \dot{\mathbf{x}} &= \hat{\mathbf{k}} + \dot{\mathbf{k}} \times \mathbf{b}\end{aligned}$$

- The $\dot{\mathbf{k}} \times \mathbf{b}$ term is the anomalous velocity (Karplus-Luttinger '54) .
- Liouville measure

$$\sqrt{\omega} = 1 + \mathbf{b} \cdot \mathbf{B} \quad \text{(Q. Niu } \textit{et al.} \text{ '95-).}$$

- Berry monopole $\nabla \cdot \mathbf{b} = 2\pi\delta^3(\mathbf{k})$ gives source to Liouville's theorem

$$\boxed{\frac{\partial\sqrt{\omega}}{\partial t} + \frac{\partial\sqrt{\omega}\dot{k}^i}{\partial k^i} + \frac{\partial\sqrt{\omega}\dot{x}^i}{\partial x^i} = 2\pi\delta^3(\mathbf{k})(\mathbf{E}\cdot\mathbf{B})}$$

Boltzmann equation ⇒ Anomaly

- Phase space density: $f(\mathbf{x}, \mathbf{k}, t)$
- Chiral current:

$$J^\mu(\mathbf{x}, t) = \int \dot{x}^\mu f(\mathbf{x}, \mathbf{k}, t)\sqrt{\omega}\frac{d^3k}{(2\pi)^3}$$

- plus Boltzmann

$$\left(\frac{\partial}{\partial t} + \dot{\mathbf{k}}\cdot\nabla_{\mathbf{k}} + \dot{\mathbf{x}}\cdot\nabla_{\mathbf{x}}\right) f = 0$$

- plus anomalous Liouville

$$\frac{\partial\sqrt{\omega}}{\partial t} + \frac{\partial\sqrt{\omega}\dot{k}^i}{\partial k^i} + \frac{\partial\sqrt{\omega}\dot{x}^i}{\partial x^i} = 2\pi\delta^3(\mathbf{k})(\mathbf{E}\cdot\mathbf{B}).$$

- ⇒ Anomaly

$$\boxed{\partial_\mu J^\mu = \frac{1}{(2\pi)^2}(\mathbf{E}\cdot\mathbf{B}) f(\mathbf{x}, \mathbf{0}, t)}$$

Our Goal:

Generalize to non-abelian groups and all dimensions

Strategy:

- Set up a classical dynamics so that

  Lie algebra commutator $\rightarrow$ Poisson bracket

- Generalize the 4d abelian example by using the group dynamics.
- See what we get!

Classical Mechanics

- Hamiltonian action for phase space $\boldsymbol{\xi} \equiv (\xi^1, \dots \xi^{2n})$.

$$S[\boldsymbol{\xi}] = \int \left\{ \sum_{i=1}^{2n} \eta_i(\boldsymbol{\xi}, t)\dot{\xi}^i - H(\boldsymbol{\xi}, t) \right\} dt.$$

- Equations of motion

$$\left(\frac{\partial \eta_j}{\partial \xi^i} - \frac{\partial \eta_i}{\partial \xi^j} \right) \dot{\xi}^j = \left(\frac{\partial H}{\partial \xi^i} + \frac{\partial \eta_i}{\partial t} \right).$$

- Symplectic matrix

$$\omega_{ij} = \frac{\partial \eta_j}{\partial \xi^i} - \frac{\partial \eta_i}{\partial \xi^j}.$$

Form Language

- Two-forms

$$\begin{aligned}\omega &= \frac{1}{2}\omega_{ij} d\xi^i d\xi^j \\ \omega_H &= \omega - \left(\frac{\partial \eta_i}{\partial t} + \frac{\partial H}{\partial \xi^i}\right) d\xi^i \, dt.\end{aligned}$$

- Vector field

$$\mathbf{v} = \frac{\partial}{\partial t} + \dot{\xi}^j \frac{\partial}{\partial \xi^j}.$$

- Equation of motion

$$i_{\mathbf{v}} \omega_H = 0.$$

Liouville's Theorem

- Define $2n+1$ form

$$\Omega = \frac{1}{n!}\omega_H^n dt = \frac{1}{n!}\omega^n dt$$

- Liouville states that the Lie derivative

$$\mathcal{L}_{\mathbf{v}}\Omega = 0.$$

- In co-ordinates, with $\sqrt{\omega} = \mathrm{Pf}(\omega_{ij})$, this is

$$\frac{\partial\sqrt{\omega}}{\partial t} + \frac{\partial\sqrt{\omega}\dot{\xi}^i}{\partial \xi^i} = 0.$$

Co-adjoint orbit dynamics

- Hamiltonian action

$$S[g] = \int dt\, \left(\mathrm{tr}\left\{ \alpha_\Lambda g^{-1} \frac{dg}{dt} \right\} - \mathcal{H}(g) \right)$$

where $g \in G$, $\mathcal{H}(g) = \mathrm{tr}\,\{\alpha_\Lambda g^{-1} X g\}$, and $\alpha_\Lambda, X \in \mathrm{Lie}(G)$

- Equation of motion

$$[\alpha, g^{-1}(\partial_t - X)g] = 0 \quad \Rightarrow \quad g^{-1}(\partial_t - X)g + h(t) = 0$$

where $[h(t), \alpha] = 0$.

- Solution

$$g(t) = \mathcal{T} \exp \left\{ \int_0^t X dt \right\} H(t) \in G/H$$

- The phase space is G/H— a co-adjoint orbit (Kirillov) determined by α_Λ, that is in turn determined by the representation Λ of interest.

Kostant-Kirillov Bracket

- For functions on G/H define a Poisson Bracket by

$$\{\mathcal{H}_1, \mathcal{H}_2\} \stackrel{\text{def}}{=} \left.\frac{d\mathcal{H}_2}{dt}\right|_{\mathcal{H}_1}.$$

.

- Let $\lambda_a \in \mathrm{Lie}(G)$ obey $[\lambda_a, \lambda_b] = i f_{ab}{}^c \lambda_c$.
- Define $Q_a(g) = \mathrm{tr}\{\alpha_\Lambda g^{-1} \lambda_a g\}$. It is a function on G/H.
- Find that

$$\{Q_a, Q_b\} = i f_{ab}{}^c Q_c.$$

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Dequantization!

Wong-Kirillov dynamics

For non-abelian groups and $2N$-dimensional space-time

$$S[\mathbf{x},\mathbf{k},g,\sigma] = \int\left(k^i\dot{x}^i - |k| + i\mathrm{tr}\left\{\alpha_\Lambda g^{-1}\left(\frac{d}{dt} - i(A_0 + \dot{x}^i A_i)\right) g\right\} -i\mathrm{tr}\left\{\beta_{\mathfrak{s}}\sigma^{-1}\left(\frac{d}{dt} - i\dot{k}^i\mathfrak{a}_i\right)\sigma\right\}\right) dt.$$

- $\{Q \equiv g\alpha_\Lambda g^{-1}; g \in G\}$ is co-adjoint orbit for gauge group G
- $\{\mathfrak{S} \equiv \sigma\beta_{\mathfrak{s}}\sigma^{-1} : \sigma \in \mathrm{Spin}(2N-2)\}$ is co-adjoint orbit for spin.

Equations of motion

- Classical Lie algebra $\hat{\lambda}_a \to Q_a$, and

$$\{Q_a, Q_b\} = i f_{ab}{}^c Q_c, \quad \text{(Poisson bracket)}$$

- Internal degrees of freedom

$$\begin{aligned} \dot{Q}^c &= f_{ab}{}^c Q^a (A_0^b + \dot{x}^i A_i^b), \\ \dot{\mathfrak{S}}^{(c)} &= f_{(a)(b)}{}^{(c)} \mathfrak{S}^{(a)} \mathfrak{a}_i^{(b)} \dot{k}^i. \end{aligned}$$

- Velocity and momentum

$$\begin{aligned} \dot{k}^i &= Q_a (F_{i0}^a + F_{ij}^a \dot{x}^j), \\ \dot{x}^i &= \hat{k}^i - \mathfrak{S}_{(a)} \mathfrak{F}_{ij}^{(a)} \dot{k}^j. \end{aligned}$$

Liouville Measure

The Liouville measure is

$$\frac{1}{M!}\omega_H^M = \frac{1}{(2N+1)!}(dp^i dx^i + \widetilde{F} - \widetilde{\mathfrak{F}})^{2N+1} d\mu_\Lambda d\mu_{\mathfrak{s}},$$

where

$$\begin{aligned} d\mu_\Lambda &= \frac{1}{m_\Lambda!}[i\mathrm{tr}\,\{\alpha_\Lambda(\omega_L^Q)^2\}]^{m_\Lambda} \\ d\mu_{\mathfrak{s}} &= \frac{1}{m_{\mathfrak{s}}!}[-i\mathrm{tr}\,\{\beta_{\mathfrak{s}}(\omega_L^{\mathfrak{S}})^2)\}]^{m_{\mathfrak{s}}}, \end{aligned}$$

are the Kirillov-Kostant measures on the co-adjoint orbits.

Liouville Measure

Measure contains a double Pfaffian

$$\begin{aligned} \sqrt{\omega} &= \sqrt{\det(1-\widetilde{F}\widetilde{\mathfrak{F}})} = \mathrm{Pf}(\widetilde{\mathfrak{F}}, \widetilde{F}) \\ &\stackrel{\text{def}}{=} \sum_{k=0}^{N}\sum_{I_{2k}} \mathrm{Pf}(\widetilde{\mathfrak{F}}_{I_{2k}})\mathrm{Pf}(\widetilde{F}_{I_{2k}}). \end{aligned}$$

with

$$\begin{aligned} d\left(\frac{1}{N!}\mathrm{tr}\left(\frac{\mathfrak{F}}{2\pi}\right)^{N}\right) &= (-1)^N \delta^{2N+1}(\mathbf{p})dp^1\cdots dp^{2N+1}, \\ d\left(\frac{1}{(N+1)!}\mathrm{tr}\left(\frac{F}{2\pi}\right)^{N+1}\right) &= 0 \end{aligned}$$

Quantum *versus* Classical Anomaly

In $d = 2N$ space-time

- Quantum

$$\mathrm{tr}\left\{\lambda_a \nabla_\mu J^\mu\right\} = \frac{1}{(4\pi)^N N!}\mathrm{str}_\Lambda\{\hat{\lambda}_a F_{\mu_1\mu_2}\cdots F_{\mu_{N-1}\mu_N}\}\epsilon^{\mu_1\ldots\mu_N}$$

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- Classical

$$\nabla_\mu J_a^\mu = \frac{1}{(4\pi)^N N!}\mathrm{ctr}_\Lambda\{Q_a \widetilde{F}_{\mu_1\mu_2}\cdots \widetilde{F}_{\mu_{N-1}\mu_N}\}\epsilon^{\mu_1\ldots\mu_N}$$

Classical and quantum traces

We want matrix traces, but instead have integrals over the co-adjoint phase spaces — *i.e.* *classical traces* with

$$\hat{\lambda}_a \to Q_a, \quad F_{\mu\nu} = \hat{\lambda}_a F^a_{\mu\nu} \to \tilde{F}_{\mu\nu} = Q_a F^a_{\mu\nu}$$

- Quantum matrix traces

$$\dim(\Lambda) \equiv \mathrm{tr}_{\Lambda}(\mathbb{I}), \quad \mathrm{tr}_{\Lambda}(\hat{\lambda}_a \hat{\lambda}_b), \quad \frac{1}{2}\mathrm{tr}_{\Lambda}(\hat{\lambda}_a \{\hat{\lambda}_b, \hat{\lambda}_c\}).$$

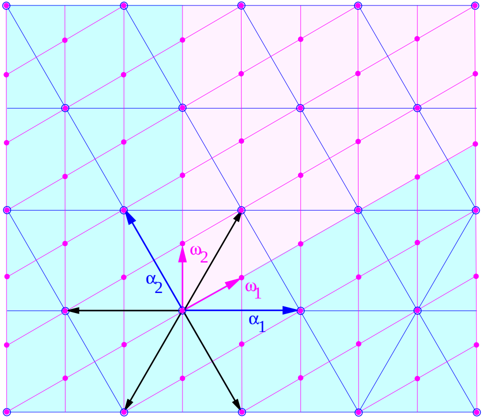
- Classical traces

$$\int_{\mathcal{O}_\Lambda} d\mu_\Lambda, \quad \int_{\mathcal{O}_\Lambda} Q_a Q_b \, d\mu_\Lambda \quad \int_{\mathcal{O}_\Lambda} Q_a Q_b Q_c \, d\mu_\Lambda$$

Classical *versus* Quantum traces

Example: $\mathrm{SU}(3)$, $\mathbf{\Lambda} = p\boldsymbol{\omega}_1 + q\boldsymbol{\omega}_2$

- Quantum
- $\dim(p,q) = \mathrm{tr}\,\mathbb{I} = (p+1)(q+1)(p+q+2)/2.$
- $\hat{C}_2 = \hat{\lambda}_a\hat{\lambda}_a = \frac{2}{3}(p^2+pq+q^2+3p+3q)\mathbb{I}$
- $\hat{C}_3 = d_{abc}\hat{\lambda}_a\hat{\lambda}_b\hat{\lambda}_c = \frac{2}{9}(p-q)(2p+q+3)(2q+p+3)\mathbb{I}$



$\mathrm{SU}(3)$ weights and Weyl chamber

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- $\dim(p,q) \stackrel{?}{=} \mathrm{Vol}\,\mathcal{O}_\Lambda = pq(p+q)$
- $\hat{C}_2^{\mathrm{geom}} = Q_a Q_a = \frac{2}{3}(p^2 + pq + q^2)$
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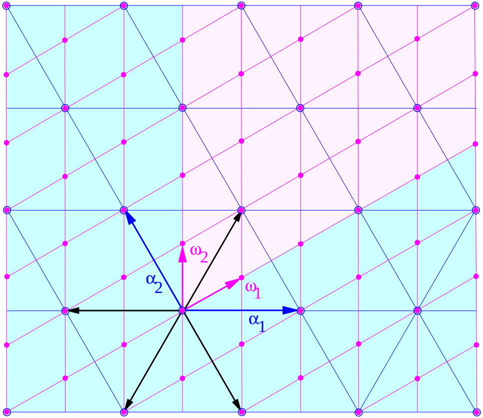

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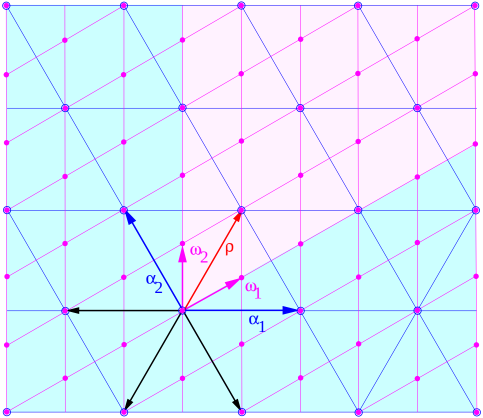

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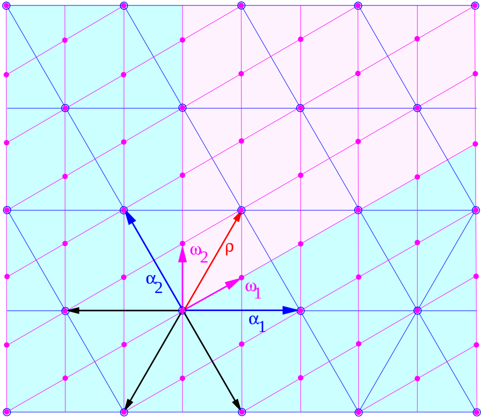

$\mathrm{SU}(3)$ weights and Weyl chamber

Get correct result with Weyl shift $\boldsymbol{\Lambda} \to \boldsymbol{\Lambda} + \boldsymbol{\rho}$, *i.e.* $p \to p+1$ $q \to q+1$.

Conclusions and future work

- The classical derivation brings out the rôle of the Berry phase in accounting for the gyroscopic effect of spin.
- To get the correct coefficients for “small” representations of the gauge or spin group, we need to Weyl shift the weight.
- Currently working on the gravity ($\hat{A}$-genus) contribution.
- Is there a similar route to the gravitational (energy-momentum) anomaly?

SO(6) roots, weights and Weyl Chamber