NOTES ON KURANISHI ATLASES

DUSA MCDUFF

DRAFT – comments welcome

Contents

1. Introduction

These notes aim to explain a joint project with Katrin Wehrheim that uses finite dimensional reductions to construct a virtual fundamental class (VFC) for the Gromov–Witten moduli space of closed genus zero curves. Our method is based on work by

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Fukaya–Ono [FO] and Fukaya–Oh–Ohta–Ono [FOOO]; see also [FOOO12]. However we have reformulated their ideas in order to clarify the formal structures underlying the construction and make explicit all important choices (of tamings, shrinkings and reductions), thus creating tools with which to give an explicit proof that the virtual class $[X]^{vir}_{\mathcal{K}}$ is independent of these choices.

Our ultimate aim is to prove the following theorems.

Theorem A. *Let (M^{2n},ω,J) be a $2n$-dimensional symplectic manifold with tame almost complex structure J, let $\overline{\mathcal{M}}_{0,k}(A,J)$ be the compact space of nodal J-holomorphic genus zero stable maps in class A with k marked points modulo reparametrization, and let $d = 2n + 2c_1(A) + 2k - 6$. Then $X := \overline{\mathcal{M}}_{0,k}(A,J)$ has an oriented, d-dimensional, weak, effective SS Kuranishi atlas $\mathcal{K}$ that is well defined modulo oriented cobordism.*

Theorem B. *Let $\mathcal{K}$ be an oriented, d-dimensional, weak, effective SS Kuranishi atlas on a compact metrizable space X. Then $\mathcal{K}$ determines a cobordism class of oriented, compact weighted branched topological manifolds, and an element $[X]^{vir}_{\mathcal{K}}$ in the Čech homology group $\check{H}_d(X;\mathbb{Q})$. Both depend only on the oriented cobordism class of $\mathcal{K}$.*

If the curves in X have no isotropy and smooth (i.e. non nodal) domains, we construct the invariant as an oriented cobordism class of compact smooth manifolds, and then take an appropriate inverse limit to get the Čech homology class. If there is isotropy we use the theory of weighted branched (smooth) manifolds in [M07] that are modelled by weighted nonsingular branched (wnb) groupoids. In the general case we analyse what happens when nodes are resolved by means of a gluing theorem. One aim of our project is to prove Theorem A using the approach to gluing in [MS]. This provides continuity of the gluing map as the gluing parameters $\mathbf{a}$ converge to zero, but gives no control over derivatives with respect to these parameters $\mathbf{a}$. With this approach, the charts are only weakly stratified smooth (abbreviated SS), i.e. they are topological spaces that are unions of even dimensional, smooth strata. As we explain in §3.3, this introduces various complications into the arguments, and specially into the construction of perturbation sections for Kuranishi atlases of dimension > 1.[1] On the positive side it means that there is no need to change the usual smooth structure of Deligne-Mumford space or of the moduli spaces X of J-holomorphic curves by choosing a gluing profile, which is the approach both of Fukaya et al and Hofer–Wysocki–Zehnder. This part of the project is not yet complete. Hence in these notes we will either restrict to the case $d = 0$ or will assume the existence of a gluing theorem that provides at least $\mathcal{C}^1$ control.

We begin by developing the abstract theory of Kuranishi atlases, that is on proving theorem B for smooth atlases. The first two sections of these notes give precise statements of the main definitions and results from [MW12, MW14], and sketches of the most important proofs. For simplicity we first discuss the smooth case with trivial isotropy and then the case of nontrivial isotropy. We end Section 3 with some notes

[1] In the Gromov–Witten case all lower strata have codimension ≥ 2, which means that in most situations one can avoid these complications by cutting down dimensions via intersections with appropriate cycles; cf. §5.3.

on the nodal case. We do not explain the full theory here, restricting consideration to so-called weakly SS maps since they are much easier to understand. Thus our proof of Theorem B applies to the smooth case in all dimensions and to the weakly SS case in dimension $d = 0$.

The rest of these notes are more informal, explaining how the theory can be used in practice. In Section 4 we discuss some modifications of the basic definitions that are useful when considering products. The point here is that the product of two Kuranishi atlases is not an atlas in the sense of our original definition. However, the theory can deal with products if one weakens the so-called additivity requirements. Section 5 outlines the proof of Theorem A, explaining the set up in detail but omitting most analytic details. Many of these can be found in [MW12, MW14], though gluing will be treated in [MWss]; see also [C] that will complete the construction of a $\mathcal{C}^1$-atlas. We restrict to genus zero here since in this case the relevant Deligne-Mumford space $\overline{\mathcal{M}}_{0,k}$ can be understood simply in terms of cross ratios, which makes the equation easier to understand explicitly. However, the argument should easily adapt to the higher genus case.

Finally we discuss some examples. The following result is proved in §6.1. As explained there, we think of an orbifold as the realization of an ep groupoid.

Proposition C: *Each compact orbifold has a Kuranishi atlas with trivial obstruction spaces. Moreover, there is a bijective correspondence between commensurability classes of such Kuranishi atlases and Morita equivalence classes of ep groupoids.*

We show in §6.2 that Kuranishi atlases give the expected results in situations when X has specially nice form. For example, if the space X of equivalence classes of stable maps is a compact orbifold with obstruction bundle E then the invariant is simply the Euler class of E. Finally in §6.3 we use Kuranishi atlases to prove a result claimed in [M00] about the vanishing of certain two point GW invariants of the product manifold $S^2 \times M$.

1.1. **Outline of the main ideas.** The space X whose fundamental class we want to understand is given as the solution set of a Fredholm operator (such as the Cauchy–Riemann operator) on the space of sections of a bundle over a nodal Riemann surface. In the Gromov–Witten setting X can fail to be an orbifold for two reasons: the zero set of the operator is not in general cut out transversally and the topological type of the Riemann surface may change. Because the operator is Fredholm and the changes in the Riemann surface can be understood via gluing, there is a good notion of a finite dimensional reduction, which allows us to build a **basic chart $\mathbf{K}$** that models some open subset set $F \subset X$, called its footprint. A **Kuranishi atlas** $\mathcal{K}$ is made from a finite covering family of these charts. Since typically there is no direct map from one basic chart to another we relate them via **sum charts and coordinate changes**. The needed abstract structure is explained in §2.1, as is the relation between Kuranishi atlases and the Kuranishi structures of [FO, FOOO12].

Our first aim is to unite all these charts into an **étale category $\mathbf{B}_{\mathcal{K}}$**, akin to the étale proper groupoids often used to model orbifolds. If we ignore questions of smoothness and suppose that all isotropy groups are trivial, the set of objects $\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$ of such a topological category is the disjoint union $\bigsqcup_I U_I$ of smooth manifolds of different dimensions. There are at most a finite number of morphisms between any two points. Therefore the space $|\mathcal{K}|$ obtained by quotienting $\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$ by the equivalence relation generated by the morphisms looks something like a manifold. In fact, in good cases this space, called the **virtual neighbourhood of** X, is a finite union of (non disjoint) manifolds; cf. Remark 2.2.7. It supports a "bundle" $\mathrm{pr} : |\mathbf{E}_{\mathcal{K}}| \to |\mathcal{K}|$ with canonical section $s : |\mathcal{K}| \to |\mathbf{E}_{\mathcal{K}}|$. The latter is the finite dimensional remnant of the original Fredholm operator, and its zero set can be canonically identified with a copy $\iota_{\mathcal{K}}(X)$ of X. Hence the idea is that the virtual moduli cycle $[X]^{vir}_{\mathcal{V}}$ should be represented by the zero set of a perturbed section $s + \nu$ that is chosen to be transverse to zero.

We now outline the main steps in this construction.

- The first difficulty in realizing this idea is that in practice one cannot actually construct atlases; instead one constructs a weak atlas, which is like an atlas except that one has less control of the domains of the charts and coordinate changes. But a weak atlas does not even define a a category, let alone one whose realization $|\mathbf{B}_{\mathcal{K}}| =: |\mathcal{K}|$ has good topological properties. For example, we would like $|\mathcal{K}|$ to be Hausdorff and (in order to make local constructions possible) for the projection $\pi_{\mathcal{K}} : U_I \to |\mathcal{K}|$ to be a homeomorphism to its image. In §2.2 we formulate the **taming conditions** for an atlas, and show in Proposition 2.2.6 that tame atlases do have well behaved realizations. Then in Proposition 2.3.4 we sketch the construction of a tame atlas starting from a weak atlas. As explained in Remark 2.3.6, the notion of additivity is crucial here. (Cf. §4 where this notion of additivity is weakened to a notion that is compatible with products.) Theorem 2.3.1 summarizes the main topological facts about $\mathcal{K}$ that are needed for subsequent constructions.
- The taming procedure gives us two categories $\mathbf{B}_{\mathcal{K}}$ and $\mathbf{E}_{\mathcal{K}}$ with a projection functor $\mathrm{pr} : \mathbf{E}_{\mathcal{K}} \to \mathbf{B}_{\mathcal{K}}$ and section functor $s : \mathbf{B}_{\mathcal{K}} \to \mathbf{E}_{\mathcal{K}}$. However the category has too many morphisms (i.e. the chart domains overlap too much) for us to be able to construct a perturbation functor $\nu : \mathbf{B}_{\mathcal{K}} \to \mathbf{E}_{\mathcal{K}}$ such that $s + \nu \pitchfork 0$. We therefore pass to a full subcategory $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}$ of $\mathbf{B}_{\mathcal{K}}$ with objects $\mathcal{V} := \bigsqcup V_I$ that does support suitable functors $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$. This subcategory $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}$ is called a **reduction** of $\mathcal{K}$; cf. Definition 2.4.2. Constructing it is akin to passing from the covering of a triangulated space by the stars of its vertices to the covering by the stars of its first barycentric subdivision. In §2.4 we say rather little about how to carry out such construction since we discuss a more general result in §4.1; cf. Lemma 4.1.12.
- We next define the notion of a reduced section of $\mathcal{K}$ (cf. Definition 2.4.6), and show that, if ν is precompact in a suitable sense, the zero set $(s|_{\mathcal{V}} + \nu)^{-1}(0)$ is compact. Proposition 2.4.10 sketches the construction of the section ν in considerable detail, though still does not do quite enough for a complete proof. In the trivial isotropy case the zero set is a closed submanifold of $|\mathcal{K}|$ lying in the precompact "neighbourhood"[2]

[2] In fact, $\iota_{\mathcal{K}}(X)$ does *not* have a compact neighbourhood in $|\mathcal{K}|$; as explained in Remark 2.4.5 we should think of $|\mathcal{V}|$ as the closest we can come to a compact neighbourhood of $\iota_{\mathcal{K}}(X)$.

$|\mathcal{V}| = \bigcup_I \pi_{\mathcal{K}}(V_I) \cup |\mathcal{K}|$ of $\iota_{\mathcal{K}}(X)$. The final step is to construct the fundamental class $[X]^{vir}_{\mathcal{K}}$ from this zero set. This class lies in rational Čech homology because this is a homology theory with the needed continuity properties under inverse limits.

- As we will see in §3 the above ideas adapt readily to the case of nontrivial isotropy via the notion of the **intermediate category**. Further, we can use the action of the isotropy groups Γ_I of the charts to generate the different branches of perturbation section ν, which now must be multivalued. Therefore we get a very precise description of the zero set $(s|_{\mathcal{V}} + \Gamma\nu)^{-1}(0)$; cf. equation 3.2.6.

Of course, to obtain a fundamental class one also needs to discuss orientations, and in order to prove uniqueness of this class one also needs to set up an adequate cobordism theory. For these the reader should consult the original papers since we only mention these aspects of the argument in passing.

Remark 1.1.1. (i) Note that although the cobordism relation is all one needs when proving the uniqueness of the VFC, it does not seem to be the "correct" relation, in the sense that rather different moduli problems might well give rise to cobordant atlases; cf. [MW12, Remark 4.3.2(iv)]. There is a (possibly) stricter equivalence relation for atlases on a fixed space X that is called **commensurability**; cf. Definition 5.1.5. This comes closer to characterizing the essential features of a Gromov–Witten moduli space X. The construction in §5.1 for GW moduli spaces X builds an atlas whose commensurability class is independent of all choices. However, the method involves the use of some geometric procedures (formalized in Definition 5.2.1 as the notion of a GW atlas) that have no abstract description. Therefore this is probably not the correct relation either. It may be that Joyce's notion of a d-manifold [J12] best captures the Fredholm index condition on X; see also Yang [Y14]. The aim of our work is not to tackle such an abstract problem, but to develop a complete and explicit theory that can be used in practice to calculate GW invariants.
(ii) Pardon's very interesting approach to the construction of the GW virtual fundamental class uses atlases that have many of the features of the theory presented here. In particular, his notion of implicit atlas includes sum charts and coordinate changes that are essentially the same as ours. However he avoids making choices by considering *all* charts, and he avoids the taming problems we encounter firstly by considering *all* solutions to the given equation and secondly by using a different more topological way to define the VFC (via a version of sheaf theory) that does not involved considering the quotient space $|\mathcal{K}|$.

The lectures [M14] give an overview of the whole construction.

2. The smooth case with trivial isotropy

Throughout this section, X is assumed to be a compact and metrizable space. We assume (usually without explicit mention) that the isotropy is trivial. The proof of Theorem B in this case is completed at the end of §2.4. For the general case see §3.

2.1. **Kuranishi charts and coordinate changes.** In this section we give basic definitions, and make some comparisons with the notion of Kuranishi structure in [FOOO].

Definition 2.1.1. *Let $F \subset X$ be a nonempty open subset. A* **Kuranishi chart** *for X with* **footprint** *F (and trivial isotropy) is a tuple $\mathbf{K} = (U, E, s, \psi)$ consisting of*

- *the* **domain** *U, which is an open smooth k-dimensional manifold;*
- *the* **obstruction space** *E, which is a finite dimensional real vector space;*
- *the* **section** *$U \to U \times E, x \mapsto (x, s(x))$ which is given by a smooth map $s : U \to E$;*
- *the* **footprint map** *$\psi : s^{-1}(0) \to X$, which is a homeomorphism to the footprint $\psi(s^{-1}(0)) = F$, which is an open subset of X.*

The **dimension** *of* $\mathbf{K}$ *is* $\dim \mathbf{K} := \dim U - \dim E$.

Definition 2.1.2. *A* **map** *$\widehat{\Phi} : \mathbf{K} \to \mathbf{K}'$ between Kuranishi charts is a pair $(\phi, \widehat{\phi})$ consisting of an embedding $\phi : U \to U'$ and a linear injection $\widehat{\phi} : E \to E'$ such that*

(i) *the embedding restricts to $\phi|_{s^{-1}(0)} = \psi'^{-1} \circ \psi : s^{-1}(0) \to s'^{-1}(0)$, the transition map induced from the footprints in X;*

(ii) *the embedding intertwines the sections, $s' \circ \phi = \widehat{\phi} \circ s$, on the entire domain U.*

That is, the following diagrams commute:

$$
\begin{array}{ccc} U \times E & \xrightarrow{\phi\times\widehat{\phi}} & U' \times E' \\ \uparrow s & & \uparrow s' \\ U & \xrightarrow{\phi} & U' \end{array}
\qquad\qquad
\begin{array}{ccc} s^{-1}(0) & \xrightarrow{\phi} & s'^{-1}(0) \\ \downarrow \psi & & \downarrow \psi' \\ X & \xrightarrow{\mathrm{Id}} & X. \end{array}
\tag{2.1.1}
$$

The dimension of the obstruction space E typically varies as the footprint $F \subset X$ changes. Indeed, the maps $\phi, \widehat{\phi}$ need not be surjective. However, as we will see in Definition 2.1.5, the maps allowed as coordinate changes are carefully controlled in the normal direction. Since we only defined maps of Kuranishi charts that induce an inclusion of footprints, we now need to define a notion of restriction of a Kuranishi chart to a smaller subset of its footprint.

Definition 2.1.3. *Let $\mathbf{K}$ be a Kuranishi chart and $F' \subset F$ an open subset of the footprint. A* **restriction of $\mathbf{K}$ to F'** *is a Kuranishi chart of the form*

$$\mathbf{K}' = \mathbf{K}|_{U'} := \big(U' \,,\, E' = E \,,\, s' = s|_{U'} \,,\, \psi' = \psi|_{s'^{-1}(0)} \big)$$

given by a choice of open subset $U' \subset U$ of the domain such that $U' \cap s^{-1}(0) = \psi^{-1}(F')$. In particular, $\mathbf{K}'$ has footprint $\psi'(s'^{-1}(0)) = F'$.

By [MW12, Lemma 5.1.4], we may restrict to any open subset of the footprint. If moreover $F' \sqsubset F$ is precompact, then U' can be chosen to be precompact in U, written $U' \sqsubset U$.

The next step is to construct a coordinate change $\widehat{\Phi}_{IJ} : \mathbf{K}_I \to \mathbf{K}_J$ between two charts with nested footprints $F_I \supset F_J$. For simplicity we will formulate the definition in the situation that is relevant to Kuranishi atlases. That is, we suppose that a finite set of Kuranishi charts $(\mathbf{K}_i)_{i\in\{1,\dots,N\}}$ is given such that for each $I \subset \{1, \dots, N\}$ with

$F_I := \bigcap_{i\in I} F_i \neq \emptyset$ we have another Kuranishi chart $\mathbf{K}_I$ (informally called a **sum chart**) with

$$E_I = \prod_{i\in I} E_i, \quad \text{and} \quad \text{footprint } F_I := \bigcap_{i\in I} F_i. \tag{2.1.2}$$

(2.1.2) obstruction space $E_I = \prod_{i\in I} E_i$, and footprint $F_I := \bigcap_{i\in I} F_i$.

Remark 2.1.4. Since we assume in an atlas that

$$\dim U_I - \dim E_I =: \dim \mathbf{K}_I = \dim \mathbf{K}_i = \dim U_i - \dim E_i, \quad \forall i \in I,$$

in general the domain of the sum chart U_I has dimension strictly larger than $\dim U_i$ for $i \in I$. Further, U_I usually cannot be built in some topological way from the U_i (e.g. by taking products). Indeed in the Gromov–Witten situation U_I consists (very roughly speaking) of the solutions to an equation of the form $\overline{\partial}_J u = \sum_{i\in I} \lambda(e_i)$, and so cannot be made directly from the U_i, which are solutions to the individual equations $\big(\overline{\partial}_J u = \lambda(e_i)\big)_{i\in I}$. Note also that we choose the obstruction spaces E_i to cover the cokernel of the linearization of $\overline{\partial}_J$ at the points in U_i. Thus each domain U_i is a manifold that is cut out transversally by the equation. Since the function $s_I : U_I \to E_I$ is the finite dimensional reduction of $\overline{\partial}_J$, its derivative $\mathrm{d}_x s_I$ at a point $x \in \operatorname{im} \phi_{IJ}$ has kernel contained in $\mathrm{T}_x(\operatorname{im} \phi_{IJ})$ and cokernel that is covered by $\widehat{\phi}_{IJ}(E_I)$. This explains the index condition in Definition 2.1.5 below. See §5.1(VI) for more details.

When $I \subset J$ we write $\widehat{\phi} := \widehat{\phi}_{IJ} : E_I \to E_J$ for the natural inclusion, omitting it where no confusion is possible.[3]

Definition 2.1.5. *For $I \subset J$, let $\mathbf{K}_I$ and $\mathbf{K}_J$ be Kuranishi charts as above, with domains U_I, U_J and footprints $F_I \supset F_J$. A* **coordinate change** *from $\mathbf{K}_I$ to $\mathbf{K}_J$ with* **domain** *U_{IJ} is a map $\widehat{\Phi} : \mathbf{K}_I|_{U_{IJ}} \to \mathbf{K}_J$, which satisfies the* **index condition** *in (i),(ii) below, and whose domain is an open subset $U_{IJ} \subset U_I$ such that $\psi_I(s_I^{-1}(0) \cap U_{IJ}) = F_J$.*

(i) *The embedding $\phi : U_{IJ} \to U_J$ underlying the map $\widehat{\Phi}$ identifies the kernels,*

$$\mathrm{d}_u\phi\big(\ker \mathrm{d}_u s_I\big) = \ker \mathrm{d}_{\phi(u)} s_J \qquad \forall u \in U_{IJ};$$

(ii) *the linear embedding $\widehat{\phi} : E_I \to E_J$ given by the map $\widehat{\Phi}$ identifies the cokernels,*

$$\forall u \in U_{IJ} : \qquad E_I = \operatorname{im} \mathrm{d}_u s_I \oplus C_{u,I} \quad \Longrightarrow \quad E_J = \operatorname{im} \mathrm{d}_{\phi(u)} s_J \oplus \widehat{\phi}(C_{u,I}).$$

Remark 2.1.6. By [MW12, Lemma 5.2.2] the index condition is equivalent to the **tangent bundle condition**, which requires isomorphisms for all $v = \phi(u) \in \phi(U_{IJ})$,

$$\mathrm{d}_v s_J : \; {}^{\mathrm{T}_v U_J}\big/_{\mathrm{d}_u\phi(\mathrm{T}_u U_I)} \xrightarrow{\cong} {}^{E_J}\big/_{\widehat{\phi}(E_I)}, \tag{2.1.3}$$

or equivalently at all (suppressed) base points as above

$$E_J = \operatorname{im} \mathrm{d}s_J + \operatorname{im} \widehat{\phi}_{IJ} \qquad \text{and} \qquad \operatorname{im} \mathrm{d}s_J \cap \operatorname{im} \widehat{\phi}_{IJ} = \widehat{\phi}_{IJ}(\operatorname{im} \mathrm{d}s_I). \tag{2.1.4}$$

Moreover, the index condition implies that $\phi(U_{IJ})$ is an open subset of $s_J^{-1}(\widehat{\phi}(E_I))$, and that the charts $\mathbf{K}_I, \mathbf{K}_J$ have the same dimension.

[3] Note that the assumption $E_I = \prod_{i\in I} E_i$ means that the family is **additive** in the sense of [MW12, Definition 6.2.2]. Therefore all the atlases that we now consider are additive, and for simplicity we no longer mention this condition explicitly. We discuss a weakened version in §4.

Definition 2.1.7. *Let X be a compact metrizable space.*

- *A* **covering family of basic charts** *for X is a finite collection $(\mathbf{K}_i)_{i=1,\dots,N}$ of Kuranishi charts for X whose footprints cover $X = \bigcup_{i=1}^N F_i$.*
- **Transition data** *for a covering family $(\mathbf{K}_i)_{i=1,\dots,N}$ is a collection of Kuranishi charts $(\mathbf{K}_J)_{J\in\mathcal{I}_\mathcal{K},|J|\ge 2}$ and coordinate changes $(\widehat{\Phi}_{IJ})_{I,J\in\mathcal{I}_\mathcal{K},I\subsetneq J}$ as follows:*
  - (i) *$\mathcal{I}_\mathcal{K}$ denotes the set of subsets $I \subset \{1,\dots,N\}$ for which the intersection of footprints is nonempty,*
  $$F_I := \textstyle\bigcap_{i\in I} F_i \ \neq\ \emptyset\ ;$$
  - (ii) *$\mathbf{K}_J$ is a Kuranishi chart for X with footprint $F_J = \bigcap_{i\in J} F_i$ for each $J \in \mathcal{I}_\mathcal{K}$ with $|J| \ge 2$, and for one element sets $J = \{i\}$ we denote $\mathbf{K}_{\{i\}} := \mathbf{K}_i$;*
  - (iii) *$\widehat{\Phi}_{IJ}$ is a coordinate change $\mathbf{K}_I \to \mathbf{K}_J$ for every $I, J \in \mathcal{I}_\mathcal{K}$ with $I \subsetneq J$.*

The transition data for a covering family automatically satisfies a cocycle condition on the zero sets since, due to the footprint maps to X, we have for $I \subset J \subset K$:

$$\phi_{JK}\circ\phi_{IJ} = \psi_K^{-1}\circ\psi_J\circ\psi_J^{-1}\circ\psi_I = \psi_K^{-1}\circ\psi_I = \phi_{IK} \qquad \text{on } s_I^{-1}(0)\cap U_{IK}.$$

Further, the composite maps $\phi_{JK}\circ\phi_{IJ}, \widehat{\phi}_{JK}\circ\widehat{\phi}_{IJ} = \widehat{\phi}_{IK}$ automatically satisfy the intertwining relations in Definition 2.1.2. Hence one can always define a composite coordinate change $\widehat{\Phi}_{JK}\circ\widehat{\Phi}_{IJ}$ from $\mathbf{K}_I$ to $\mathbf{K}_K$ with domain $U_{IJ}\cap\phi_{IJ}^{-1}(U_{JK})$. But in general this domain may have little relation to the domain U_{IK} of ϕ_{IK}, apart from the fact that these two sets have the same intersection with the zero set $s_I^{-1}(0)$. Since there is no natural ambient topological space into which the entire domains of the Kuranishi charts map, the cocycle condition on the complement of the zero sets has to be added as axiom. There are three natural notions of cocycle condition with varying requirements on the domains of the coordinate changes.

Definition 2.1.8. *Let $\mathcal{K} = (\mathbf{K}_I, \widehat{\Phi}_{IJ})_{I,J\in\mathcal{I}_\mathcal{K},I\subsetneq J}$ be a tuple of basic charts and transition data. Then for any $I,J,K \in \mathcal{I}_K$ with $I \subsetneq J \subsetneq K$ we define the composed coordinate change $\widehat{\Phi}_{JK}\circ\widehat{\Phi}_{IJ} : \mathbf{K}_I \to \mathbf{K}_K$ as above with domain $\phi_{IJ}^{-1}(U_{JK}) \subset U_I$. We say that the triple of coordinate changes $\widehat{\Phi}_{IJ}, \widehat{\Phi}_{JK}, \widehat{\Phi}_{IK}$ satisfies the*

- **weak cocycle condition** *if $\widehat{\Phi}_{JK}\circ\widehat{\Phi}_{IJ} \approx \widehat{\Phi}_{IK}$, i.e. the coordinate changes are equal on the overlap; in particular if*
$$\phi_{JK}\circ\phi_{IJ} = \phi_{IK} \qquad \textit{on}\ \ \phi_{IJ}^{-1}(U_{JK})\cap U_{IK};$$
- **cocycle condition** *if $\widehat{\Phi}_{JK}\circ\widehat{\Phi}_{IJ} \subset \widehat{\Phi}_{IK}$, i.e. $\widehat{\Phi}_{IK}$ extends the composed coordinate change; in particular if*
$$\phi_{JK}\circ\phi_{IJ} = \phi_{IK} \qquad \textit{on}\ \ \phi_{IJ}^{-1}(U_{JK}) \subset U_{IK}; \tag{2.1.5}$$
- **strong cocycle condition** *if $\widehat{\Phi}_{JK}\circ\widehat{\Phi}_{IJ} = \widehat{\Phi}_{IK}$ are equal as coordinate changes; in particular if*
$$\phi_{JK}\circ\phi_{IJ} = \phi_{IK} \qquad \textit{on}\ \ \phi_{IJ}^{-1}(U_{JK}) = U_{IK}. \tag{2.1.6}$$

The relevance of the these versions is that the weak cocycle condition can be achieved in practice by constructions of finite dimensional reductions for holomorphic curve moduli spaces, whereas the strong cocycle condition is needed for our construction of a virtual moduli cycle from perturbations of the sections in the Kuranishi charts. The cocycle condition is an intermediate notion which is too strong to be constructed in practice and too weak to induce a VMC, but it does allow us to formulate Kuranishi atlases categorically. This in turn gives rise, via a topological realization of a category, to a virtual neighbourhood of X into which all Kuranishi domains map.

Definition 2.1.9. *A* **weak Kuranishi atlas of dimension d** *on a compact metrizable space X is a tuple*

$$\mathcal{K} = \big(\mathbf{K}_I, \widehat{\Phi}_{IJ}\big)_{I,J\in\mathcal{I}_\mathcal{K}, I\subsetneq J}$$

consisting of a covering family of basic charts $(\mathbf{K}_i)_{i=1,\dots,N}$ of dimension d and transition data $(\mathbf{K}_J)_{|J|\ge 2}$, $(\widehat{\Phi}_{IJ})_{I\subsetneq J}$ for $(\mathbf{K}_i)$ as in Definition 2.1.7, that satisfy the weak cocycle condition $\widehat{\Phi}_{JK}\circ\widehat{\Phi}_{IJ}\approx\widehat{\Phi}_{IK}$ for every triple $I,J,K\in\mathcal{I}_K$ with $I\subsetneq J\subsetneq K$. A weak Kuranishi atlas $\mathcal{K}$ is called a **Kuranishi atlas** *if it satisfies the cocycle condition of* (2.1.5).

Remark 2.1.10. (i) Very similar definitions apply if the isotropy groups are nontrivial, or if X is stratified (for example, it consists of nodal J-holomorphic curves). In the former case we must modify the coordinate changes (cf Definition 3.1.10), while in the latter case the domains of the charts are stratified smooth (SS) spaces, which means that we must develop an adequate theory of SS maps.
(ii) The basic definitions above are also rather close to those in [FOOO12]. In fact, in our view, the notion of a weak Kuranishi atlas simply makes explicit the assumptions of their construction of a Kuranishi structure. However that may be, it is very easy to obtain a Kuranishi structure from a Kuranishi atlas by restriction. Recall that to define a Kuranishi structure one needs to specify a family of Kuranishi charts $(\mathbf{K}_p)_{p\in X}$ with footprints $F_p\ni p$, together with coordinate changes $(\phi_{qp}:\mathbf{K}_q\to\mathbf{K}_p)_{q\in F_p}$ that satisfy the weak cocycle condition. Even though there could be uncountably many charts $\mathbf{K}_p$, Fukaya et al. construct them from a finite covering family in much the same way that we now describe. In fact, when the isotropy groups are trivial this is precisely what they do; cf. Remark 3.2.12 for a comment on the case with nontrivial isotropy.

- First choose a precompact "shrinking" $\{G_i\sqsubset F_i\}_{i=1,\dots,N}$ of the footprints. Set $G_I:=\bigcap_{i\in I}G_i$, and for $p\in X$, define $I_p:=\{i\,|\,p\in G_i\}$.
- For $p\in X$ define $\mathbf{K}_p$ by choosing a restriction of sum chart $\mathbf{K}_{I_p}$ to

$$F_p := \big(\cap_{i\in I_p}F_i\big)\smallsetminus\big(\cup_{j\notin I_p}G_j\big).$$

  (Note that $p\in F_p$.)
- For $q\in F_p$ define the coordinate change $\phi_{pq}:\mathbf{K}_q\to\mathbf{K}_p$ to be a suitable restriction of $\widehat{\Phi}_{I_qI_p}$. Then the compatibility $\phi_{pq}\circ\phi_{qr}=\phi_{pr}$ follows from the weak cocycle condition for $\mathcal{K}$, which can be checked by a finite process.

This process of passing to small charts loses information that turns out to be crucial in the SS (nodal) case. It also seems a little inefficient, in that one needs to rebuild larger charts in order to get a "good coordinate system". Although it might be possible to simplify our constructions by doing taming and reduction simultaneously (which is one way of formulating what is done in [FOOO12]) it is actually very useful to have the intermediate object of a Kuranishi atlas since this captures all the needed information about the coordinate changes in the simplest possible form. This atlas corresponds to a category $\mathbf{B}_{\mathcal{K}}$, which is a nice kind of object to deal with (e.g. we can explain the needed compatibility conditions in the language of functors). After reduction, we get an object that is best thought of as a subcategory $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}$ of $\mathbf{B}_{\mathcal{K}}$ rather than as a category (or atlas) in its own right since in the former set up we do not need to add lots of extra morphisms. However, there is a corresponding (nonadditive) atlas $\mathcal{K}^{\mathcal{V}}$ which is defined in [MW12, Proposition 7.1.15].)

2.2. **The Kuranishi category and virtual neighbourhood $|\mathcal{K}|$.** After defining the Kuranishi category $\mathbf{B}_{\mathcal{K}}$ of a Kuranishi atlas $\mathcal{K}$ and the associated realization $|\mathcal{K}|$, we show in Proposition 2.2.6 that when $\mathcal{K}$ is tame its realization $|\mathcal{K}|$ has good topological properties, for example, it is Hausdorff.

It is useful to think of the domains and obstruction spaces of a Kuranishi atlas as forming the following categories.

Definition 2.2.1. *Given a Kuranishi atlas $\mathcal{K}$ we define its* **domain category $\mathbf{B}_{\mathcal{K}}$** *to consist of the space of objects*[4]

$$\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}} := \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} U_I \;=\; \bigl\{(I,x)\,\big|\, I\in\mathcal{I}_{\mathcal{K}}, x\in U_I\bigr\}$$

and the space of morphisms

$$\mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}} := \bigsqcup_{I,J\in\mathcal{I}_{\mathcal{K}}, I\subset J} U_{IJ} \;=\; \bigl\{(I,J,x)\,\big|\, I,J\in\mathcal{I}_{\mathcal{K}}, I\subset J, x\in U_{IJ}\bigr\}.$$

Here we denote $U_{II} := U_I$ for $I=J$, and for $I\subsetneq J$ use the domain $U_{IJ}\subset U_I$ of the restriction $\mathbf{K}_I|_{U_{IJ}}$ to F_J that is part of the coordinate change $\widehat{\Phi}_{IJ}:\mathbf{K}_I|_{U_{IJ}}\to\mathbf{K}_J$.

Source and target of these morphisms are given by

$$(I,J,x)\in\mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}}\bigl((I,x),(J,\phi_{IJ}(x))\bigr),$$

where $\phi_{IJ}:U_{IJ}\to U_J$ is the embedding given by $\widehat{\Phi}_{IJ}$, and we denote $\phi_{II}:=\mathrm{id}_{U_I}$. Composition is defined by

$$\bigl(J,K,y\bigr)\circ\bigl(I,J,x\bigr) := \bigl(I,K,x\bigr)$$

for any $I\subset J\subset K$ and $x\in U_{IJ}, y\in U_{JK}$ such that $\phi_{IJ}(x)=y$.

[4] When forming categories such as $\mathbf{B}_{\mathcal{K}}$, we take always the space of objects to be the disjoint union of the domains U_I, even if we happen to have defined the sets U_I as subsets of some larger space such as $\mathbb{R}^2$ or a space of maps as in the Gromov–Witten case. Similarly, the morphism space is a disjoint union of the U_{IJ} even though $U_{IJ}\subset U_I$ for all $J\supset I$.

The **obstruction category** $\mathbf{E}_{\mathcal{K}}$ *is defined in complete analogy to* $\mathbf{B}_{\mathcal{K}}$ *to consist of the spaces of objects* $\mathrm{Obj}_{\mathbf{E}_{\mathcal{K}}} := \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} U_I \times E_I$ *and morphisms*

$$\mathrm{Mor}_{\mathbf{E}_{\mathcal{K}}} := \big\{(I,J,x,e)\,\big|\, I,J\in\mathcal{I}_{\mathcal{K}}, I\subset J, x\in U_{IJ}, e\in E_I\big\}.$$

We may also express the further parts of a Kuranishi atlas in categorical terms:

- The obstruction category $\mathbf{E}_{\mathcal{K}}$ is a bundle over $\mathbf{B}_{\mathcal{K}}$ in the sense that there is a functor $\mathrm{pr}_{\mathcal{K}} : \mathbf{E}_{\mathcal{K}} \to \mathbf{B}_{\mathcal{K}}$ that is given on objects and morphisms by projection $(I,x,e)\mapsto (I,x)$ and $(I,J,x,e)\mapsto(I,J,x)$ with locally trivial fiber E_I.
- The sections s_I induce a smooth section of this bundle, i.e. a functor $s_{\mathcal{K}} : \mathbf{B}_{\mathcal{K}} \to \mathbf{E}_{\mathcal{K}}$ which acts smoothly on the spaces of objects and morphisms, and whose composite with the projection $\mathrm{pr}_{\mathcal{K}} : \mathbf{E}_{\mathcal{K}} \to \mathbf{B}_{\mathcal{K}}$ is the identity. More precisely, it is given by $(I,x)\mapsto(I,x,s_I(x))$ on objects and by $(I,J,x)\mapsto(I,J,x,s_I(x))$ on morphisms.
- The zero sets of the sections $\bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}}\{I\}\times s_I^{-1}(0)\subset \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$ form a very special strictly full subcategory $s_{\mathcal{K}}^{-1}(0)$ of $\mathbf{B}_{\mathcal{K}}$. Namely, $\mathbf{B}_{\mathcal{K}}$ splits into the subcategory $s_{\mathcal{K}}^{-1}(0)$ and its complement (given by the full subcategory with objects $\{(I,x)\,|\,s_I(x)\neq 0\}$) in the sense that there are no morphisms of $\mathbf{B}_{\mathcal{K}}$ between the underlying sets of objects. (This holds because, given any morphism (I,J,x), we have $s_I(x)=0$ if and only if $s_J(\phi_{IJ}(x)) = \widehat{\phi}_{IJ}(s_I(x)) = 0$.)
- The footprint maps ψ_I give rise to a surjective functor $\psi_{\mathcal{K}} : s_{\mathcal{K}}^{-1}(0)\to\mathbf{X}$ to the category $\mathbf{X}$ with object space X and trivial morphism spaces. It is given by $(I,x)\mapsto \psi_I(x)$ on objects and by $(I,J,x)\mapsto \mathrm{id}_{\psi_I(x)}$ on morphisms.

We denote the **topological realization of the category** $\mathbf{B}_{\mathcal{K}}$ by $|\mathbf{B}_{\mathcal{K}}|$, often abbreviated to $|\mathcal{K}|$. This is the space formed as the quotient of $\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}} = \bigsqcup_I U_I$ by the equivalence relation generated by the morphisms, and is given the quotient topology. Thus, for example, if X is compact the realization of the category $\mathbf{X}$ is the space X itself. The categories $\mathbf{B}_{\mathcal{K}}, \mathbf{E}_{\mathcal{K}}$ are étale, i.e. the spaces of objects and morphisms are smooth manifolds and all structural maps (such as the source map, composition and so on) are local diffeomorphisms. They are very similar to the topological groupoids that are used to model orbifolds (cf. e.g. [M07]), except that in a groupoid all morphisms are invertible, while here we do *not* add inverses to the morphisms $(I,J,x), I\subsetneq J$, since doing so would in general destroy the étale property. The difficulty is that because the images $\mathrm{im}\,\phi_{IJ}, \mathrm{im}\,\phi_{KJ}$ might not intersect transversally, the set of morphisms from U_I to U_K via U_J of the form $x\mapsto \phi_{JK}^{-1}(\phi_{IJ}(x))^{-1}$ do not usually form a manifold. In fact, such a composite can in general only be formed if $s_I(x)=0$, so that locally this set of morphisms is homeomorphic to the footprint $F_I\cap F_K = F_{I\cup K}$.

Let $\preceq$ denote the partial order on $\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$ given by

$$(I,x)\preceq(J,y) \quad\Longleftrightarrow\quad \mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}}((I,x),(J,y))\neq\emptyset.$$

That is, we have $(I,x)\preceq(J,y)$ iff $x\in U_{IJ}$ and $y=\phi_{IJ}(x)$. Then [MW12, Lemma 6.2.11] shows that $(I,x)\sim(J,y)$ iff there are elements (I_j,x_j) such that

$$(I,x) = (I_0,x_0)\preceq(I_1,x_1)\succeq(I_2,x_2)\preceq\ldots\succeq(I_k,x_k) = (J,y). \tag{2.2.1}$$

Since $s_{\mathcal{K}}$ is a functor, this equivalence relation preserves the zero sets, and one can show that the realization $|s_{\mathcal{K}}|^{-1}(0)$ of the subcategory $s_{\mathcal{K}}^{-1}(0)$ is a subset of $|\mathcal{K}|$ that can be naturally identified with the space X. Indeed, [MW12, Lemma 6.1.9] shows that the inverse of the footprint maps $\psi_I^{-1}: F_I \to U_I$ fit together to give an injective map

$$\iota_{\mathcal{K}}: X \to |s_{\mathcal{K}}|^{-1}(0) \subset |\mathcal{K}| \tag{2.2.2}$$

that (because X is compact) is a homeomorphism to its image $|s|^{-1}(0)$. However, as is shown by the examples at the end of [MW12, §6.1], the topology on $|\mathcal{K}|$ itself can be very wild; it is not in general Hausdorff and the natural maps $\pi_{\mathcal{K}}: U_I \to |\mathcal{K}|$ need not be injective, let alone homeomorphisms to their images. Moreover the fibers of the projection $|\mathrm{pr}|: |\mathbf{E}_{\mathcal{K}}| \to |\mathcal{K}|$ need not be vector spaces.

Remark 2.2.2. Because we assumed in (2.1.2) that E_I is the direct product $\prod_{i\in I} E_I$, the compatibility condition $\widehat{\phi}_{IJ} \circ s_I = s_J \circ \phi_{IJ}$ implies that $(I,x) \sim (J,y)$ only if there is $H \subset I \cap J$ such that $s_I(x) \in E_H$ and $s_J(y) \in E_H$.[5] This means that any equivalences between elements in U_I, U_J come from "lower levels" (where we order the set U_I by the cardinality $|I|$.) This makes it possible to make inductive arguments over $k = |I|$ that start at $k = 1$. The taming construction outlined in Proposition 2.3.4 below is one such example.

We will see that in order to obtain a realization $|\mathcal{K}|$ with reasonable topological properties it is enough to **tame** $\mathcal{K}$ as follows.

Definition 2.2.3. *A weak Kuranishi atlas is* **tame** *if for all $I, J, K \in \mathcal{I}_{\mathcal{K}}$ we have*

$$U_{IJ} \cap U_{IK} \;=\; U_{I(J\cup K)} \qquad \forall I \subset J, K; \tag{2.2.3}$$

$$\phi_{IJ}(U_{IK}) \;=\; U_{JK} \cap s_J^{-1}\big(\widehat{\phi}_{IJ}(E_I)\big) \qquad \forall I \subset J \subset K. \tag{2.2.4}$$

Here we allow equalities, using the notation $U_{II} := U_I$ and $\phi_{II} := \mathrm{Id}_{U_I}$. Further, to allow for the possibility that $J \cup K \notin \mathcal{I}_{\mathcal{K}}$, we define $U_{IL} := \emptyset$ for $L \subset \{1, \dots, N\}$ with $L \notin \mathcal{I}_{\mathcal{K}}$. Therefore (2.2.3) *includes the condition*

$$U_{IJ} \cap U_{IK} \neq \emptyset \quad \Longrightarrow \quad F_J \cap F_K \neq \emptyset \qquad \big(\iff \quad J \cup K \in \mathcal{I}_{\mathcal{K}} \quad \big).$$

The first tameness condition (2.2.3) extends the identity for footprints $\psi_I^{-1}(F_J) \cap \psi_I^{-1}(F_K) = \psi_I^{-1}(F_{J\cup K})$ to the domains of the transition maps in U_I. In particular, with $J \subset K$ it implies nesting of the domains of the transition maps,

$$U_{IK} \subset U_{IJ} \qquad \forall I \subset J \subset K. \tag{2.2.5}$$

The second tameness condition (2.2.4) extends the control of transition maps between footprints and zero sets $\phi_{IJ}(\psi_I^{-1}(F_K)) = \psi_J^{-1}(F_K) = U_{JK} \cap s_J^{-1}(0)$ to the Kuranishi domains. In particular, with $J = K$ it controls the image of the transition maps,

$$\operatorname{im} \phi_{IJ} := \phi_{IJ}(U_{IJ}) = s_J^{-1}(\widehat{\phi}_{IJ}(E_I)) \qquad \forall I \subset J. \tag{2.2.6}$$

[5] To be more correct we should write $s_I(x) \in \widehat{\phi}_{HI}(E_H)$, but as usual we suppress mention of the inclusions $\widehat{\phi}_{HI}: E_H \to E_I$. Further, we define $E_\emptyset := \{0\}$ to cover the case when $H := I \cap J = \emptyset$.

This implies that the image of ϕ_{IJ} is a closed subset of U_J, and strengthens the inclusion $\operatorname{im}\phi_{IJ} \subset s_J^{-1}(\widehat{\phi}_{IJ}(E_I))$ that follows from the compatibility conditions in Definition 2.1.5. To include these identities on the footprints and zero sets into the tameness conditions, it is convenient to extend the notation U_{IJ} to the case $I = \emptyset$, defining $U_{\emptyset J} := F_J \subset X$ (when $J \neq \emptyset$) and the map $\phi_{\emptyset J}$ in (2.2.4) to be ψ_J^{-1}. Then (2.2.6) also holds in the case $I = \emptyset$.

The following result is proved in [MW12, Lemma 6.2.8].

Lemma 2.2.4. *Every tame weak Kuranishi atlas satisfies the strong cocycle condition; in particular it is a Kuranishi atlas.*

Another important result is that the equivalence relation (2.2.1) simplifies drastically when $\mathcal{K}$ is tame.

Lemma 2.2.5 (adapted from Lemma 6.2.12 in [MW12])**.** *Let $\mathcal{K}$ be a tame Kuranishi atlas.*

(a) *For $(I,x),(J,y) \in \operatorname{Obj}_{\mathbf{B}_{\mathcal{K}}}$ with $s_I(x) \neq 0$ the following are equivalent.*
  (i) *$(I,x) \sim (J,y)$;*
  (ii) *there exists $z \in U_{I\cup J}$ such that $(I,x) \preceq (I\cup J, z) \succeq (J,y)$;*
  (iii) *there exists $w \in U_{I\cap J}$ such that $(I,x) \succeq (I\cap J, w) \preceq (J,y)$.*

(b) *$\pi_{\mathcal{K}} : U_I \to |\mathcal{K}|$ is injective for each $I \in \mathcal{I}_{\mathcal{K}}$, that is $(I,x) \sim (I,y)$ implies $x = y$ In particular, the elements z and w in (a) are automatically unique.*

(c) *If $S_I \subset U_I$ is closed then $\varepsilon_J(S_I) \subset U_J$ is also closed for all $J \in \mathcal{I}_K$ with $I \cap J \neq J$, where*
$$\varepsilon_J(S_I) = U_J \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(S_I)).$$

Sketch of proof. The key step is to show that the taming conditions imply the equivalence of (a:ii) and (a:iii). For example, if w exists as in (a:iii) then $w \in U_{(I\cap J)I} \cap U_{(I\cap J)J}$ which is a subset of $U_{(I\cap J)(I\cup J)}$ by (2.2.3). But then
$$x = \phi_{(I\cap J)I}(w) \in \phi_{(I\cap J)I}(U_{(I\cap J)(I\cup J)}) = U_{I(I\cup J)} \cap s_I^{-1}(\widehat{\phi}_{(I\cap I)I}(E_{I\cap J})$$
by (2.2.4), so that $\phi_{I(I\cup J)}(x)$ is defined. Moreover,
$$z := \phi_{I(I\cup J)}(x) = \phi_{I(I\cup J)} \circ \phi_{(I\cap J)I}(w) = \phi_{(I\cap J)(I\cup J)}(w) \in U_{I\cup J},$$
by the cocycle condition. A similar argument shows that $z = \phi_{(I\cap J)(I\cup J)}(w) = \phi_{J(I\cup J)}(y)$. Hence (a:ii) holds. Conversely, if z exists as in (a:ii) then tameness (2.2.4) and the additivity condition on the obstruction spaces in (2.1.2) imply that with $K := I \cup J$ we have
$$\begin{aligned} z \in \phi_{IK}(U_{IK}) \cap \phi_{JK}(U_{JK}) &= s_K^{-1}(\operatorname{im}(\widehat{\phi}_{IK})) \cap s_K^{-1}(\operatorname{im}(\widehat{\phi}_{JK})) \\ &= s_K^{-1}(\operatorname{im}(\widehat{\phi}_{(I\cap J)K})) = \phi_{(I\cap J)K}(U_{(I\cap J)K}), \end{aligned} \tag{2.2.7}$$
which implies the existence of suitable $w \in U_{(I\cap J)K}$. From this, and the injectivity of the maps $\phi_{\bullet\bullet}$, it is easy to show that (a:iii) holds. Once we know the equivalence of (a:ii) and (a:iii), it follows easily that every chain (2.2.1) can be shortened to have at most three elements, which gives the equivalence to (a:i).

Statement (b) then holds by applying (i) with $I = J$. Finally, to prove (c) note that because (a:i) implies (a:iii) we have

$$\varepsilon_J(S_I) = \phi_{(I\cap J)J}\big(\phi^{-1}_{(I\cap J)I}(S_I)\big) \subset \mathrm{im}\,(\phi_{(I\cap J)J}),$$

which is closed when $I \cap J \neq J$ because each map ϕ_{HK} is a homeomorphism from U_{HK} onto a relatively closed subset $s_K^{-1}(E_H)$ of U_K. □

The above lemma is the basis for the proof of the following result, taken from [MW12, Proposition 6.2.13 and 6.2.14].

Proposition 2.2.6. *Suppose that the Kuranishi atlas $\mathcal{K}$ is tame. Then $|\mathcal{K}|$ and $|\mathbf{E}_{\mathcal{K}}|$ are Hausdorff, and for each $I \in \mathcal{I}_{\mathcal{K}}$ the quotient maps $\pi_{\mathcal{K}}|_{U_I} : U_I \to |\mathcal{K}|$ and $\pi_{\mathcal{K}}|_{U_I\times E_I} : U_I \times E_I \to |\mathbf{E}_{\mathcal{K}}|$ are homeomorphisms onto their image. Further there is a unique linear structure on the fibers of $|\mathrm{pr}_{\mathcal{K}}| : |\mathbf{E}_{\mathcal{K}}| \to |\mathcal{K}|$ such that for every $I \in \mathcal{I}_{\mathcal{K}}$ the embedding $\pi_{\mathcal{K}} : U_I \times E_I \to |\mathbf{E}_{\mathcal{K}}|$ is linear on the fibers.*

Sketch of proof. We sketch the proofs of the claims about $|\mathcal{K}|$. To see that $|\mathcal{K}|$ is Hausdorff, note first that the equivalence relation on $O := \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}} = \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} U_I$ is closed, i.e. the subset

$$R := \big\{\big((I, x), (J, y)\big) \mid (I, x) \sim (J, y)\big\} \subset O \times O$$

is closed. Since $\mathcal{I}_{\mathcal{K}}$ is finite and $O \times O$ is the disjoint union of the second countable sets $U_I \times U_J$, this will follow if we show that for all pairs I, J and all convergence sequences $x^\nu \to x^\infty$ in U_I, $y^\nu \to y^\infty$ in U_J with $(I, x^\nu) \sim (J, y^\nu)$ for all ν, we have $(I, x^\infty) \sim (J, y^\infty)$. For that purpose denote $H := I \cap J$, then by Lemma 2.2.5(a) there is a sequence $w^\nu \in U_H$ such that $x^\nu = \phi_{HI}(w^\nu)$ and $y^\nu = \phi_{HJ}(w^\nu)$. Now it follows from the tameness condition (2.2.6) that x^∞ lies in the relatively closed subset $\phi_{HI}(U_{HI}) = s_I^{-1}(E_H) \subset U_I$, and since ϕ_{HI} is a homeomorphism to its image we deduce convergence $w^\nu \to w^\infty \in U_{HI}$ to a preimage of $x^\infty = \phi_{HI}(w^\infty)$. Then by continuity of the transition map we obtain $\phi_{HJ}(w^\infty) = y^\infty$, so that $(I, x^\infty) \sim (J, y^\infty)$ as claimed.

We then use a general result from [Bbk] (cf. Exercise 19, §11, Chapter 1) stating that whenever a space O is locally compact, Hausdorff and countable at infinity, its quotient by a closed relation is Hausdorff. The proof goes as follows. Choose an increasing family of precompact open sets $O_k \subset O_{k+1}$ with $O = \bigcup O_k$. Let us say that the set $A_k \subset \overline{O}_k$ is k**-saturated** if it contains all points $y \in \overline{O}_k$ such that $y \sim x$ for some $x \in A_k$. Thus the k-saturation $\mathrm{Sat}_k(A_k)$ of $A_k \subset \overline{O}_k$ is

$$\mathrm{Sat}_k(A_k) = pr_2\big((A_k \times \overline{O}_k) \cap R\big).$$

The key point is that, because R is closed and O is Hausdorff, the k-saturation of a closed (and hence compact) set $A_k \subset \overline{O}_k$ is compact, and hence closed. Further, if $C \subset \overline{O}_k$ is disjoint from a k-saturated set S we also have $\mathrm{Sat}_k(C) \cap S = \emptyset$.

To see that $O/{\sim}$ is Hausdorff we need to show that any two distinct equivalence classes A, B have disjoint saturated neighbourhoods $\mathcal{N}(A), \mathcal{N}(B)$ in O. Note that $A_k := A \cap \overline{O}_k$ is the k-saturation of one of its points and so is compact, and similarly for B_k. It suffices to find closed subsets $\mathcal{N}_k(A), \mathcal{N}_k(B) \subset \overline{O}_k$ for each $k \geq 1$ such that the following holds for all k:

- $\mathcal{N}_k(A) \cap \overline{O}_{k-1} = \mathcal{N}_{k-1}(A)$ for all k;
- $\mathcal{N}_k(A)$ is a closed k-saturated neighbourhood of $A_k := A \cap \overline{O}_k$ in $\overline{O}_k$;
- $\mathcal{N}_k(B)$ has similar properties;
- $\mathcal{N}_k(A) \cap \mathcal{N}_k(B) = 0$ for all k.

We may construct such sets by induction on k. At the kth step, consider the set $\mathrm{Sat}_k(\mathcal{N}_{k-1}(A) \cup A_k)$. Since $\mathcal{N}_{k-1}(A) \cup A_k$ is compact and disjoint from $\mathcal{N}_{k-1}(B) \cup B_k$, its k-saturation $\mathrm{Sat}_k(\mathcal{N}_{k-1}(A) \cup A_k)$ is also compact. Moreover, the added points lie in

$$\mathrm{Sat}_k(\mathcal{N}_{k-1}(A) \cup A_k) \setminus (\mathcal{N}_{k-1}(A) \cup A_k) \quad \subset \quad \overline{O}_k \setminus \overline{O}_{k-1}$$

and do not intersect B_k. Therefore the k-saturated compact set $S_1 := \mathrm{Sat}_k(\mathcal{N}_{k-1}(A) \cup A_k)$ is disjoint from the closed set $\mathcal{N}_{k-1}(B) \cup B_k$, and hence also disjoint from its k-saturation $S_2 := \mathrm{Sat}_k(\mathcal{N}_{k-1}(B) \cup B_k)$. It remains to check that any two disjoint k-saturated compact subsets S_1, S_2 of $\overline{O}_k$ have disjoint k-saturated compact neighbourhoods $\mathcal{N}_k(S_1), \mathcal{N}_k(S_2)$ (Take $\mathcal{N}_k(S_1)$ to be the k-saturation of a compact neighbourhood of S_1 in $\overline{O}_k \setminus S_2$, and then take $\mathcal{N}_k(S_2)$ to be the k-saturation of a compact neighbourhood of S_2 in $\overline{O}_k \setminus \mathcal{N}_k(S_1)$.)

This shows that $|\mathcal{K}|$ is Hausdorff. Since the projection $\pi_{\mathcal{K}} : U_I \to |\mathcal{K}|$ is continuous and injective by Lemma 2.2.5 (b), to show that it is a homeomorphism to its image it suffices to construct for each open $W \subset U_I$ an open subset $\mathcal{W}$ of $|\mathcal{K}|$ such that $U_I \cap \pi_{\mathcal{K}}^{-1}(\mathcal{W}) = W$. Thus we need $W_J := U_J \cap \pi_{\mathcal{K}}^{-1}(\mathcal{W})$ to be open for each J. List the elements $J \subset I$ as I_ℓ for $\ell = -p, \dots, 0$ where $I_0 := I$, and define $W_\ell := U_{I_\ell} \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(W))$ for these ℓ. Then list the remaining elements $\{J \in \mathcal{I}_{\mathcal{K}} \mid J \not\subset I\}$ as $I_1, \dots, I_m$ in any order such that $|I_j| \le |I_k|$ for $1 \le j < k$. By induction, it suffices to choose open subsets $W_k \subset U_{I_k}$ for $k \ge 1$ so that if $\mathcal{W}_k := \bigcup_{-p \le j \le k} \pi_{\mathcal{K}}(W_j)$, we have

$$U_{I_j} \cap \pi_{\mathcal{K}}^{-1}(\mathcal{W}_k) = W_j, \quad \forall \ -p \le j \le k.$$

Since this identity holds when $k = 0$, it remains to check that when $k > 0$ we may take

$$W_k := U_{I_k} \setminus \bigcup_{0 \le j < k} \varepsilon_{I_k}(U_{I_j} \setminus W_{I_j}).$$

Here we use the fact that $\varepsilon_{I_k}(U_{I_j} \setminus W_{I_j})$ is closed by Lemma 2.2.5 (c), since $U_{I_k} \not\subset U_{I_j}$ when $0 \le j < k$ by our choice of ordering. For more details see [MW12]. $\square$

Remark 2.2.7. (i) The above construction gives a rather nice picture of the virtual neighbourhood $|\mathcal{K}|$ for a tame atlas. It is a union of sets $\pi_{\mathcal{K}}(U_I)$, each of which is a homeomorphic image of a manifold and has "boundary" $\overline{\pi_{\mathcal{K}}(U_I)} \setminus \pi_{\mathcal{K}}(U_I)$ contained in the union of the lower dimensional sets $\bigcup_{H \subsetneq I} \pi_{\mathcal{K}}(U_H)$. A pairwise intersection $\pi_{\mathcal{K}}(U_I) \cap \pi_{\mathcal{K}}(U_J)$ is nonempty only if the corresponding footprint intersection $F_I \cap F_J = F_{I \cup J}$ is nonempty, in which case we have $\pi_{\mathcal{K}}(U_I) \cap \pi_{\mathcal{K}}(U_J) \subset \pi_{\mathcal{K}}(U_{I \cup J})$. If also $I \cap J \ne \emptyset$, then $\pi_{\mathcal{K}}(U_I) \cap \pi_{\mathcal{K}}(U_J)$ may be identified with the submanifold $\pi_{\mathcal{K}}\big(s_{I \cup J}^{-1}(E_{I \cap J})\big)$ of $\pi_{\mathcal{K}}(U_{I \cup J})$, which implies that the intersection of $\pi_{\mathcal{K}}(U_I)$ with $\pi_{\mathcal{K}}(U_J)$ can be considered to be transverse. However, if $I \cap J = \emptyset$ then these two sets intersect only along the zero set $\iota_{\mathcal{K}}(X)$, where $\iota_{\mathcal{K}}$ is as in (2.2.2). For example, the domains of two basic charts $\pi_{\mathcal{K}}(U_1)$ and $\pi_{\mathcal{K}}(U_2)$ will in general intersect nontransversally in $\iota_{\mathcal{K}}(F_{12})$, while the two sum domains $\pi_{\mathcal K}(U_{12})$ and $\pi_{\mathcal K}(U_{23})$ intersect transversally in the submanifold $\pi_{\mathcal K}(U_2)\cap\pi_{\mathcal K}(U_{123})$ of $\pi_{\mathcal K}(U_{123})$.

(ii) Notice also that the effect of the taming condition is to reduce the equivalence relation to a two step process: $(I,x)\sim(J,y)$ iff we can write $(I,x)\preceq(I\cup J,z)\succeq(J,y)$, or equivalently $(I,x)\succeq(I\cap J,w)\preceq(J,y)$. The reduction process described in §2.4 below will simplify the equivalence relation even further to a single step. In fact, this process discards all the elements in $U_I\smallsetminus V_I$, for suitable choice of open sets $V_I\subset U_I$, so that when $x\in V_I, y\in V_J$ we have $(I,x)\sim(J,y)$ only if $(I,x)\preceq(J,y)$ or $(I,x)\succeq(J,y)$.

(iii) See [MW12, Example 6.1.11] for a (non tame) atlas for which the map $\pi_{\mathcal K}$ is not injective on U_I and [MW12, Example 6.1.12] for a case where the fibers of $|\mathrm{pr}|:|\mathbf{E}_{\mathcal K}|\to|\mathcal K|$ have no linear structure.

2.3. **Taming weak atlases.** We saw above that the realization of a tame atlas has good topological properties. We now explain how to construct a tame atlas from a weak atlas, and give other background needed to understand the following result.

Theorem 2.3.1 (cf. Theorem 6.2.6 in [MW12])**.** *Let $\mathcal K$ be a weak Kuranishi atlas (with trivial isotropy) on a compact metrizable space X. Then an appropriate shrinking of $\mathcal K$ provides a metrizable tame Kuranishi atlas $\mathcal K'$ with domains $(U_I'\subset U_I)_{I\in\mathcal I_{\mathcal K'}}$ such that the realizations $|\mathcal K'|$ and $|\mathbf{E}_{\mathcal K'}|$ are Hausdorff in the quotient topology. In addition, for each $I\in\mathcal I_{\mathcal K'}=\mathcal I_{\mathcal K}$ the projection maps $\pi_{\mathcal K'}:U_I'\to|\mathcal K'|$ and $\pi_{\mathcal K'}:U_I'\times E_I\to|\mathbf{E}_{\mathcal K'}|$ are homeomorphisms onto their images and fit into a commutative diagram*

$$\begin{array}{ccc} U_I'\times E_I & \overset{\pi_{\mathcal K'}}{\hookrightarrow} & |\mathbf{E}_{\mathcal K'}| \\ \downarrow & & \downarrow|\mathrm{pr}_{\mathcal K'}| \\ U_I' & \overset{\pi_{\mathcal K'}}{\hookrightarrow} & |\mathcal K'| \end{array}$$

where the horizontal maps intertwine the vector space structure on E_I with a vector space structure on the fibers of $|\mathrm{pr}_{\mathcal K'}|$.

Moreover, any two such shrinkings are cobordant by a metrizable tame Kuranishi cobordism whose realization also has the above Hausdorff, homeomorphism, and linearity properties.

We begin by explaining shrinkings, first for the footprint cover and then for an atlas. We will write $V'\sqsubset V$ to denote that V' is precompact in V, i.e. the closure (written $\overline{V'}$ or $cl_V(V')$) of V' in V is compact.

Definition 2.3.2. *Let $(F_i)_{i=1,\dots,N}$ be an open cover of a compact space X. We say that $(F_i')_{i=1,\dots,N}$ is a* **shrinking** *of (F_i) if $F_i'\sqsubset F_i$ are precompact open subsets, which cover $X=\bigcup_{i=1,\dots,N}F_i'$, and are such that for all subsets $I\subset\{1,\dots,N\}$ we have*

$$F_I:=\textstyle\bigcap_{i\in I}F_i\neq\emptyset\qquad\Longrightarrow\qquad F_I':=\bigcap_{i\in I}F_i'\neq\emptyset. \tag{2.3.1}$$

Definition 2.3.3. *Let $\mathcal K=\big(\mathbf{K}_I,\widehat\Phi_{IJ}\big)_{I,J\in\mathcal I_{\mathcal K},I\subsetneq J}$ be a weak Kuranishi atlas. We say that a weak Kuranishi atlas $\mathcal K'=(\mathbf{K}_I',\widehat\Phi_{IJ}')_{I,J\in\mathcal I_{\mathcal K'},I\subsetneq J}$ is a* **shrinking** *of $\mathcal K$ if*

(i) *the footprint cover* $(F_i')_{i=1,\dots,N'}$ *is a shrinking of the cover* $(F_i)_{i=1,\dots,N}$*, in particular the numbers* $N = N'$ *of basic charts agree, and so do the index sets* $\mathcal{I}_{\mathcal{K}'} = \mathcal{I}_{\mathcal{K}}$*;*

(ii) *for each* $I \in \mathcal{I}_{\mathcal{K}}$ *the chart* $\mathbf{K}_I'$ *is the restriction of* $\mathbf{K}_I$ *to a precompact domain* $U_I' \subset U_I$ *as in Definition 2.1.3;*

(iii) *for each* $I, J \in \mathcal{I}_{\mathcal{K}}$ *with* $I \subsetneq J$ *the coordinate change* $\widehat{\Phi}'_{IJ}$ *is the restriction of* $\widehat{\Phi}_{IJ}$ *to the open subset* $U'_{IJ} := \phi_{IJ}^{-1}(U_J') \cap U_I'$ *(cf.* [MW12, Lemma 5.2.3]*).*

Note that any shrinking of an additive weak Kuranishi atlas preserves the weak cocycle condition (since it only requires equality on overlaps). Moreover, a shrinking is determined by the choice of the domains $U_I' \sqsubset U_I$ of the sum charts (since condition (iii) then specifies the domains of the coordinate changes), and so can be considered as the restriction of $\mathcal{K}$ to the subset $\bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} U_I' \subset \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$. However, for a shrinking to satisfy a stronger form of the cocycle condition (such as tameness) the domains $U'_{IJ} := \phi_{IJ}^{-1}(U_J') \cap U_I'$ of the coordinate changes must satisfy appropriate compatibility conditions, so that the domains U_I' can no longer be chosen independently of each other. Since the relevant conditions are expressed in terms of the U'_{IJ}, we next show that the construction of a tame shrinking can be achieved by iterative choice of these sets U'_{IJ}.

Here is the main result of [MW12, §6.3]. We explained in Remark 2.2.2 above why the basic strategy of its proof (upwards induction on $|I|$) works.

Proposition 2.3.4. *Every weak Kuranishi atlas* $\mathcal{K}$ *has a shrinking* $\mathcal{K}'$ *that is a tame Kuranishi atlas – for short called a* **tame shrinking**.

Sketch of proof. Since X is compact and metrizable and the footprint open cover (F_i) is finite, it has a shrinking (F_i') in the sense of Definition 2.3.2. In particular we can ensure that $F_I' \neq \emptyset$ whenever $F_I \neq \emptyset$ by choosing $\delta > 0$ so that every nonempty F_I contains some ball $B_\delta(x_I)$ and then choosing the F_i' to contain $B_{\delta/2}(x_I)$ for each $I \ni i$ (i.e. $F_I \subset F_i$). Then we obtain $F_I' \neq \emptyset$ for all $I \in \mathcal{I}_{\mathcal{K}}$ since $B_{\delta/2}(x_I) \subset \bigcap_{i\in I} F_i' = F_I'$.

In another preliminary step, we find precompact open subsets $U_I^{(0)} \sqsubset U_I$ and open sets $U_{IJ}^{(0)} \subset U_{IJ} \cap U_I^{(0)}$ for all $I, J \in \mathcal{I}_{\mathcal{K}}$ such that

$$U_I^{(0)} \cap s_I^{-1}(0) = \psi_I^{-1}(F_I'), \qquad U_{IJ}^{(0)} \cap s_I^{-1}(0) = \psi_I^{-1}(F_I' \cap F_J'). \tag{2.3.2}$$

Here we choose any suitable $U_I^{(0)}$ (which is possible by [MW12, Lemma 5.1.4]), and then define the $U_{IJ}^{(0)}$ by restriction:

$$U_{IJ}^{(0)} := U_{IJ} \cap U_I^{(0)} \cap \phi_{IJ}^{-1}(U_J^{(0)}).$$

We then construct the required shrinking $\mathcal{K}'$ by choosing possibly smaller domains $U_I' \subset U_I^{(0)}$ and $U'_{IJ} \subset U_{IJ}^{(0)}$ with the same footprints F_I'. We also arrange $U'_{IJ} = U_I' \cap \phi_{IJ}^{-1}(U_J')$, so that $\mathcal{K}'$ is a shrinking of the original $\mathcal{K}$. Therefore we just need to make sure that $\mathcal{K}'$ satisfies the tameness conditions (2.2.3) and (2.2.4).

We construct the domains U_I', U'_{IJ} by a finite iteration, starting with $U_I^{(0)}, U_{IJ}^{(0)}$. Here we streamline the notation by setting $U_I^{(k)} := U_{II}^{(k)}$ and extend the notation to all pairs of subsets $I\subset J\subset\{1,\dots,N\}$ by setting $U^{(k)}_{IJ}=\emptyset$ if $J\notin\mathcal{I}_{\mathcal{K}}$. (Note that $J\in\mathcal{I}_{\mathcal{K}}$ and $I\subset J$ implies $I\in\mathcal{I}_{\mathcal{K}}$.) Then in the k-th step we construct open subsets $U^{(k)}_{IJ}\subset U^{(k-1)}_{IJ}$ for all $I\subset J\subset\{1,\dots,N\}$ so that the following holds.

(i) The zero set conditions $U^{(k)}_{IJ}\cap s_I^{-1}(0)=\psi_I^{-1}(F'_J)$ hold for all $I\subset J$.

(ii) The first tameness condition (2.2.3) holds for all $I\subset J,K$ with $|I|\le k$, that is

$$U^{(k)}_{IJ}\cap U^{(k)}_{IK}=U^{(k)}_{I(J\cup K)}.$$

In particular, we have $U^{(k)}_{IK}\subset U^{(k)}_{IJ}$ for $I\subset J\subset K$ with $|I|\le k$.

(iii) The second tameness condition (2.2.4) holds for all $I\subset J\subset K$ with $|I|\le k$, that is

$$\phi_{IJ}(U^{(k)}_{IK})=U^{(k)}_{JK}\cap s_J^{-1}(E_I).$$

In particular we have $\phi_{IJ}(U^{(k)}_{IJ})=U^{(k)}_{J}\cap s_J^{-1}(E_I)$ for all $I\subset J$ with $|I|\le k$.

In other words, we need the tameness conditions to hold up to level k.

The above choice of the domains $U^{(0)}_{IJ}$ completes the 0-th step since conditions (ii) (iii) are vacuous. Now suppose that the $(k-1)$-th step is complete for some $k\ge 1$. We then define $U^{(k)}_{IJ}:=U^{(k-1)}_{IJ}$ for all $I\subset J$ with $|I|\le k-1$. For $|I|=k$ we also set $U^{(k)}_{II}:=U^{(k-1)}_{II}$. This ensures that (i) and (ii) continue to hold for $|I|<k$. In order to preserve (iii) for triples $H\subset I\subset J$ with $|H|<k$ we then require that the intersection $U^{(k)}_{IJ}\cap s_I^{-1}(E_H)=U^{(k-1)}_{IJ}\cap s_I^{-1}(E_H)$ is fixed. In case $H=\emptyset$, this is condition (i), and since $U^{(k)}_{IJ}\subset U^{(k-1)}_{IJ}$ it can generally be phrased as inclusion (i′) below. With that it remains to construct the open sets $U^{(k)}_{IJ}\subset U^{(k-1)}_{IJ}$ as follows.

(i′) For all $H\subsetneq I\subset J$ with $|H|<k$ and $|I|\ge k$ we have $U^{(k-1)}_{IJ}\cap s_I^{-1}(E_H)\subset U^{(k)}_{IJ}$. Here we include $H=\emptyset$, in which case the condition says that $U^{(k-1)}_{IJ}\cap s_I^{-1}(0)\subset U^{(k)}_{IJ}$ (which implies $U^{(k)}_{IJ}\cap s_I^{-1}(0)=\psi_I^{-1}(F'_J)$, as explained above).

(ii′) For all $I\subset J,K$ with $|I|=k$ we have $U^{(k)}_{IJ}\cap U^{(k)}_{IK}=U^{(k)}_{I(J\cup K)}$.

(iii′) For all $I\subsetneq J\subset K$ with $|I|=k$ we have $\phi_{IJ}(U^{(k)}_{IK})=U^{(k)}_{JK}\cap s_J^{-1}(E_I)$.

The construction is then completed in two steps.

Step A constructs $U^{(k)}_{IK}$ for $|I|=k$ and $I\subsetneq K$ satisfying (i′),(ii′) and

(iii″) $U^{(k)}_{IK}\subset\phi_{IJ}^{-1}(U^{(k-1)}_{JK})$ for all $I\subsetneq J\subset K$.

Step B constructs $U^{(k)}_{JK}$ for $|J|>k$ and $J\subset K$ satisfying (i′) and (iii′).

Step A uses the following nontrivial result to show that the required sets exist.

Lemma 2.3.5 (Lemma 6.3.5 in [MW12]). *Let U be a complete metric space, $U'\subset U$ a precompact open set, and $Z\subset U'$ a relatively closed subset. Suppose we are given a finite collection of relatively open subsets $Z_i\subset Z$ for $i=1,\dots,N$ and open subsets $W_K\subset U'$ with*

$$W_K\cap Z=Z_K:=\textstyle\bigcap_{i\in K}Z_i$$

for all index sets $K \subset \{1,\dots,N\}$. Then there exist open subsets $U_K \subset W_K$ with $U_K \cap Z = Z_K$ and $U_J \cap U_K = U_{J\cup K}$ for all $J, K \subset \{1,\dots,N\}$.

We apply this with $U = U_I^{(k)}$ for some $|I| = k$ with Z given by:

$$Z := \bigcup_{H \subsetneq I} \big(U_{II}^{(k-1)} \cap s_I^{-1}(\mathrm{im}\,(\widehat{\phi}_{HI})\big) \;=\; \bigcup_{H\subsetneq I} \phi_{HI}\big(U_{HI}^{(k-1)}\big) \;\subset\; U'$$

We take $W_K = U_{IK}^{(k-1)}$ and $Z_i = W_{I\cup\{i\}} \cap Z$ for $i \notin I$, and then define $U_{IK}^{(k)} := U_K$. It is not hard to check that the required conditions hold.

Step B then modifies the sets $U_{JK}^{(k-1)}$ by removing the extra parts that contradict (iii′). In other words we define

$$U_{JK}^{(k)} \;:=\; U_{JK}^{(k-1)} \smallsetminus \bigcup_{I\subset J, |I|=k} \big(s_J^{-1}(E_I) \smallsetminus \phi_{IJ}(U_{IJ}^{(k)})\big).$$

For further details, see the proof of [MW12, Proposition 6.3.4]. □

Remark 2.3.6. To understand the crucial role of additivity in the above proof, consider a weak atlas that contains just three charts $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_{12}$ each with obstruction space E so that $\widehat{\phi}_{i(12)} = \mathrm{id}$ for $i = 1, 2$. Then when $k = 1$ we must construct sets $U^{(1)}_{i(12)}$ for $i = 1, 2$ that both satisfy $\phi_{i(12)}\big(U^{(1)}_{i(12)}\big) = U^{(1)}_{12} \cap s_{12}^{-1}(E) = U^{(1)}_{12}$. Hence the choices of the two level one sets $U^{(1)}_{1(12)}$ and $U^{(1)}_{2(12)}$ are not independent. In an additive situation, one can only have $E_1 = E_{12} = E$ if $E_2 = \{0\}$. In this case we still need $\phi_{1(12)}\big(U^{(1)}_{1(12)}\big) = U^{(1)}_{12}$. However, the condition for $i = 2$ is $\phi_{2(12)}\big(U^{(1)}_{2(12)}\big) = s_2^{-1}(0)$, which has been arranged at level 0.

Even though $|\mathcal{K}|$ is Hausdorff when $\mathcal{K}$ is tame, its topology is still not very nice. For example, it is never metrizable in the quotient topology unless all obstruction spaces vanish.

Example 2.3.7 (Failure of metrizability and local compactness)**.** For simplicity we will give an example with noncompact $X = \mathbb{R}$. (A similar example can be constructed with $X = S^1$.) We construct a Kuranishi atlas $\mathcal{K}$ on X by two basic charts, $\mathbf{K}_1 = (U_1 = \mathbb{R}, E_1 = \{0\}, s = 0, \psi_1 = \mathrm{id})$ and

$$\mathbf{K}_2 = \big(U_2 = (0,\infty)\times\mathbb{R},\ E_2 = \mathbb{R},\ s_2(x,y) = y,\ \psi_2(x,y) = x\big),$$

one sum chart $\mathbf{K}_{12} = \mathbf{K}_2|_{U_{12}}$ with domain $U_{12} := U_2$, and the coordinate changes $\widehat{\Phi}_{i,12}$ induced by the natural embeddings of the domains $U_{1,12} := (0,\infty) \hookrightarrow (0,\infty)\times\{0\}$ and $U_{2,12} := U_2 \hookrightarrow U_2$. Then as a set $|\mathcal{K}| = \big(U_1 \sqcup U_2 \sqcup U_{12}\big)/\sim$ can be identified with $\big(\mathbb{R}\times\{0\}\big) \cup \big((0,\infty)\times\mathbb{R}\big) \subset \mathbb{R}^2$. However, the quotient topology at $(0,0) \in |\mathcal{K}|$ is strictly stronger than the subspace topology. That is, for any $O \subset \mathbb{R}^2$ open the induced subset $O \cap |\mathcal{K}| \subset |\mathcal{K}|$ is open, but some open subsets of $|\mathcal{K}|$ cannot be represented in this way. In fact, for any $\varepsilon > 0$ and continuous function $f : (0,\varepsilon) \to (0,\infty)$, the set

$$U_{f,\varepsilon} \;:=\; \big\{[x] \,\big|\, x \in U_1, |x| < \varepsilon\big\} \;\cup\; \big\{[(x,y)] \,\big|\, (x,y) \in U_2, |x| < \varepsilon, |y| < f(x)\big\} \;\subset\; |\mathcal{K}|$$

is open in the quotient topology. It is shown in [MW12, Example 6.1.15] that these sets form a basis for the neighbourhoods of $[(0,0)]$ in the quotient topology.

Notice that this atlas $\mathcal{K}$ is tame. Therefore taming by itself does not give a quotient with manageable topology. On the other hand, the only bad point $|\mathcal{K}|$ is $(0,0)$. Indeed, according to Proposition 2.3.10 the realization of any shrinking $\mathcal{K}'$ of $\mathcal{K}$ injects into $|\mathcal{K}|$ and is metrizable with the corresponding subspace topology. For example, we could take $U_1' := (-\infty, 2) \sqsubset U_1$, $U_2' := (1,\infty)\times\mathbb{R} \sqsubset U_2$ and $U_{12}' := U_2'$.

Remark 2.3.8. As we show at the end of [MW12, §6.2], shrinkings are helpful in understanding the different topologies on precompact subsets of $|\mathcal{K}|$. However, tame shrinkings are even better. To see why this is so, note that if $\mathcal{K}'$ is a shrinking of $\mathcal{K}$ then, even though $\mathbf{B}_{\mathcal{K}}'$ is a full subcategory of $\mathbf{B}_{\mathcal{K}}$, the natural map $|\mathcal{K}'| \to |\mathcal{K}|$ need not be injective. (Two elements $(I.x),(J,y) \in \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}'}$ might be related via some element $(K,z) \in \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$ that has been removed from $\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}'}$.) However, this does not happen if $\mathcal{K}'$ is also tame; cf. [MW12, Lemma 6.3.6]. Moreover, if $\mathcal{K}$ is tame, the topology induced on $|\mathcal{K}'|$ by considering it as a subspace of $|\mathcal{K}|$ is metrizable. This means that $\mathcal{K}'$ is metrizable in the following sense: cf. Definition 6.2.4 [MW12].

Definition 2.3.9. *A Kuranishi atlas $\mathcal{K}$ is said to be* **metrizable** *if there is a bounded metric d on the set $|\mathcal{K}|$ such that for each $I \in \mathcal{I}_{\mathcal{K}}$ the pullback metric $d_I := (\pi_{\mathcal{K}}|_{U_I})^*d$ on U_I induces the given topology on the manifold U_I. In this situation we call d an* **admissible metric** *on $|\mathcal{K}|$. A* **metric Kuranishi atlas** *is a pair $(\mathcal{K}, d)$ consisting of a metrizable Kuranishi atlas together with a choice of admissible metric d.*

In order to construct metric tame Kuranishi atlases, we will find it useful to consider tame shrinkings $\mathcal{K}_{sh}$ of a weak Kuranishi atlas $\mathcal{K}$ that are obtained as shrinkings of an intermediate tame shrinking $\mathcal{K}'$ of $\mathcal{K}$. For short we will call such $\mathcal{K}_{sh}$ a **preshrunk tame shrinking** of $\mathcal{K}$ and write $\mathcal{K}_{sh} \sqsubset \mathcal{K}' \sqsubset \mathcal{K}$. The proof of the next result is not hard. It combines [MW12, Proposition 6.3.7] with [MW12, Proposition 6.2.18].

Proposition 2.3.10. *Let $\mathcal{K}$ be a weak Kuranishi atlas. Then every preshrunk tame shrinking of $\mathcal{K}$ is metrizable. In particular, $\mathcal{K}$ has a metrizable tame shrinking $\mathcal{K}_{sh}$. Moreover, if $\mathcal{K}_{sh} \sqsubset \mathcal{K}'' \sqsubset \mathcal{K}$, where $\mathcal{K}$"$'$ is an arbitrary tame shrinking of $\mathcal{K}$, then the metric topology on $|\mathcal{K}_{sh}|$ equals its topology as a subspace of $|\mathcal{K}''|$ with the quotient topology.*

The final concept used in Theorem 2.3.1 is that of cobordism. We develop an appropriate theory of **cobordism Kuranishi atlases** in [MW12, §6.4]. This reference deals only with the theory of atlases over the trivial cobordism $X \times I$, but the theory would easily generalize. The essential feature of our definitions (cf. [MW12, §6.4]) is that the charts are now manifolds with collared boundary, i.e. we require that there is a product structure near the boundary of the domains U_I (which are now manifolds with boundary), and require compatibility of the product structure with coordinate changes and all other structures, such as metrics. Thus a metric Kuranishi cobordism $(\mathcal{K}, d)$ from $\mathcal{K}^0$ to $\mathcal{K}^1$ is a metric atlas $(\mathcal{K}, d)$ over $X \times [0,1]$ that for $\alpha = 0,1$ restricts to the atlas $\mathcal{K}^\alpha =: \partial^\alpha \mathcal{K}^{[0,1]}$ on X, and near each boundary has an isometric identification with the product $\mathcal{K}^\alpha \times A^\alpha_\varepsilon$ where $A^0_\varepsilon = [0,\varepsilon), A^1_\varepsilon = (1-\varepsilon, 1]$; see [MW12, Definition 6.4.13]. It is surprisingly hard to show that one can interpolate between two given metrics in this way. (Note that we are *not* considering Riemannian metrics.) The necessary details are given in [MW12, Proposition 6.4.15]. All the above ideas and results are summarized in Theorem 2.3.1 stated at the beginning of this subsection.

2.4. **Reductions and the construction of perturbation sections.** We now assume that $\mathcal{K}$ is tame atlas, and explain how to construct the corresponding virtual moduli cycle $[X]^{vir}_{\mathcal{K}}$.

The cover of X by the footprints $(F_I)_{I\in\mathcal{I}_\mathcal{K}}$ of all the Kuranishi charts (both the basic charts and those that are part of the transitional data) is closed under intersection. This makes it easy to express compatibility of the charts, since the overlap of footprints of any two charts $\mathbf{K}_I$ and $\mathbf{K}_J$ is covered by another chart $\mathbf{K}_{I\cup J}$. However, this yields so many compatibility conditions that a construction of compatible perturbations in the Kuranishi charts may not be possible. For example, a choice of perturbation (in E_I) in the chart $\mathbf{K}_I$ also fixes the perturbation in each chart $\mathbf{K}_J$ over $\phi^{-1}_{J(I\cup J)}\bigl(\operatorname{im}\phi_{I(I\cup J)}\bigr) \subset U_J$, whenever $I \cup J \subset \mathcal{I}_\mathcal{K}$. Since we do not assume transversality of the coordinate changes, this subset of U_J need not be a submanifold, [6] and hence the perturbation may not extend smoothly to a map from U_J to E_J. Moreover, for such an extension to exist at all, the perturbation would have to take values in the intersection of $\widehat{\phi}_{I(I\cup J)}(E_I) \cap \widehat{\phi}_{J(I\cup J)}(E_J) \subset \widehat{\phi}_{I\cap J(I\cup J)}E_{I\cap J}$, a very restrictive condition. In fact $I\cap J = \emptyset$, this would mean that the perturbation would have to vanish over $F_{I\cup J}$. We will avoid these difficulties, and also make a first step towards compactness, by reducing the domains of the Kuranishi charts to precompact subsets $V_I \sqsubset U_I$ such that all compatibility conditions between $\mathbf{K}_I|_{V_I}$ and $\mathbf{K}_J|_{V_J}$ are given by direct coordinate changes $\widehat{\Phi}_{IJ}$ or $\widehat{\Phi}_{JI}$. As explained more fully in [MW12] the reduction process is analogous to replacing the star cover of a simplicial set by the star cover of its first barycentric subdivision.

Remark 2.4.1. Reductions are the closest we come to the notion of a “good coordinate system” as used in [FOOO, FOOO12]. This also has the feature that the equivalence relation is induced by direct coordinate changes. However, each of the finite number of charts in their good coordinate system has to be built from the Kuranishi neighbourhoods $\mathbf{K}_p$ by amalgamating the domains of a given dimension, which (in the presence of isotropy) is possible only on the orbifold level. However, our reduction is built on the level of the category itself instead of on the level of the intermediate category, and so we can retain complete information on the group actions; cf. Definition 3.2.8 and the subsequent discussion.

Definition 2.4.2. *A* **reduction** *of a tame Kuranishi atlas $\mathcal{K}$ is an open subset $\mathcal{V} = \bigsqcup_{I\in\mathcal{I}_\mathcal{K}} V_I \subset \mathrm{Obj}_{\mathbf{B}_\mathcal{K}}$ i.e. a tuple of (possibly empty) open subsets $V_I \subset U_I$, satisfying the following conditions:*

(i) $V_I \sqsubset U_I$ *for all $I \in \mathcal{I}_\mathcal{K}$, and if $V_I \neq \emptyset$ then $V_I \cap s_I^{-1}(0) \neq \emptyset$;*

[6] As explained in Remark 2.2.7, it will be a submanifold if $I \cap J \neq \emptyset$, but not otherwise.

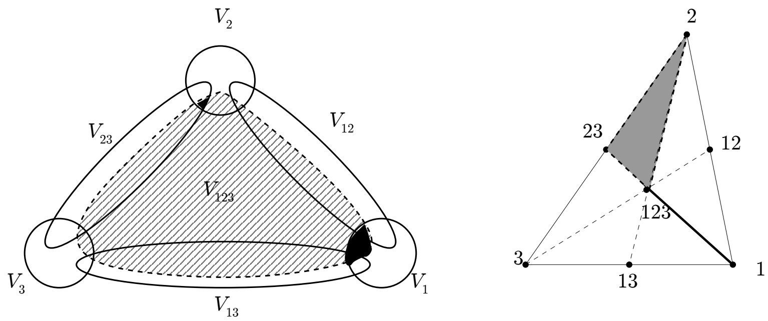

FIGURE 2.4.1. The right diagram shows the first barycentric subdivision of the triangle with vertices $1, 2, 3$. It has three new vertices labelled ij at the barycenters of the three edges and one vertex labelled 123 at the barycenter of the triangle. The left is a schematic picture of the cover by the stars of the vertices of this barycentric subdivision. The black sets are examples of multiple intersections of the new cover, which correspond to the simplices in the barycentric subdivision. E.g. $V_2 \cap V_{23} \cap V_{123}$ corresponds to the triangle with vertices $2, 23, 123$, whereas $V_1 \cap V_{123}$ corresponds to the edge between 1 and 123. This new cover has the same intersection properties as the reduction of the original cover by the stars U_1, U_2, U_3 of the three vertices.

(ii) *if* $\pi_{\mathcal{K}}(\overline{V_I}) \cap \pi_{\mathcal{K}}(\overline{V_J}) \neq \emptyset$ *then* $I \subset J$ *or* $J \subset I$*;*
(iii) *the zero set* $\iota_{\mathcal{K}}(X) = |s_{\mathcal{K}}|^{-1}(0)$ *is contained in* $\pi_{\mathcal{K}}(\mathcal{V}) = \bigcup_{I \in \mathcal{I}_{\mathcal{K}}} \pi_{\mathcal{K}}(V_I)$.

Given a reduction $\mathcal{V}$, we define the **reduced domain category** $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}$ *and the* **reduced obstruction category** $\mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ *to be the full subcategories of* $\mathbf{B}_{\mathcal{K}}$ *and* $\mathbf{E}_{\mathcal{K}}$ *with objects* $\bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} V_I$ *resp.* $\bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} V_I \times E_I$*, and denote by* $s|_{\mathcal{V}} : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ *the section given by restriction of* $s_{\mathcal{K}}$*.*

We show in [MW12, Lemma 7.1.5] that the realization $|\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}|$ of the subcategory $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}$ (i.e. its object space modulo the equivalence relation generated by its morphisms) injects into $|\mathbf{B}_{\mathcal{K}}| =: |\mathcal{K}|$.[7]

There is a related notion of cobordism reduction (cf. [MW12, Definition 7.1.3]), which is just as you would imagine, keeping in mind that all sets have product form near the boundary.

Here is the main existence result. It is proved by first constructing a reduction of the footprint cover (a process well understood in algebraic topology), and then extending this suitably. The proof requires care, but is not intrinsically hard. See Lemma 4.1.12 and Corollary 4.1.13 below for proofs of related results.

Proposition 2.4.3 (Proposition 7.1.11 in [MW12])**.** *The following statements hold.*

(a) *Every tame Kuranishi atlas $\mathcal{K}$ has a reduction $\mathcal{V}$.*

[7]This is not automatic: cf. the discussion before Definition 2.3.9.

(b) *Every tame Kuranishi cobordism* $\mathcal{K}^{[0,1]}$ *has a cobordism reduction* $\mathcal{V}^{[0,1]}$.

(c) *Let* $\mathcal{V}^0, \mathcal{V}^1$ *be reductions of a tame Kuranishi atlas* $\mathcal{K}$. *Then there exists a cobordism reduction* $\mathcal{V}$ *of* $\mathcal{K} \times [0,1]$ *such that* $\partial^\alpha \mathcal{V} = \mathcal{V}^\alpha$ *for* $\alpha = 0, 1$.

Example 2.4.4. A reduction of the atlas $\mathcal{K}$ in Example 2.3.7 has three sets V_1, V_2, V_{12} that cover the zero set and have the property that $\pi_{\mathcal{K}}(V_1) \cap \pi_{\mathcal{K}}(V_2) = \emptyset$. For instance, we can take $V_1 = (-\infty, 2) \sqsubset U_1, V_2 = (3,\infty)\times\mathbb{R} \sqsubset U_2$ and $V_{12} = (1,3)\times\mathbb{R} \subset U_{12}(= U_2)$

Remark 2.4.5. Sets of the form $\pi_{\mathcal{K}}(\mathcal{V})$ contain the zero set $\iota_{\mathcal{K}}(X)$ and are the analog in the virtual neighbourhood $|\mathcal{K}|$ of "precompact neighbourhoods of the zero set". Since $|\mathcal{K}|$ is not locally compact even in the metric topology (cf. Example 2.3.7), there are no compact neighbourhoods of the zero set. On the other hand, because $V_I \sqsubset U_I$, the subset $\pi_{\mathcal{K}}(\mathcal{V})$ is precompact in $|\mathcal{K}|$, and is "open" to the extent that it is the image by $\pi_{\mathcal{K}}$ of the open set $\bigsqcup_I V_I$. (But, of course, it is not open.) We can interpret Figure 2.4.1 as a schematic picture of the subsets $\pi_{\mathcal{K}}(V_I)$ in $|\mathcal{K}|$, though it is not very accurate since the dimensions of the V_I change. Notice that there are points $x \in V_1 \cap U_{12}$ whose image under $\phi_{1,12}$ lies in $U_{12}\smallsetminus V_{12}$, so that $\pi_{\mathcal{K}}(\mathcal{V})$ intersects a neighbourhood of $\pi_{\mathcal{K}}(x)$ in $|\mathcal{K}|$ in the proper submanifold $\pi_{\mathcal{K}}(U_{12})$ of $\pi_{\mathcal{K}}(U_2)$.) Another good property of $\mathcal{V}$ is that if any intersection $\pi_{\mathcal{K}}(V_I) \cap \pi_{\mathcal{K}}(V_J)$ is nonempty then we always have $I \subset J$ or $J \subset I$ so that the intersection is a the image of a submanifold. Contrast this with the equivalence relation for by a tame atlas which, as explained in Remark 2.2.7 (ii), is given by a two step process.

We now introduce the notion of a section from [MW12, §7.2].

Definition 2.4.6. *A* **reduced section** *of* $\mathcal{K}$ *is a smooth functor* $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ *between the reduced domain and obstruction categories of some reduction* $\mathcal{V}$ *of* $\mathcal{K}$, *such that* $\mathrm{pr}_{\mathcal{K}} \circ \nu$ *is the identity functor. That is,* $\nu = (\nu_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ *is given by a family of smooth maps* $\nu_I : V_I \to E_I$ *such that for each* $I \subsetneq J$ *we have a commuting diagram*

$$\begin{array}{ccc} V_I \cap \phi_{IJ}^{-1}(V_J) & \xrightarrow{\nu_I} & E_I \\ {\scriptstyle \phi_{IJ}}\downarrow & & \downarrow{\scriptstyle \widehat{\phi}_{IJ}} \\ V_J & \xrightarrow{\nu_J} & E_J. \end{array}$$

We say that a reduced section ν *is an* **admissible perturbation** *of* $s_{\mathcal{K}}|_{\mathcal{V}}$ *if*

$$\mathrm{d}_y \nu_J(\mathrm{T}_y V_J) \subset \mathrm{im}\,\widehat{\phi}_{IJ} \qquad \forall\, I \subsetneq J,\ y \in V_J \cap \phi_{IJ}(V_I \cap U_{IJ}). \tag{2.4.1}$$

Each reduced section $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ induces a continuous map $|\nu| : |\mathcal{V}| \to |\mathbf{E}_{\mathcal{K}}|$ such that $|\mathrm{pr}_{\mathcal{K}}| \circ |\nu| = \mathrm{id}$, where $|\mathrm{pr}_{\mathcal{K}}|$ is as in Theorem 2.3.1. Each such map has the further property that $|\nu|\big|_{\pi_{\mathcal{K}}(V_I)}$ takes values in $\pi_{\mathcal{K}}(U_I \times E_I)$. Note that the zero section $0_{\mathcal{K}}$, given by $U_I \to 0 \in E_I$, restricts to an admissible perturbation $0_{\mathcal{V}} : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ in the sense of the above definition. Similarly, the canonical section $s := s_{\mathcal{K}}$ of the Kuranishi atlas restricts to a section $s|_{\mathcal{V}} : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ of any reduction. However, the canonical section is generally not admissible. In fact, it follows from the index condition that for all $y \in V_J \cap \phi_{IJ}(V_I \cap U_{IJ})$ the map

$$\mathrm{pr}^{\perp}_{E_I} \circ \mathrm{d}_y s_J \;:\; {}^{\mathrm{T}_y U_J}\Big/_{\mathrm{T}_y(\phi_{IJ}(U_{IJ}))} \;\longrightarrow\; {}^{E_J}\Big/_{\widehat{\phi}_{IJ}(E_I)}$$

is an isomorphism, while for an admissible section it is identically zero. So for any reduction $\mathcal{V}$ and admissible perturbation ν, the sum

$$s + \nu := (s_I|_{V_I} + \nu_I)_{I \in \mathcal{I}_{\mathcal{K}}} \;:\; \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \;\to\; \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}} \tag{2.4.2}$$

is a reduced section that satisfies the index condition.

Here are some further definitions.

- We say that two reductions $\mathcal{C}, \mathcal{V}$ are **nested** and write $\mathcal{C} \sqsubset \mathcal{V}$ if $C_I \sqsubset V_I$ for all $I \in \mathcal{I}_{\mathcal{K}}$. One can show that any two such pairs $\mathcal{C}_0 \sqsubset \mathcal{V}_0, \mathcal{C}_1 \sqsubset \mathcal{V}_1$ are cobordant via a nested cobordism $\mathcal{C}^{01} \sqsubset \mathcal{V}^{01}$.
- A section $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ is called **precompact** if there is a nested reduction $\mathcal{C} \sqsubset \mathcal{V}$ such that
$$\pi_{\mathcal{K}}\big((s+\nu)^{-1}(0)\big) \;\subset\; \pi_{\mathcal{K}}(\mathcal{C}).$$
- It is called **transverse** if for all $z \in V_I \cap (s_I|_{V_I} + \nu_I)^{-1}(0)$ the map $\mathrm{d}_z(s_I + \nu_I) : TU_I \to E_I$ is surjective.

It is not hard to see using (2.4.2) that if ν is admissible, transversality of the local sections $s_I|_{V_I} + \nu_I$ is preserved under coordinate changes. More precisely, if $z \in V_I \cap U_{IJ}$ and $w \in V_J$ are such that $\phi_{IJ}(z) = w$, then z is a transverse zero of $s_I|_{V_I} + \nu_I$ if and only if w is a transverse zero of $s_J|_{V_J} + \nu_J$. Here is the main result about the zero sets, from [MW12, Proposition 7.2.7].

Proposition 2.4.7. *Let $\mathcal{K}$ be a tame d-dimensional Kuranishi atlas with trivial isotropy and a reduction $\mathcal{V} \sqsubset \mathcal{K}$, and suppose that $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ is a precompact transverse perturbation. Then $|\mathbf{Z}_\nu| = |(s+\nu)^{-1}(0)|$ is a smooth closed d-dimensional manifold. Moreover, its quotient topology agrees with the subspace topology on $|(s+\nu)^{-1}(0)| \subset |\mathcal{K}|$.*

The only tricky part of the proof is to show compactness. But this holds because by assumption the zero set maps into the precompact subset $\pi_{\mathcal{K}}(\mathcal{C})$ of $|\mathcal{K}|$. There are similar result for cobordisms. Thus the main remaining problem is to construct suitable sections ν. Even though the constructions are fiddly, the statements of the main results in Propositions 2.4.10 and 2.4.11 below are very precise. Also our language gives us great control over all aspects of the construction, so that it can be adapted for example to other situations.

The construction involves the choice of an **admissible metric** d on $|\mathcal{K}|$ as in Definition 2.3.9, i.e. a metric whose pullback d_I to each domain U_I is compatible with its topology. We denote the δ-neighbourhoods of subsets $Q \subset |\mathcal{K}|$ resp. $A \subset U_I$ for $\delta > 0$ by

$$B_\delta(Q) \;:=\; \big\{w \in |\mathcal{K}| \mid \exists q \in Q : d(w,q) < \delta\big\},$$
$$B^I_\delta(A) \;:=\; \big\{x \in U_I \mid \exists a \in A : d_I(x,a) < \delta\big\}.$$

Note that $\phi_{IJ}(B^I_\delta(A)) = \operatorname{im}\phi_{IJ}\big(B^J_\delta(\phi_{IJ}(A)\big)$ because all coordinate changes are isometries. Similarly $U_I \cap \pi_{\mathcal K}(B_\delta(Q)) = B^I_\delta\big(U_I \cap \pi_{\mathcal K}^{-1}(Q)\big)$.

The situation is this. We are given a nested reduction $\mathcal C \sqsubset \mathcal V$ of a metric tame Kuranishi structure $(\mathcal K, d)$, and want to construct an admissible and transverse section $\nu : \mathbf{B}_{\mathcal K}|_{\mathcal V} \to \mathbf{E}_{\mathcal K}|_{\mathcal V}$ whose zero set projects into $\pi_{\mathcal K}(\mathcal C)$. We do this by an intricate induction in which we construct suitable functions ν_I on sets slightly larger than V_I. Thus we consider a decreasing sequence of nested reductions $\mathcal V^k := (V^k_I)_{I\in\mathcal I_{\mathcal K}} \sqsupset \mathcal V^{k+1} := (V^{k+1}_I)_{I\in\mathcal I_{\mathcal K}}$, where

$$V^k_I \;:=\; B^I_{2^{-k}\delta}(V_I) \;\sqsubset\; U_I \qquad \text{for } k \geq 0, \tag{2.4.3}$$

and $\delta > 0$ is chosen so that

$$B_\delta\big(\pi_{\mathcal K}(V^k_I)\big) \cap B_\delta\big(\pi_{\mathcal K}(V^k_J)\big) \subset B_{\delta+2^{-k}\delta}\big(\pi_{\mathcal K}(V_I)\big) \cap B_{\delta+2^{-k}\delta}\big(\pi_{\mathcal K}(V_J)\big) = \emptyset. \tag{2.4.4}$$

This implies that when $I \subsetneq J$,

$$\begin{aligned} V^k_I \cap \pi_{\mathcal K}^{-1}(\pi_{\mathcal K}(V^k_J)) \;&=\; V^k_I \cap \phi_{IJ}^{-1}(V^k_J),\\ V^k_J \cap \pi_{\mathcal K}^{-1}(\pi_{\mathcal K}(V^k_I)) \;&=\; V^k_J \cap \phi_{IJ}(V^k_I \cap U_{IJ}) \;=:\; N^k_{JI} \end{aligned} \tag{2.4.5}$$

for the sets on which we will require compatibility of the perturbations ν_I and ν_J. Similarly, we have precompact inclusions for any $k' > k \geq 0$

$$N^{k'}_{JI} \;=\; V^{k'}_J \cap \phi_{IJ}(V^{k'}_I \cap U_{IJ}) \;\sqsubset\; V^k_J \cap \phi_{IJ}(V^k_I \cap U_{IJ}) \;=\; N^k_{JI}. \tag{2.4.6}$$

We abbreviate

$$N^k_J \;:=\; \textstyle\bigcup_{J\supsetneq I} N^k_{JI} \;\subset\; V^k_J,$$

and call the union $N^{|J|}_J$ the **core** of $V^{|J|}_J$, since it is the part of this set on which we will prescribe ν_J by compatibility with the ν_I for $I \subsetneq J$. We then define constants $\delta_{\mathcal V} > 0$ and $\sigma(\delta, \mathcal V, \mathcal C) > 0$ that depend only on the indicated data as follows.

Definition 2.4.8. *Given a reduction $\mathcal V$ of a metric Kuranishi atlas $(\mathcal K, d)$, we set $\delta_{\mathcal V} > 0$ to be the maximal constant such that any $2\delta < 2\delta_{\mathcal V}$ satisfies*

$$B_{2\delta}(V_I) \sqsubset U_I \qquad \forall I \in \mathcal I_{\mathcal K}, \tag{2.4.7}$$

$$B_{2\delta}(\pi_{\mathcal K}(\overline{V_I})) \cap B_{2\delta}(\pi_{\mathcal K}(\overline{V_J})) \neq \emptyset \qquad \Longrightarrow \qquad I \subset J \text{ or } J \subset I. \tag{2.4.8}$$

Given a nested reduction $\mathcal C \sqsubset \mathcal V$ of a metric Kuranishi atlas $(\mathcal K, d)$ and $0 < \delta < \delta_{\mathcal V}$, we set

$$\eta_0 := (1 - 2^{-\frac14})\delta, \qquad \eta_{|J|-\frac12} := 2^{-|J|+\frac12}\eta_0$$

and

$$\sigma(\delta, \mathcal V, \mathcal C) \;:=\; \min_{J\in\mathcal I_{\mathcal K}} \inf\Big\{ \, \|s_J(x)\| \;\Big|\; x \in \overline{V^{|J|}_J} \smallsetminus \Big(\widetilde C_J \cup \textstyle\bigcup_{I\subsetneq J} B^J_{\eta_{|J|-\frac12}}\big(N^{|J|-\frac14}_{JI}\big)\Big)\Big\}, \tag{2.4.9}$$

where

$$\widetilde C_J \;:=\; \textstyle\bigcup_{K\supset J} \phi_{JK}^{-1}(C_K) \;\subset\; U_J.$$

Note that $\sigma(\delta, \mathcal V, \mathcal C) > 0$ by [MW12, Lemma 7.3.2]. Here is a slightly shortened version of [MW12, Definition 7.3.3].

Definition 2.4.9. *Given a nested reduction $\mathcal{C} \sqsubset \mathcal{V}$ of a metric tame Kuranishi atlas $(\mathcal{K}, d)$ and constants $0 < \delta < \delta_{\mathcal{V}}$ and $0 < \sigma \le \sigma(\delta, \mathcal{V}, \mathcal{C})$, we say that a perturbation ν of $s_{\mathcal{K}}|_{\mathcal{V}}$ is $(\mathcal{V}, \mathcal{C}, \delta, \sigma)$-**adapted** if the sections $\nu_I : V_I \to E_I$ extend to sections over $V_I^{|I|}$ (also denoted ν_I) so that the following conditions hold for every $k = 1, \dots, M_{\mathcal{K}}$ with*

$$M_{\mathcal{K}} := \max_{I \in \mathcal{I}_{\mathcal{K}}} |I|, \qquad \eta_k := 2^{-k}\eta_0 = 2^{-k}(1 - 2^{-\frac{1}{4}})\delta.$$

a) *The perturbations are compatible on $\bigcup_{|I| \le k} V_I^k$, that is*

$$\nu_I \circ \phi_{HI}|_{V_H^k \cap \phi_{HI}^{-1}(V_I^k)} \;=\; \widehat{\phi}_{HI} \circ \nu_H|_{V_H^k \cap \phi_{HI}^{-1}(V_I^k)} \qquad \text{for all } H \subsetneq I, |I| \le k.$$

b) *The perturbed sections are transverse, that is $(s_I|_{V_I^k} + \nu_I) \pitchfork 0$ for each $|I| \le k$.*

c) *The perturbations are strongly admissible with radius η_k, that is for all $H \subsetneq I$ and $|I| \le k$ we have*

$$\nu_I(B^I_{\eta_k}(N^k_{IH})) \;\subset\; \widehat{\phi}_{HI}(E_H) \qquad \text{with } \; N^k_{IH} = V_I^k \cap \phi_{HI}(V_H^k \cap U_{HI}).$$

d) *The perturbed zero sets are contained in $\pi_{\mathcal{K}}^{-1}\big(\pi_{\mathcal{K}}(\mathcal{C})\big)$; more precisely $s_I + \nu_I \ne 0$ on $V_I^k \smallsetminus \pi_{\mathcal{K}}^{-1}\big(\pi_{\mathcal{K}}(\mathcal{C})\big)$.*

e) *The perturbations are small, that is $\sup_{x \in V_I^k} \|\nu_I(x)\| < \sigma$ for $|I| \le k$.*

The above conditions are more than needed to ensure that every $(\mathcal{V}, \mathcal{C}, \delta, \sigma)$-adapted perturbation ν of $s_{\mathcal{K}}|_{\mathcal{V}}$ is an admissible, transverse perturbation with $\pi_{\mathcal{K}}((s+\nu)^{-1}(0)) \subset \pi_{\mathcal{K}}(\mathcal{C})$. In fact, the definition of σ and condition (e) imply that the zero set of $s_I|_{V_I^{|I|}} + \nu_I$ must either lie in $\widetilde{C}_I$ and hence project to $\pi_{\mathcal{K}}(\mathcal{C})$ or lie in the extended core, and hence project to $\pi_{\mathcal{K}}(\mathcal{C})$ by the inductive nature of the construction.

We now explain the argument that such perturbations ν exist. For full details see [MW12, Proposition 7.3.5].

Proposition 2.4.10. *Let $(\mathcal{K}, d)$ be metric tame Kuranishi atlas with nested reduction $\mathcal{C} \sqsubset \mathcal{V}$. Then for any $0 < \delta < \delta_{\mathcal{V}}$ and $0 < \sigma \le \sigma(\delta, \mathcal{V}, \mathcal{C})$ there exists a $(\mathcal{V}, \mathcal{C}, \delta, \sigma)$-adapted perturbation ν of $s_{\mathcal{K}}|_{\mathcal{V}}$.*

Proof. The construction is by an inductive process that constructs the required sections ν_I on sets larger than V_I. Namely, this proposition constructs functions $\nu_I : V_I^{|I|} \to E_I$ by an iteration over $k = 0, \dots, M = \max_{I \in \mathcal{I}_{\mathcal{K}}} |I|$, where in step k we will define $\nu_I : V_I^k \to E_I$ for all $|I| = k$ that, together with the $\nu_I|_{V_I^k}$ for $|I| < k$ obtained by restriction from earlier steps, satisfy conditions a)-e) of Definition 2.4.9. Restriction to $V_I \subset V_I^{|I|}$ then yields a $(\mathcal{V}, \mathcal{C}, \delta, \sigma)$-adapted perturbation ν of $s_{\mathcal{K}}|_{\mathcal{V}}$. A key point in the construction is that because the different sets $V_I^{|I|}, |I| = k$, are disjoint (by (2.4.4)), at the kth step the needed functions ν_I can be constructed independently of each other.

Assume inductively that suitable $\nu_I : V_I^{|I|} \to E_I$ have been found for $|I| \le k$, and consider the construction of ν_J for some J with $|J| = k+1$. We construct ν_J as a sum $\widetilde{\nu}_J + \nu_{\pitchfork}$ where

- $\widetilde{\nu}_J|_{N_J^{k+1}} = \mu_J|_{N_J^{k+1}}$ where $\mu_J : N_J^k \to E_J$ is defined on the extended core N_J^k by the compatibility conditions, and
- $\nu_\pitchfork$ is a final perturbation chosen so as to achieve transversality.

We construct the extension $\widetilde{\nu}_J$ by extended each component $\mu_J^j, j \in J$, of μ_J separately. In turn, we construct each μ_J^j by an elaborate iterative process over the increasing family of sets $W_\ell, 1 \le \ell \le k$, defined in equation (7.3.19) of [MW12]. Here W_ℓ is a carefully chosen neighbourhood of the part $\bigcup_{|H|\le \ell} N_{JH}^{k+\frac12}$ of the core $N_J^{k+\frac12}$ defined by sets H with $|H| \le \ell$. In particular, when $\ell = k$ we define

$$W_k := B^J_{\eta_{k+\frac12}}(N_J^{k+\frac12}) =: W_J. \tag{2.4.10}$$

Thus, omitting J from the notation, we need to construct for $1 \le \ell \le k$ functions $\widetilde{\mu}_\ell^j : W_\ell \to E_j$ that extend μ_J^j and satisfy certain vanishing and size conditions that will guarantee (a-e). Again simplifying by omitting j from the notation, it turns out to suffice to construct the extension $\widetilde{\mu}_\ell$ on a certain set B'_ℓ, that is a union of disjoint sets B'_L, one for each $L \subset J$ with $|L| = \ell$. For each L, we localize the latter extension problem, reducing it to the construction of extensions $\widetilde{\mu}_z$ near each point z in the set B'_L defined in [MW12, Equation (7.3.20)], that we then sum up using a partition of unity. In most cases we can choose $\widetilde{\mu}_z$ either to be zero or to be given by the compatibility conditions. In fact, the only case in which this extension is nontrivial is when z is in the core, more precisely the case $z \in \overline{N_{JL}^{k+\frac12}}$. In that case we define $\widetilde{\mu}_z$ by extending $\mu_J^j|_{B^J_{r_z}(z)\cap N^k_{JL}}$ to be "constant in the normal directions".

When these extensions have all been constructed for $k \le \ell$ and $j \in J$, we define

$$\widetilde{\nu}_J := \beta \cdot \left(\textstyle\sum_{j\in J} \widetilde{\mu}_k^j\right), \tag{2.4.11}$$

where $\beta : U_J \to [0,1]$ is a smooth cutoff function with $\beta|_{N_J^{k+\frac12}} \equiv 1$ and $\operatorname{supp}\beta \subset W_J$, so that $\widetilde{\nu}_J$ extends trivially to $U_J \smallsetminus W_J$. Here are some important points.

(A) By [MW12, (7.3.9)] the constants η_k are chosen so that $W_J \cap N_J^k$ is compactly contained in $N_J^{k+\frac14}$. Further, by the discussion proving [MW12, (7.3.21)][8] we also know that $s_J^j \ne 0$ on $cl\big(B^J_{\eta_{k+\frac12}}(N_{JI}^{k+\frac12})\big)\smallsetminus N^k_{JI}$ whenever $j \notin I, I \subsetneq J$. The latter condition gives control over zero sets as in (D) below, for all sufficiently small perturbations $\widetilde{\nu}_J$ satisfying the admissibility conditions in (C).

(B) The section $s_J + \widetilde{\nu}_J$ is transverse to 0 on both $B := B^J_{\eta_{k+\frac12}}(N_J^{k+\frac34}) \subset W_J$ and on $N_J^{k+\frac12}$.

[8]This argument actually concerns the open set $B^J_{\eta_{k+\frac12}}(N_{JI}^{k+\frac12})$ rather than its closure. However, because the inclusions $V_J^k \subset V_J^0$ are precompact, it applies equally well to the closure.

(C) The following strong admissibility condition holds: if $I \subsetneq J$ and $j \in J \smallsetminus I$ then $\widetilde{\mu}_J^j = 0$ on $B^J_{\eta_{k+\frac12}}(N_{JI}^{k+\frac12}) \subset W_J$ and on $N_{JI}^{|J|-1}$.

(D) For any section $\widetilde{\nu}_J$ with support in W_J, $\|\widetilde{\nu}_J\| < \sigma$, and satisfying the admissibility condition (C), we have

$$V_J^{k+1} \cap (s_J + \widetilde{\nu}_J)^{-1}(0) \subset N_J^{k+\frac14} \cup (\widetilde{C}_J \smallsetminus \overline{W_J}).$$

The set B in (B) above compactly contains the neighbourhood $B' := B^J_{\eta_{k+1}}(N_J^{k+1})$ of the core $N := N_J^{|J|}$ on which compatibility requires $\nu_J|_N = \mu_J|_N = \widetilde{\nu}_J|_N$.

At this stage conditions a), c), d), e) hold, so that we only need work to achieve transversality b) while keeping $\nu_\pitchfork$ so small that e) and hence d) remain true. We first choose a relatively open subset $W \subset \overline{V_J^{k+1}} \smallsetminus B'$ so that $(s_J + \widetilde{\nu}_J)^{-1}(0) \cap W \subset O := \overline{V_J^{k+1}} \cap \widetilde{C}_J$, which is possible by (D) and the fact $B' \sqsubset B$. Because transversality holds on B' by (B), there is an open precompact subset P of W so that transversality holds on $W \smallsetminus P$. Finally we choose $\nu_\pitchfork$ to be a very small smooth function with values in E_J that achieves transversality and is such that $(s_J + \widetilde{\nu}_J + \nu_\pitchfork)^{-1}(0) \subset O \subset \widetilde{C}_J$. This completes the inductive step, and hence the construction of ν. $\square$

To show that different choices lead to cobordant zero sets we also need a relative version of this construction. Here the constant $\sigma_{\rm rel}(\delta, \mathcal{V}, \mathcal{C})$ depends on the given data, and in particular on the constants $\sigma(\delta, \mathcal{V}^\alpha, \mathcal{C}^\alpha), \alpha = 0,1$ that occur in Proposition 2.4.10. (See [MW12, Definition 7.3.6] for a precise formula.)

Proposition 2.4.11. *Let $(\mathcal{K}, d)$ be a metric tame Kuranishi cobordism with nested cobordism reduction $\mathcal{C} \sqsubset \mathcal{V}$, let $0 < \delta < \min\{\varepsilon, \delta_{\mathcal{V}}\}$, where ε is the collar width of $(\mathcal{K}, d)$ and the reductions $\mathcal{C}, \mathcal{V}$. Then we have $\sigma_{\rm rel}(\delta, \mathcal{V}, \mathcal{C}) > 0$ and the following holds.*

(i) *Given any $0 < \sigma \le \sigma_{\rm rel}(\delta, \mathcal{V}, \mathcal{C})$, there exists an admissible, precompact, transverse cobordism perturbation ν of $s_{\mathcal{K}}|_{\mathcal{V}}$ with $\pi_{\mathcal{K}}\big((s+\nu)^{-1}(0)\big) \subset \pi_{\mathcal{K}}(\mathcal{C})$, whose restrictions $\nu|_{\partial^\alpha \mathcal{V}}$ for $\alpha = 0,1$ are $(\partial^\alpha \mathcal{V}, \partial^\alpha \mathcal{C}, \delta, \sigma)$-adapted perturbations of $s_{\partial^\alpha \mathcal{K}}|_{\partial^\alpha \mathcal{V}}$.*

(ii) *Given any perturbations ν^α of $s_{\partial^\alpha \mathcal{K}}|_{\partial^\alpha \mathcal{V}}$ for $\alpha = 0,1$ that are $(\partial^\alpha \mathcal{V}, \partial^\alpha \mathcal{C}, \delta, \sigma)$-adapted with $\sigma \le \sigma_{\rm rel}(\delta, \mathcal{V}, \mathcal{C})$, the perturbation ν of $s_{\mathcal{K}}|_{\mathcal{V}}$ in (i) can be constructed to have boundary values $\nu|_{\partial^\alpha \mathcal{V}} = \nu^\alpha$ for $\alpha = 0,1$.*

This is proved by making minor modifications in the construction given above.

Proof of Theorem B. For this, we must discuss orientations both on Kuranishi atlases and on Kuranishi cobordisms. This is done by constructing two versions of the determinant line bundle over $\mathcal{K}$, one that restricts on a chart to the line $\Lambda^{\max}(\mathrm{T}U_I) \otimes \big(\Lambda^{\max}(E_I)\big)^*$ and the other with restriction given by the determinant bundle of ${\rm d}s_I : \mathrm{T}U_I \to E_I$ as defined in [MS, Theorem A.2.2].[9] We say that $\mathcal{K}$ is oriented if this line bundle has a nonvanishing section, and show in [MW12, Proposition 7.4.13] that an

[9]One must take care when defining the effect of coordinate changes using the second definition; the orderings chosen in [MS, Exercise A.2.3] are inconsistent. For full details see [MW12, §7.4].

orientation of $\mathcal{K}$ induces an orientation on any zero set of the form $|\mathbf{Z}_\nu|$. The upshot is that each ν determines an oriented closed manifold $|\mathbf{Z}_\nu|$ whose oriented cobordism class is independent of all choices. Moreover because $|\mathbf{Z}_\nu|$ lies in a δ-neighbourhood of the zero set $|s|^{-1}(0) = \iota_{\mathcal{K}}(X)$ of $|\mathcal{K}|$, we get a well defined element in the Čech homology group $\check{H}_d(X;\mathbb{Q})$ by taking an inverse limit. For more details see [MW12, §7.5].

3. Kuranishi atlases with nontrivial isotropy

The main change in this case is that the domains of the charts are no longer smooth manifolds, but rather group quotients (U,Γ) where Γ is a finite group acting on U. We will begin by assuming that U is smooth, considering more general domains in §3.3. Our definitions are chosen so that the quotient of a Kuranishi chart $\mathbf{K}$ with isotropy group Γ can be thought of as a Kuranishi chart $\underline{\mathbf{K}}$ with trivial isotropy that we call the **intermediate** chart. Similarly the quotient of a weak Kuranishi atlas by its finite isotropy groups is (essentially) a weak Kuranishi atlas without isotropy. This means that we can apply the taming procedures explained above to tame the intermediate weak Kuranishi atlas, and then lift this to a taming of $\mathcal{K}$.

The other key new idea is that the coordinate changes $\mathbf{K}_I \to \mathbf{K}_J$ should no longer be given by inclusions ϕ_{IJ} of an open subset $U_{IJ} \subset U_I$ into U_J. These inclusions exist on the intermediate level as $\underline{\phi}_{IJ} : \underline{U}_{IJ} \to \underline{U}_J$. However, it is the inverse $\underline{\phi}_{IJ}^{-1}$ that lifts to the charts themselves: there is a Γ_J-invariant submanifold $\widetilde{U}_{IJ} \subset s_J^{-1}(E_I)$ on which the kernel of the natural projection $\Gamma_J \to \Gamma_I$ acts freely with quotient homeomorphic to a Γ_I-invariant subset U_{IJ} of U_I. This gives a **covering map** $\rho_{IJ} : \widetilde{U}_{IJ} \to U_{IJ} \subset U_I$ that descends on the intermediate level to the inverse $\underline{\phi}_{IJ}^{-1}$ of $\underline{\phi}_{IJ}$. In the Gromov–Witten setting, these maps ρ_{IJ} occur very naturally as maps that forget certain sets of added marked points. (Cf. the end of Lecture 2 in [M14], and §5 below.)

Thus, most of the proofs are routine generalizations of those in §2; the only real difficulty is to make appropriate definitions. This section therefore consists mostly of notation and definitions. The main reference is [MW14], still under construction.

3.1. **Kuranishi atlases.**

Definition 3.1.1. *A* **group quotient** *is a pair (U,Γ) consisting of a smooth manifold U and a finite group Γ together with a smooth action $\Gamma \times U \to U$. We will denote the quotient space by*

$$\underline{U} := {}^{U}\!/_{\Gamma},$$

giving it the quotient topology, and write $\pi : U \to \underline{U}$ for the associated projection. Moreover, we denote the **stabilizer** *of each $x \in U$ by*

$$\mathrm{Stab}_x := \{\gamma \in \Gamma \,|\, \gamma x = x\} \subset \Gamma.$$

Both the basic and transition charts of Kuranishi atlases will be group quotients, related by coordinate changes that are composites of the following kinds of maps.

Definition 3.1.2. *Let $(U,\Gamma),(U',\Gamma')$ be group quotients. A* **group embedding**

$$(\phi,\phi^\Gamma) : (U,\Gamma) \to (U',\Gamma')$$

is a smooth embedding $\phi : U \to U'$ that is equivariant with respect to an injective group homomorphism $\phi^\Gamma : \Gamma \to \Gamma'$ and induces an injection $\underline{\phi} : \underline{U} \to \underline{U}'$ on the quotient spaces.

In a Kuranishi atlas we often consider embeddings $(\phi, \phi^\Gamma) : (U, \Gamma) \to (U', \Gamma)$ where $\dim U < \dim U'$ and $\phi^\Gamma : \Gamma \to \Gamma' := \Gamma$ is the identity map. On the other hand, group quotients of the same dimension are usually related either by restriction or by coverings as follows.

Definition 3.1.3. *Let (U, Γ) be a group quotient and $\underline{V} \subset \underline{U}$ an open subset. Then the* **restriction of** (U, Γ) **to** $\underline{V}$ *is the group quotient $(\pi^{-1}(\underline{V}), \Gamma)$.*

Note that the inclusion $\pi^{-1}(\underline{V}) \to U$ induces an equidimensional group embedding $(\pi^{-1}(\underline{V}), \Gamma) \to (U, \Gamma)$ that covers the inclusion $\underline{V} \to \underline{U}$. The third kind of map that occurs in a coordinate change is a group covering. This notion is less routine; notice in particular the requirement in (ii) that $\ker \rho^\Gamma$ act freely. Further, the two domains $\widetilde{U}, U$ will necessarily have the same dimension since they are related by a regular covering ρ.

Definition 3.1.4. *Let (U, Γ) be a group quotient. A* **group covering** *of (U, Γ) is a tuple $(\widetilde{U}, \widetilde{\Gamma}, \rho, \rho^\Gamma)$ consisting of*

- (i) *a surjective group homomorphism $\rho^\Gamma : \widetilde{\Gamma} \to \Gamma$,*
- (ii) *a group quotient $(\widetilde{U}, \widetilde{\Gamma})$ where $\ker \rho^\Gamma$ acts freely,*
- (iii) *a regular covering $\rho : \widetilde{U} \to U$ that is the quotient map $\widetilde{U} \to {}^{\widetilde{U}}\!/\!_{\ker \rho^\Gamma}$ composed with a diffeomorphism ${}^{\widetilde{U}}\!/\!_{\ker \rho^\Gamma} \cong U$ that is equivariant with respect to the induced $\Gamma = \operatorname{im} \rho^\Gamma$ action on both spaces.*

Thus $\rho : \widetilde{U} \to U$ is equivariant with respect to $\rho^\Gamma : \widetilde{\Gamma} \to \Gamma$. We denote by $\underline{\rho} : \underline{\widetilde{U}} \to \underline{U}$ the induced map on quotients.

Here is an elementary but important lemma ([MW14, Lemma 2.1.6]). (Part (ii) is well known from orbifold theory.)

Lemma 3.1.5. *Let (U, Γ) be a group quotient.*

- (i) *The projection $\pi : U \to \underline{U}$ is open and proper (i.e. the inverse image of a compact set is compact).*
- (ii) *Every point $x \in U$ has a neighbourhood U_x that is invariant under Stab_x and is such that inclusion $U_x \hookrightarrow U$ induces a homeomorphism from ${}^{U_x}\!/\!_{\mathrm{Stab}_x}$ to $\pi(U_x)$. In particular, the inclusion $(U_x, \mathrm{Stab}_x) \to (U, \Gamma)$ is a group embedding.*
- (iii) *If $(\widetilde{U}, \widetilde{\Gamma}, \rho, \rho^\Gamma)$ is a group covering of (U, Γ), then $\underline{\rho} : \underline{\widetilde{U}} \to \underline{U}$ is a homeomorphism.*

Definition 3.1.6. *A Kuranishi chart for X is a tuple $\mathbf{K} = (U, E, \Gamma, s, \psi)$ made up of*

- *the* **domain** *U which is a smooth finite dimensional manifold;*
- *a finite dimensional vector space E called the* **obstruction space***;*
- *a finite* **isotropy group** *Γ with a smooth action on U and a linear action on E;*

- *a smooth* Γ*-equivariant function* $s : U \to E$*, called the* **section***;*
- *a continuous map* $\psi : s^{-1}(0) \to X$ *that induces a homeomorphism*

$$\underline{\psi} : \underline{s^{-1}(0)} := {}^{s^{-1}(0)}\!/_{\Gamma} \;\to\; F$$

with open image $F \subset X$*, called the* **footprint** *of the chart.*

The **dimension** *of* $\mathbf{K}$ *is* $\dim(\mathbf{K}) := \dim U - \dim E$*, and we will say that the chart is*

- **minimal** *if there is a point* $x \in s^{-1}(0)$ *at which* ψ *is injective, i.e.* $x = \psi^{-1}(\psi(x))$*, or equivalently* $\Gamma x = x$*;*
- **effective** *if the diagonal action on* $U' \times E$ *is effective for any* Γ*-invariant open subset* $U' \subset U$*.*

In order to extend topological constructions from §2 to the case of nontrivial isotropy, we will also consider the following notion of intermediate Kuranishi charts which have trivial isotropy but less smooth structure.

Definition 3.1.7. *We associate to each Kuranishi chart* $\mathbf{K} = (U, E, \Gamma, s, \psi)$ *the* **intermediate chart** $\underline{\mathbf{K}} := (\underline{U}, \underline{U \times E}, \underline{S}, \underline{\psi})$*, where* $\underline{U \times E}$ *is the quotient by the diagonal action of* Γ *and* $\underline{S}$ *is the section of the bundle* $\underline{\mathrm{pr}} : \underline{U \times E} \to \underline{U}$ *induced by* $S = \mathrm{id}_U \times s : U \to U \times E$*.*

We view $\underline{\mathbf{K}}$ as a Kuranishi chart with trivial isotropy group as in Definition 2.1.1, with the exception that $\underline{\mathrm{pr}} : \underline{U \times E} \to \underline{U}$ is an orbibundle [10] rather than a trivialized vector bundle. We write $\pi : U \to \underline{U} := {}^{U}\!/_{\Gamma}$ for the projection from the Kuranishi domain to the intermediate domain, and will distinguish everything connected to the intermediate charts by underlines. Moreover if a chart $\mathbf{K}_I = (U_I, E_I, \Gamma_I, s_I, \psi_I)$ has the label I, the corresponding projection is denoted $\pi_I : U_I \to \underline{U}_I$.

We will find that many results (in particular the taming constructions) from §2.1 carry over to nontrivial isotropy via the intermediate charts, since precompact subsets of $\underline{U}$ lift to precompact subsets of U by Lemma 3.1.5 (i). An important exception is the construction of perturbations which must be done on the smooth spaces U rather than on their quotients $\underline{U}$.

Definition 3.1.8. *Let* $\mathbf{K} = (U, E, \Gamma, s, \psi)$ *be a Kuranishi chart and* $F' \subset F$ *an open subset of its footprint. A* **restriction of K to** $\boldsymbol{F'}$ *is a Kuranishi chart of the form*

$$\mathbf{K}' = \mathbf{K}|_{\underline{U}'} := \big(U', E, \Gamma, s' = s|_{U'}\,,\, \psi' = \psi|_{s'^{-1}(0)}\big), \qquad U' = \pi^{-1}(\underline{U}')$$

given by a choice of open subset $\underline{U}' \subset \underline{U}$ *such that* $\underline{U}' \cap \underline{\psi}^{-1}(F) = \underline{\psi}^{-1}(F')$*. We call* $\underline{U}'$ *the* **intermediate domain** *of the restriction and* U' *its* **domain**.

Note that the restriction $\mathbf{K}'$ in the above definition has footprint $\psi'(s'^{-1}(0)) = F'$, and its domain group quotient (U', Γ) is the restriction of (U, Γ) to $\underline{U}'$ in the sense of Definition 3.1.3.

---

[10] Roughly speaking, an orbibundle $E \to Y$ over an orbifold Y is the realization of a functor $\mathrm{pr} : \mathcal{E} \to \mathcal{Y}$ between a pair of ep groupoids whose restriction to the spaces of objects $\mathrm{pr} : \mathrm{Obj}_{\mathcal{E}} \to \mathrm{Obj}_{\mathcal{Y}}$ is a locally trivial vector bundle; cf. the discussion relating to (6.2.5).

Moreover, because the restriction of a chart is determined by a subset of the intermediate domain $\underline{U}$, all results about restrictions are easy to generalize to the case of nontrivial isotropy. In particular the following result holds, where we use the notation $\sqsubset$ to denote a precompact inclusion and $\mathrm{cl}_V(V')$ to denote the closure of a subset $V' \subset V$ in the relative topology of V.

Lemma 3.1.9. *Let* $\mathbf{K}$ *be a Kuranishi chart. Then for any open subset* $F' \subset F$ *there is a restriction* $\mathbf{K}'$ *to* F' *with domain* U' *such that* $\mathrm{cl}_U(U') \cap s^{-1}(0) = \psi^{-1}(\mathrm{cl}_X(F'))$*. Moreover, if* $F' \sqsubset F$ *is precompact, then* $U' \sqsubset U$ *can be chosen precompact, and if* $\mathbf{K}$ *is effective, so is* $\mathbf{K}'$*.*

Most definitions in §2 extend with only minor changes to the case of nontrivial isotropy. However, the notion of coordinate change needs to be generalized significantly to include a covering map. We will again formulate the definition in the situation that is relevant to Kuranishi atlases. That is, we suppose that a finite set of Kuranishi charts $(\mathbf{K}_i)_{i\in\{1,\dots,N\}}$ is given such that for each $I \subset \{1,\dots,N\}$ with $F_I := \bigcap_{i\in I} F_i \neq \emptyset$ we have another Kuranishi chart $\mathbf{K}_I$ with

- group $\Gamma_I = \prod_{i\in I} \Gamma_i$,
- obstruction space $E_I = \prod_{i\in I} E_i$ on which Γ_I acts with the product action,
- footprint $F_I := \bigcap_{i\in I} F_i$.

Then for $I \subset J$ note that the natural inclusion $\widehat{\phi} : E_I \to E_J$ is equivariant with respect to the inclusion $\Gamma_I \hookrightarrow \Gamma_I \times \{\mathrm{id}\} \subset \Gamma_J$ and we have a natural splitting $\Gamma_J \cong \Gamma_I \times \Gamma_{J\setminus I}$, so that the complement $\Gamma_{J\setminus I}$ acts trivially on the image $\widehat{\phi}(E_I) \subset E_J$.

Definition 3.1.10. *Given* $I \subset J \subset \{1,\dots,N\}$ *let* $\mathbf{K}_I$ *and* $\mathbf{K}_J$ *be Kuranishi charts as above, so that* $F_I \supset F_J$*. A* **coordinate change** $\widehat{\Phi}$ *from* $\mathbf{K}_I$ *to* $\mathbf{K}_J$ *with* **domain** $\underline{U}_{IJ} \subset \underline{U}_I$ *consists of*

- *a choice of restriction* $\mathbf{K}_I|_{\underline{U}_{IJ}}$ *of* $\mathbf{K}_I$ *to* F_J,
- *the splitting* $\Gamma_J \cong \Gamma_I \times \Gamma_{J\setminus I}$ *as above, and the induced inclusion* $\Gamma_I \hookrightarrow \Gamma_J$ *and projection* $\rho^\Gamma : \Gamma_J \to \Gamma_I$,
- *a* Γ_J*-invariant submanifold* $\widetilde{U}_{IJ} \subset U_J$ *on which* $\Gamma_{J\setminus I}$ *acts freely,*
- *a group covering* $(\widetilde{U}_{IJ}, \Gamma_J, \rho, \rho^\Gamma)$ *of* (U_{IJ}, Γ_I)*, where* $U_{IJ} := \pi_I^{-1}(\underline{U}_{IJ}) \subset U_I$,
- *the linear equivariant injection* $\widehat{\phi} : E_I \to E_J$ *as above,*

such that the Γ_J*-equivariant inclusion* $\widetilde{\phi} : \widetilde{U}_{IJ} \hookrightarrow U_J$ *intertwines the sections and footprint maps,*

$$s_J \circ \widetilde{\phi} = \widehat{\phi} \circ s_I \circ \rho \quad \text{on } \widetilde{U}_{IJ}, \qquad \psi_J \circ \widetilde{\phi} = \psi_I \circ \rho \quad \text{on } \rho^{-1}(s_I^{-1}(0)). \tag{3.1.1}$$

Moreover, we denote $s_{IJ} := s_I \circ \rho : \widetilde{U}_{IJ} \to E_I$ *and require the* **index condition***:*

(i) *the embedding* $\widetilde{\phi} : \widetilde{U}_{IJ} \hookrightarrow U_J$ *identifies the kernels,*

$$\mathrm{d}_u\widetilde{\phi}\big(\ker \mathrm{d}_u s_{IJ}\big) = \ker \mathrm{d}_{\widetilde{\phi}(u)} s_J \qquad \forall u \in \widetilde{U}_{IJ};$$

(ii) *the linear embedding* $\widehat{\phi} : E_I \to E_J$ *identifies the cokernels,*

$$\forall u \in \widetilde{U}_{IJ} : \qquad E_I = \mathrm{im}\, \mathrm{d}_u s_{IJ} \oplus C_{u,I} \quad \Longrightarrow \quad E_J = \mathrm{im}\, \mathrm{d}_{\widetilde{\phi}(u)} s_J \oplus \widehat{\phi}(C_{u,I}).$$

Remark 3.1.11. (i) If the isotropy and covering $\rho =: \phi^{-1}$ are both trivial, this definition agrees with that in §2.1 with $\widetilde{U}_{IJ} = \phi(U_{IJ})$.

(ii) The following diagram of group embeddings and group coverings is associated to each coordinate change:

$$
\begin{array}{ccc}
 & (\widetilde{U}_{IJ}, \Gamma_J) & \stackrel{(\widetilde{\phi},\mathrm{id})}{\longrightarrow} & (U_J, \Gamma_J) \\
 & \downarrow (\rho, \rho^\Gamma) & & \\
(U_I, \Gamma_I) \;\longleftarrow & (U_{IJ}, \Gamma_I) & &
\end{array}
\tag{3.1.2}
$$

(iii) Since $\underline{\rho} : \underline{\widetilde{U}}_{IJ} \to \underline{U}_{IJ}$ is a homeomorphism by Lemma 3.1.5 (iii), each coordinate change $(\phi, \widehat{\phi}, \rho) : \mathbf{K}_I|_{\underline{U}_{IJ}} \to \mathbf{K}_J$ induces an injective map

$$\underline{\phi} := \underline{\widetilde{\phi}} \circ \underline{\rho}^{-1} : \underline{U}_{IJ} \to \underline{U}_J$$

on the domains of the intermediate charts. In fact, there is an induced orbifold coordinate change $\widehat{\underline{\Phi}} : \underline{\mathbf{K}}_I|_{\underline{U}_{IJ}} \to \underline{\mathbf{K}}_J$ on the level of the intermediate charts, given by the bundle map $\widehat{\underline{\Phi}} : \underline{U_{IJ} \times E_I} \to \underline{U_J \times E_J}$ which is induced by the multivalued map $(\widetilde{\phi} \circ \rho^{-1}) \times \widehat{\phi}$ and hence covers $\underline{\widetilde{\phi}} \circ \underline{\rho}^{-1} =: \underline{\phi}$. This behaves exactly like the coordinate changes in §2.1, except that the domain is now an orbifold, rather than a manifold, and the bundle is now an orbibundle.

(iv) Conversely, suppose we are given an orbifold coordinate change $\widehat{\underline{\Phi}} : \underline{\mathbf{K}}_I \to \underline{\mathbf{K}}_J$ with domain $\underline{U}_{IJ}$. Then any coordinate change from $\mathbf{K}_I$ to $\mathbf{K}_J$ that induces $\widehat{\underline{\Phi}}$ is determined by the Γ_J-invariant set $\widetilde{U}_{IJ} := \pi_J^{-1}(\underline{\phi}(\underline{U}_{IJ}))$ and a choice of Γ_I-equivariant diffeomorphism between ${}^{\widetilde{U}_{IJ}}/_{\Gamma_{J\setminus I}}$ and $U_{IJ} := \pi_I^{-1}(\underline{U}_{IJ})$. When constructing coordinate changes in the Gromov–Witten setting, we will see that there is a natural choice of this diffeomorphism since the covering maps ρ are given by forgetting certain added marked points.

(v) Because $\widetilde{U}_{IJ}$ is defined to be a subset of U_J it is sometimes convenient to think of an element $x \in \widetilde{U}_{IJ}$ as an element in U_J, omitting the notation for the inclusion map $\widetilde{\phi}_{IJ} : \widetilde{U}_{IJ} \to U_J$.

The next step is to consider restrictions and composites of coordinate changes. Restrictions behave as in [MW12, Lemma 5.2.3]. Thus, for $I \subset J$, given any restrictions $\mathbf{K}'_I := \mathbf{K}_I|_{\underline{U}'_I}$ and $\mathbf{K}'_J := \mathbf{K}_J|_{\underline{U}'_J}$ whose footprints $F'_I \cap F'_J$ have nonempty intersection, and any coordinate change $\mathbf{K}_I|_{\underline{U}_{IJ}} \to \mathbf{K}_J$, there is an induced **restricted coordinate change** $\mathbf{K}'_I|_{\underline{U}'_{IJ}} \to \mathbf{K}'_J$ for any subset $\underline{U}'_{IJ} \subset \underline{U}_{IJ}$ satisfying the conditions

$$\underline{U}'_{IJ} \subset \underline{U}'_I \cap \underline{\phi}^{-1}(\underline{U}'_J), \qquad \underline{U}'_{IJ} \cap \underline{s}_I^{-1}(0) = \underline{\psi}_I^{-1}(F'_I \cap F'_J). \tag{3.1.3}$$

However, coordinate changes now do not directly compose due to the coverings involved. The induced coordinate changes on the intermediate charts still compose directly, but the analog of [MW12, Lemma 5.2.4] is the following. The proof is routine.

Lemma 3.1.12. *Let $I \subset J \subset K$ (so that automatically $F_I \cap F_K = F_J$) and suppose that $\widehat{\Phi}_{IJ} : \mathbf{K}_I \to \mathbf{K}_J$ and $\widehat{\Phi}_{JK} : \mathbf{K}_J \to \mathbf{K}_K$ are coordinate changes with domains $\underline{U}_{IJ}$ and $\underline{U}_{JK}$ respectively. Then the following holds.*

(i) *The domain $\underline{U}_{IJK} := \underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK}) \subset \underline{U}_I$ defines a restriction $\mathbf{K}_I|_{\underline{U}_{IJK}}$ to F_K with lifted domain $U_{IJK} = \pi_I^{-1}(\underline{U}_{IJK})$.*

(ii) *The composite $\rho_{IJK} := \rho_{IJ} \circ \rho_{JK} : \widetilde{U}_{IJK} \to U_{IJK}$ is defined on $\widetilde{U}_{IJK} := \pi_K^{-1}\big((\underline{\phi}_{JK} \circ \underline{\phi}_{IJ})(\underline{U}_{IJK})\big)$ and, together with the composed inclusion $\Gamma_I \hookrightarrow \Gamma_J \hookrightarrow \Gamma_K$, is a group covering $(\widetilde{U}_{IJK}, \Gamma_K, \rho_{IJK}, \rho^{\Gamma}_{IK})$ of (U_{IJK}, Γ_I).*

(iii) *The inclusion $\widetilde{\phi}_{IJK} : \widetilde{U}_{IJK} \hookrightarrow U_K$ together with $\widehat{\phi}_{IJK}$ and ρ_{IJK} satisfies* (3.1.1) *and the index condition with respect to the indices I, K.*

Hence this defines a **composite coordinate change** *$\widehat{\Phi}_{IJK} = (\widetilde{\phi}_{IJK}, \widehat{\phi}_{IJK}, \rho_{IJK})$ from $\mathbf{K}_I$ to $\mathbf{K}_K$.*

Remark 3.1.13. The induced orbifold coordinate change $\widehat{\underline{\Phi}}_{IJK} = (\underline{\phi}_{IJK}, \widehat{\underline{\phi}}_{IJK})$ between the intermediate charts $\underline{\mathbf{K}}_I$ and $\underline{\mathbf{K}}_K$ is the composite $\widehat{\underline{\Phi}}_{JK} \circ \widehat{\underline{\Phi}}_{IJ}$ as considered in §2.1. (For more detail, see [MW12, Lemma 5.2.4].)

With the notions of Kuranishi charts and coordinate changes with nontrivial isotropy in place, we can now directly extend the notion of Kuranishi atlas from §2.1. The notions of a covering family $(\mathbf{K}_i)_{i=1,\dots,N}$ of basic charts for X and of transition data $(\mathbf{K}_J)_{J \in \mathcal{I}_{\mathcal{K}}, |J| \ge 2}, (\widehat{\Phi}_{IJ})_{I,J \in \mathcal{I}_{\mathcal{K}}, I \subsetneq J}$ are as before. The cocycle conditions can now mostly be expressed in terms of the intermediate charts.

Definition 3.1.14. *Let $\mathbf{K}_\alpha$ for $\alpha = I, J, K$ be Kuranishi charts with $I \subset J \subset K$ and let $\widehat{\Phi}_{\alpha\beta} : \mathbf{K}_\alpha|_{\underline{U}_{\alpha\beta}} \to \mathbf{K}_\beta$ for $(\alpha, \beta) \in \{(I,J), (J,K), (I,K)\}$ be coordinate changes. We say that this triple $\widehat{\Phi}_{IJ}, \widehat{\Phi}_{JK}, \widehat{\Phi}_{IK}$ satisfies the*

- **weak cocycle condition** *if $\widehat{\Phi}_{JK} \circ \widehat{\Phi}_{IJ} \approx \widehat{\Phi}_{IK}$ are equal on the overlap in the sense*

$$\rho_{IK} = \rho_{IJ} \circ \rho_{JK} \quad \text{on } \widetilde{U}_{IK} \cap \rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK}); \tag{3.1.4}$$

- **cocycle condition** *if $\widehat{\Phi}_{JK} \circ \widehat{\Phi}_{IJ} \subset \widehat{\Phi}_{IK}$, i.e. $\widehat{\Phi}_{IK}$ extends the composed coordinate change in the sense that* (3.1.4) *holds and*

$$\underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK}) \subset \underline{U}_{IK}; \tag{3.1.5}$$

- **strong cocycle condition** *if $\widehat{\Phi}_{JK} \circ \widehat{\Phi}_{IJ} = \widehat{\Phi}_{IK}$ are equal as coordinate changes, that is if* (3.1.4) *holds and*

$$\underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK}) = \underline{U}_{IK}. \tag{3.1.6}$$

In fact [MW14, Lemma 2.3.4] shows that the cocycle condition (3.1.5) implies that

$$\rho_{IK} \;=\; \rho_{IJ} \circ \rho_{JK} \quad \text{on } \rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK}) \subset \widetilde{U}_{IK}. \tag{3.1.7}$$

Similarly, condition (3.1.4) implies

$$\underline{\phi}_{IK} = \underline{\phi}_{JK} \circ \underline{\phi}_{IJ} \quad \text{on } \underline{U}_{IK} \cap \big(\underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK})\big). \tag{3.1.8}$$

Therefore we are led to the following definition.

Definition 3.1.15. *A* **(weak) Kuranishi atlas of dimension d** *on a compact metrizable space X is a tuple*

$$\mathcal{K} = \big(\mathbf{K}_I, \widehat{\Phi}_{IJ}\big)_{I,J\in\mathcal{I}_\mathcal{K}, I\subsetneq J}$$

consisting of a covering family of basic charts $(\mathbf{K}_i)_{i=1,\dots,N}$ of dimension d and transition data $\big((\mathbf{K}_J)_{|J|\geq 2}, (\widehat{\Phi}_{IJ})_{I\subsetneq J}\big)$ for $(\mathbf{K}_i)_{i=1,\dots,N}$ that satisfies the (weak) cocycle condition for every triple $I,J,K \in \mathcal{I}_K$ with $I \subsetneq J \subsetneq K$.

We say that the Kuranishi atlas $\mathcal{K}$ is **effective** *if all the charts $\mathbf{K}_I$ are effective.*

Example 3.1.16. We show in Proposition 6.1.3 that every compact smooth orbifold has an atlas. As an example, consider the "football" $Y = S^2$ with two orbifold points, one at the north pole of order 2 and one at the south pole of order 3. Take charts $(U_1, \mathbb{Z}_2)$, $(U_2, \mathbb{Z}_3)$ about north/south pole with $\underline{U}_1 \cap \underline{U}_2 = \underline{U}_{\{12\}} = \underline{A}$ an annulus around the equator. Let $A_i = \pi_i^{-1}(\underline{A})$ where $\pi_i : U_i \to \underline{U}_i$ is the projection.[11] Then the restriction of $(U_1, \mathbb{Z}_2)$ over $\underline{A}$ is $(A_1, \mathbb{Z}_2)$, whereas the restriction of $(U_2, \mathbb{Z}_3)$ over $\underline{A}$ is $(A_2, \mathbb{Z}_3)$ There is no direct relation between these restrictions because the coverings $A_1 \to \underline{A}$ and $A_2 \to \underline{A}$ are incompatible. However, they do have a common free covering, namely the pullback defined by the diagram

$$\begin{tikzcd} U_{12} \arrow[r] \arrow[d] & U_1 \arrow[d, "\pi_1"] \\ U_2 \arrow[r, "\pi_2"] & Y \end{tikzcd}$$

i.e. $U_{12} := \{(x,y) \in U_1 \times U_2 \,|\, \pi_1(x) = \pi_2(y)\}$ with group $\Gamma_{12} := \Gamma_1 \times \Gamma_2$. This defines an atlas with two basic charts and one sum chart.

Remark 3.1.17. Although it seems that many choices are needed in order to construct a Kuranishi atlas, this is somewhat deceptive. For example, in the Gromov–Witten case considered in Section 5 below, the choices involved in the construction of a family of basic charts $(\mathbf{K}_i)_{i=1,\dots,N}$ essentially induce the transition data as well. Namely, for each $I \subset \{1,\dots,N\}$ such that $F_I := \bigcap_{i\in I} F_i$ is nonempty, we will construct a "sum chart" $\mathbf{K}_I$ with group $\Gamma_I := \prod_{i\in I}\Gamma_i$ and obstruction space[12] $E_I := \prod_{i\in I} E_i$. Moreover, each E_i is a product of the form $E_i = \prod_{\gamma\in\Gamma_i}(E_{0i})_\gamma$ of copies of a vector space E_{0i} that are permuted by the action of Γ_i, and Γ_I acts on E_I by the obvious product action.

More precisely, each basic chart $\mathbf{K}_i$ is constructed by adding a certain tuple $\mathbf{w}_i$ of marked points to the domains of the stable maps $[\Sigma_f, \mathbf{z}, f]$, given by the preimages of a fixed hypersurface of M. When seen on spaces of equivalence classes of maps, the action of Γ_i is easy to understand,[13] since it simply permutes this set of marked points

[11] For simplicity, we here identify each $\underline{U}_I$ with its image $F_I \subset Y$.

[12] We will use the stabilization process introduced in [MW14] that allows us to do this for *any* set of E_i; there is no need for a transversality requirement such as Sum Condition II$'$ in [MW12, Section 4.3].

[13] This point is explained in detail in §5.1 (IX), where we describe the action both on parametrized maps as in (5.1.27),(5.1.28) and on equivalence classes of maps as in the discussion after (5.1.30).

$\mathbf{w}_i$. Similarly, elements of the domains U_J of the transition charts consist of certain maps $f : \Sigma \to M$ with the given marked points $\mathbf{z}$ together with $|J|$ sets of added tuples of marked points $(\mathbf{w}_j)_{j\in J}$, each taken by f to certain hypersurfaces in M. Each factor Γ_j of the group Γ_J acts by permuting the elements of the j-th tuple of points, leaving the others alone. Moreover, the covering map $\widetilde{U}_{IJ} \to U_I$ simply forgets the extra tuples $(\mathbf{w}_j)_{j\in J\smallsetminus I}$. Thus it is immediate from the construction that the group $\Gamma_{J\smallsetminus I}$ acts freely on the subset $\widetilde{U}_{IJ}$ of U_J, and that the covering map is equivariant in the appropriate sense. Further, when $I \subset J \subset K$ the compatibility condition $\rho_{IK} = \rho_{IJ} \circ \rho_{JK}$ holds whenever both sides are defined. Therefore, just as in the case with trivial isotropy, once given the basic and sum charts, the only new choice needed to construct an atlas is that of the domains $\underline{U}_{IJ}$ of the coordinate changes which are required to intersect the zero set $\underline{s}_I^{-1}(0)$ in $\underline{\psi}_I^{-1}(F_J)$. Note that there is no simple hierarchy by which one could organize these choices to automatically fulfill the cocycle condition. Hence concrete constructions will usually only satisfy a weak cocycle condition. However, as we saw above any weak atlas can be "tamed" so that it satisfies the strong cocycle condition, and hence in particular gives a Kuranishi atlas.

Remark 3.1.18. The above definition requires that each sum chart $\mathbf{K}_I$ has group $\Gamma_I = \prod_{i\in I} \Gamma_i$. This is the easiest choice to describe. However all that is really required of an atlas is that there is a family $(\Gamma_I)_{I\in\mathcal{I}_\mathcal{K}}$ of groups such that

- Γ_I acts on each domain U_I and obstruction space E_I;
- there is a family of inclusions $\big(\iota^\Gamma_{IJ} : \Gamma_I \to \Gamma_I\big)_{I\subset J}$ and surjections $\big(\rho^\Gamma_{IJ} : \Gamma_J \to \Gamma_I\big)_{I\subset J}$ such that
  - $\operatorname{im}(\iota^\Gamma_{(I\smallsetminus J)J}) = \ker \rho^\Gamma_{IJ}$ for all $I \subset J$;
  - $\iota^\Gamma_{JK} \circ \iota^\Gamma_{IJ} = \iota^\Gamma_{IK}$ and $\rho^\Gamma_{JK} \circ \rho^\Gamma_{IJ} = \rho^\Gamma_{IK}$ for all $I \subset J \subset K$;
- for all $I \subset J$, the linear maps $\widehat{\phi}_{IJ} : E_I \to E_J$ are equivariant with respect to the inclusion $\iota^\Gamma_{IJ} : \Gamma_I \to \Gamma_J$;
- for all $I \subset J$, the projection $(\rho_{IJ}, \rho^\Gamma_{IJ}) : (\widetilde{U}_{IJ}, \Gamma_J) \to (U_{IJ}, \Gamma_I)$ is a group covering map, i.e. $\ker(\rho^\Gamma_{IJ})$ acts freely and the quotient ${}^{\widetilde{U}_{IJ}}/_{\ker(\rho^\Gamma_{IJ})}$ is Γ_I-equivariantly homeomorphic to U_{IJ}.

Such atlases are very natural when one considers products; cf. Definition 4.1.2 and Example 4.1.3 below.

3.2. **Categories, tamings, reductions and sections.** Just as in §2.2, we will associate to each Kuranishi atlas $\mathcal{K}$ two topological categories $\mathbf{B}_\mathcal{K}, \mathbf{E}_\mathcal{K}$ together with functors

$$\mathrm{pr}_\mathcal{K} : \mathbf{E}_\mathcal{K} \to \mathbf{B}_\mathcal{K}, \quad s_\mathcal{K} : \mathbf{B}_\mathcal{K} \to \mathbf{E}_\mathcal{K}, \quad \psi_\mathcal{K} : s_\mathcal{K}^{-1}(0) \to \mathbf{X},$$

where $\mathbf{X}$ is the category with objects X and only identity morphisms. Recall here that the morphism spaces will only be closed under composition (and thus generate an equivalence relation that defines the realization $|\mathcal{K}|$ as ambient space for X) if the cocycle condition holds. Thus for the following we assume that $\mathcal{K}$ is a Kuranishi atlas.

Then, as before, the domain category $\mathbf{B}_{\mathcal{K}}$ has objects

$$\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}} := \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} U_I \;=\; \bigl\{(I,x)\,\big|\, I\in\mathcal{I}_{\mathcal{K}}, x\in U_I\bigr\},$$

where we usually identify $x \in U_I$ with $(I,x) \in \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$. The morphisms in $\mathbf{B}_{\mathcal{K}}$ are composites of morphisms of the following two types.

(a) For each $I \in \mathcal{I}_{\mathcal{K}}$ the action of Γ_I gives rise to morphisms between points in U_I. These form a space $U_I \times \Gamma_I$ with source and target maps

$$\begin{aligned} s\times t: \quad U_I\times\Gamma_I \;&\longrightarrow\; \quad U_I\times U_I \qquad \subset\; \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}\times \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}\\ (x,\gamma)\;&\longmapsto\; \bigl((I,x),(I,\gamma x)\bigr), \end{aligned}$$

and inverses $(x,\gamma)^{-1} = (\gamma x, \gamma^{-1})$.

(b) For each $I \subsetneq J$ the coordinate change $\widehat{\Phi}_{IJ}$ gives rise to non-invertible morphisms from points in U_I to points in U_J given by the space $\widetilde{U}_{IJ}$ with source and target maps

$$\begin{aligned} s\times t: \quad \widetilde{U}_{IJ} \;&\longrightarrow\; \qquad U_I\times U_J \qquad \subset\; \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}\times \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}\\ y\;&\longmapsto\; \bigl((I,\rho_{IJ}(y)),(J,\widetilde{\phi}_{IJ}(y))\bigr). \end{aligned}$$

In order to determine the general morphisms in $\mathbf{B}_{\mathcal{K}}$ we will unify types (a) and (b) by allowing $I = J$, in which case we interpret $U_{II} := U_I$, $\widetilde{U}_{IJ} := U_J$, $\rho_{II} := \mathrm{id}$. Also note that for $I \subsetneq J$ we can identify $\widetilde{\phi}_{IJ}(y) = y$ since $\widetilde{\phi}_{IJ}$ is the inclusion map for $\widetilde{U}_{IJ} \subset U_J$. So the morphisms of type (b) are described by their targets $y \in U_J$ and the covering map ρ_{IJ}. In comparison, recall that in §2.1, we have no morphisms of type (a) and the morphisms of type (b) are described by their source $x \in U_{IJ} \subset U_I$ and the embedding $\phi_{IJ} : U_{IJ} \to U_J$. When the isotropy groups are all trivial, it makes no difference whether we use source or target since $\phi_{IJ} = \rho_{IJ}^{-1}$. The corresponding isomorphism of categories is given in Lemma 3.2.2 below. For nontrivial isotropy, however, the only way to obtain a continuous description of the morphism spaces is to parametrize them by the targets as follows.

Lemma 3.2.1. *Let $\mathcal{K}$ be a Kuranishi atlas. Then the space of morphisms in $\mathbf{B}_{\mathcal{K}}$ is the disjoint union*

$$\mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}} = \bigsqcup_{I\subset J} \widetilde{U}_{IJ}\times\Gamma_I = \bigl\{(I,J,y,\gamma)\,\big|\, I\subset J, y\in\widetilde{U}_{IJ}, \gamma\in\Gamma_I\bigr\},$$

with source and target maps given by

$$\begin{aligned} s\times t: \quad \widetilde{U}_{IJ}\times\Gamma_I \;&\longrightarrow\; \qquad U_I\times U_J \qquad\qquad \subset\; \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}\times \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}},\\ \bigl(I,J,y,\gamma\bigr)\;&\longmapsto\; \bigl((I,\gamma^{-1}\rho_{IJ}(y)),\ (J,y)\bigr) \end{aligned} \tag{3.2.1}$$

and composition given by the following for $x = \delta^{-1}\rho_{JK}(y)$

$$\bigl(J,K,y,\delta\bigr)\circ\bigl(I,J,x,\gamma\bigr) = \bigl(I,K,y,\rho^{\Gamma}_{IJ}(\delta)\gamma\bigr). \tag{3.2.2}$$

We form the **intermediate Kuranishi category** $\underline{\mathbf{B}}_{\mathcal{K}}$ in a similar way. Its objects

$$\mathrm{Obj}_{\underline{\mathbf{B}}_{\mathcal{K}}} := \bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} \underline{U}_I$$

are the disjoint union of the intermediate domains, and morphisms

$$\mathrm{Mor}_{\underline{\mathbf{B}}_{\mathcal{K}}} := \bigsqcup_{I,J \in \mathcal{I}_{\mathcal{K}}, I \subset J} \underline{U}_{IJ}$$

are given by the orbifold coordinate changes $\underline{\phi}_{IJ} : \underline{U}_{IJ} \to \underline{U}_J$. Thus the source and target maps are

$$s \times t : \underline{U}_{IJ} \to \underline{U}_I \times \underline{U}_J \subset \mathrm{Obj}_{\underline{\mathbf{B}}_{\mathcal{K}}} \times \mathrm{Obj}_{\underline{\mathbf{B}}_{\mathcal{K}}}, \quad (I,x) \mapsto \big((I,x),(J,\underline{\phi}_{IJ}(x))\big).$$

The identity maps $\underline{\phi}_{II}$ on $\underline{U}_{II} = \underline{U}_I$ are included, giving rise to the identity morphisms.

As before, we denote by $|\mathcal{K}|$ the **realization** of the category $\mathbf{B}_{\mathcal{K}}$, i.e. the topological space obtained as the quotient of $\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$ by the equivalence relation generated by the morphisms in $\mathbf{B}_{\mathcal{K}}$. The quotient map $\pi_{\mathcal{K}} : \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}} \to |\mathcal{K}|$, $(I,x) \mapsto [I,x]$ now factors through the intermediate category,

$$\pi_{\mathcal{K}} : \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}} \to \mathrm{Obj}_{\underline{\mathbf{B}}_{\mathcal{K}}} \to |\mathcal{K}|.$$

In particular the two categories $\mathbf{B}_{\mathcal{K}}$ and $\underline{\mathbf{B}}_{\mathcal{K}}$ have the same realization. In the latter case, we denote the natural projection by $\underline{\pi}_{\mathcal{K}} : \mathrm{Obj}_{\underline{\mathbf{B}}_{\mathcal{K}}} \to |\mathcal{K}|$. More precisely, we can formulate this as follows.

Lemma 3.2.2. *Let $\mathcal{K}$ be a Kuranishi atlas. Then there is a functor $\pi_\Gamma : \mathbf{B}_{\mathcal{K}} \to \underline{\mathbf{B}}_{\mathcal{K}}$ that is given on objects by the quotient maps $\pi_I : U_I \to \underline{U}_I$, and on morphisms by*

$$\widetilde{U}_{IJ} \times \Gamma_I \;\to\; \underline{U}_{IJ}, \qquad \big(I,J,y,\gamma\big) \;\mapsto\; \big(I,J,\underline{\rho_{IJ}(y)}\big).$$

If all isotropy groups $\Gamma_I = \{\mathrm{id}\}$ are trivial, then π_Γ is an isomorphism of categories with identical object spaces, and $\underline{\mathbf{B}}_{\mathcal{K}}$ is the category associated to the Kuranishi atlas in §2.1.

In general, the realization $|\mathcal{K}|$ of $\mathbf{B}_{\mathcal{K}}$ can be identified as topological space (with the quotient topology) with that of $\underline{\mathbf{B}}_{\mathcal{K}}$ via factoring the quotient map $\pi_{\mathcal{K}} = \underline{\pi}_{\mathcal{K}} \circ \pi_\Gamma$ into the functor π_Γ given on objects by quotienting by the group actions and the projection $\underline{\pi}_{\mathcal{K}} : \underline{\mathbf{B}}_{\mathcal{K}} \to |\mathcal{K}|$, that can be considered as a functor to a topological category with only identity morphisms. Moreover, π_Γ is proper, i.e. compact subsets of $\mathrm{Obj}_{\underline{\mathbf{B}}_{\mathcal{K}}}$ have compact preimage in $\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$.

There are similar obstruction space categories $\mathbf{E}_{\mathcal{K}}$ and $\underline{\mathbf{E}}_{\mathcal{K}}$ whose precise definition can be found in [MW14]. The projections pr_I fit together to functors

$$\mathrm{pr}_{\mathcal{K}} : \mathbf{E}_{\mathcal{K}} \to \mathbf{B}_{\mathcal{K}}, \quad \underline{\mathrm{pr}}_{\mathcal{K}} : \underline{\mathbf{E}}_{\mathcal{K}} \to \underline{\mathbf{B}}_{\mathcal{K}},$$

and the sections s_I fit together to give functors

$$s_{\mathcal{K}} : \mathbf{B}_{\mathcal{K}} \to \mathbf{E}_{\mathcal{K}}, \qquad \underline{s}_{\mathcal{K}} : \underline{\mathbf{B}}_{\mathcal{K}} \to \underline{\mathbf{E}}_{\mathcal{K}}$$

that are "sections" in the sense that $\mathrm{pr}_{\mathcal{K}} \circ s_{\mathcal{K}}$ and $\underline{\mathrm{pr}}_{\mathcal{K}} \circ \underline{s}_{\mathcal{K}}$ are the identity functors.

Proposition 3.2.3. *Let $\mathcal{K}$ be a Kuranishi atlas.*

(i) *The functors $\mathrm{pr}_{\mathcal{K}} : \mathbf{E}_{\mathcal{K}} \to \mathbf{B}_{\mathcal{K}}$ and $\underline{\mathrm{pr}}_{\mathcal{K}} : \underline{\mathbf{E}}_{\mathcal{K}} \to \underline{\mathbf{B}}_{\mathcal{K}}$ induce the same continuous map*
$$|\mathrm{pr}_{\mathcal{K}}| : |\mathbf{E}_{\mathcal{K}}| \to |\mathcal{K}|,$$
which we call the **obstruction bundle** *of $\mathcal{K}$, although its fibers generally do not have the structure of a vector space. However, it has a continuous zero section*
$$|0_{\mathcal{K}}| : \ |\mathcal{K}| \to |\mathbf{E}_{\mathcal{K}}|, \quad [I, x] \mapsto [I, x, 0].$$

(ii) *The sections $s_{\mathcal{K}} : \mathbf{B}_{\mathcal{K}} \to \mathbf{E}_{\mathcal{K}}$ and $\underline{s}_{\mathcal{K}} : \underline{\mathbf{B}}_{\mathcal{K}} \to \underline{\mathbf{E}}_{\mathcal{K}}$ descend to the same continuous section*
$$|s_{\mathcal{K}}| : |\mathcal{K}| \to |\mathbf{E}_{\mathcal{K}}|.$$
Both of these are sections in the sense that $|\mathrm{pr}_{\mathcal{K}}| \circ |s_{\mathcal{K}}| = |\mathrm{pr}_{\mathcal{K}}| \circ |0_{\mathcal{K}}| = \mathrm{id}_{|\mathcal{K}|}$.

(iii) *There is a natural homeomorphism from the realization of the subcategory $s_{\mathcal{K}}^{-1}(0)$ to the zero set of $|s_{\mathcal{K}}|$, with the relative topology induced from $|\mathcal{K}|$,*
$$\bigl|s_{\mathcal{K}}^{-1}(0)\bigr| \;=\; {s_{\mathcal{K}}^{-1}(0)}\Big/{\sim_{s_{\mathcal{K}}^{-1}(0)}} \;\xrightarrow{\cong}\; |s_{\mathcal{K}}|^{-1}(0) \;:=\; \bigl\{[I,x] \,\big|\, s_I(x) = 0\bigr\} \;\subset\; |\mathcal{K}|.$$

The proof is not difficult. The next task is to prove the following analog of Theorem 2.3.1.

Theorem 3.2.4. *Let $\mathcal{K}$ be a weak Kuranishi atlas on a compact metrizable space X. Then there is a metrizable tame shrinking $\mathcal{K}'$ of $\mathcal{K}$ with domains $(U_I' \subset U_I)_{I \in \mathcal{I}_{\mathcal{K}'}}$ such that the realizations $|\mathcal{K}'|$ and $|\mathbf{E}_{\mathcal{K}'}|$ are Hausdorff in the quotient topology. Further, for each $I \in \mathcal{I}_{\mathcal{K}'} = \mathcal{I}_{\mathcal{K}}$ the projection maps $\underline{\pi}_{\mathcal{K}'} : \underline{U_I'} \to |\mathcal{K}'|$ and $\underline{\pi}_{\mathcal{K}'} : \underline{U_I' \times E_I} \to |\mathbf{E}_{\mathcal{K}'}|$ are homeomorphisms onto their images. In addition, these projections $\pi_{\mathcal{K}'}$ fit into a commutative diagram*
$$\begin{array}{ccc} U_I' \times E_I & \xrightarrow{\pi_{\mathcal{K}'}} & |\mathbf{E}_{\mathcal{K}'}| \\ \downarrow & & \downarrow{\scriptstyle |\mathrm{pr}_{\mathcal{K}'}|} \\ U_I' & \xrightarrow{\pi_{\mathcal{K}'}} & |\mathcal{K}'| \end{array}$$
where the horizontal maps intertwine the linear structure on the fibers of $U_I' \times E_I \to U_I'$ with the induced orbibundle structure on the fibers of $|\mathrm{pr}_{\mathcal{K}'}|$.

Moreover, any two such shrinkings are cobordant by a metrizable tame Kuranishi cobordism that also has the above Hausdorff, homeomorphism, and linearity properties.

This holds essentially because we can formulate its proof in terms of the intermediate category. Since none of the relevant arguments in §2.2 used the fact that the domains U_I, U_{IJ} are manifolds rather than orbifolds, they all go through. Here are the relevant definitions. First we define tameness on the level of the intermediate category.

Definition 3.2.5 (cf. [MW12], Definition 6.2.7)**.** *A weak Kuranishi atlas is* **tame** *if for all $I, J, K \in \mathcal{I}_{\mathcal{K}}$ we have*

$$\underline{U}_{IJ} \cap \underline{U}_{IK} \;=\; \underline{U}_{I(J\cup K)} \qquad \forall I \subset J, K; \tag{3.2.3}$$

$$\underline{\phi}_{IJ}(\underline{U}_{IK}) \;=\; \underline{U}_{JK} \cap \underline{s}_J^{-1}\bigl(\widehat{\phi}_{IJ}(E_I)\bigr) \qquad \forall I \subset J \subset K. \tag{3.2.4}$$

Here we allow equalities between I, J, K, using the notation $\underline{U}_{II} := \underline{U_I}$ and $\underline{\phi}_{II} := \mathrm{Id}_{\underline{U}_I}$.

Similarly, we can define a shrinking of $\mathcal{K}$ on the level of the intermediate category.

Definition 3.2.6 (cf. [MW12], Definition 6.3.2)**.** *Let $\mathcal{K} = (\mathbf{K}_I, \widehat{\Phi}_{IJ})_{I,J\in\mathcal{I}_{\mathcal{K}}, I\subsetneq J}$ be a weak Kuranishi atlas. We say that a weak Kuranishi atlas $\mathcal{K}' = (\mathbf{K}'_I, \widehat{\Phi}'_{IJ})_{I,J\in\mathcal{I}_{\mathcal{K}'}, I\subsetneq J}$ is a* **shrinking** *of $\mathcal{K}$ if*

(i) *the footprint cover $(F'_i)_{i=1,\dots,N}$ is a shrinking of the cover $(F_i)_{i=1,\dots,N}$;*
(ii) *for each $I \in \mathcal{I}_{\mathcal{K}}$ the chart $\mathbf{K}'_I$ is the restriction of $\mathbf{K}_I$ to a precompact domain $\underline{U}'_I \sqsubset \underline{U}_I$ as in Definition 3.1.8;*
(iii) *for each $I, J \in \mathcal{I}_{\mathcal{K}}$ with $I \subsetneq J$ the coordinate change $\widehat{\Phi}'_{IJ}$ is the restriction of $\widehat{\Phi}_{IJ}$ to the open subset $\underline{U}'_{IJ} := \underline{\phi}_{IJ}^{-1}(\underline{U}'_J) \cap \underline{U}'_I$ as in equation* (3.1.3).

Note that because the maps $\pi_I : U_I \to \underline{U}_I$ are proper by Lemma 3.1.5 (i), the domain $U'_I := \pi_I^{-1}(\underline{U}'_I)$ of the shrinking $\mathbf{K}'_I$ is precompactly contained in U_I.

Next, we make a similar modification to the notion of metrizability. Note that in the presence of isotropy $\Gamma_I \neq \mathrm{id}$ it makes no sense to try to pull this metric d on $|\mathcal{K}|$ back to U_I since the pullback of a metric by a noninjective map is no longer a metric.

Definition 3.2.7. *A Kuranishi atlas $\mathcal{K}$ is called* **metrizable** *if there exists a bounded metric d on the set $|\mathcal{K}|$ such that for each $I \in \mathcal{I}_{\mathcal{K}}$ the pullback metric $\underline{d}_I := (\underline{\pi}_{\mathcal{K}}|_{\underline{U}_I})^* d$ on $\underline{U}_I$ induces the given quotient topology on $\underline{U}_I = {}^{U_I}\!/_{\Gamma_I}$.*

Granted these definitions, Theorem 3.2.4 follows by the arguments that prove Theorem 2.3.1 since we may work on the level of the intermediate category.

Construction of sections. The next task is to construct suitable sections. Here we do have more work to do. However, the notion of reduction is essentially the same as before.

Definition 3.2.8 (cf. Definition 2.4.2)**.** *A* **reduction** *of a tame Kuranishi atlas $\mathcal{K}$ is an open subset $\mathcal{V} = \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} V_I \subset \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$ i.e. a tuple of (possibly empty) open subsets $V_I \subset U_I$, satisfying the following conditions:*

(i) *$V_I = \pi_I^{-1}(\underline{V}_I)$ for each $I \in \mathcal{I}_{\mathcal{K}}$, i.e. V_I is pulled back from the intermediate category and so is Γ_I-invariant;*
(ii) *$V_I \sqsubset U_I$ for all $I \in \mathcal{I}_{\mathcal{K}}$, and if $V_I \neq \emptyset$ then $V_I \cap s_I^{-1}(0) \neq \emptyset$;*
(iii) *if $\pi_{\mathcal{K}}(\overline{V_I}) \cap \pi_{\mathcal{K}}(\overline{V_J}) \neq \emptyset$ then $I \subset J$ or $J \subset I$;*
(iv) *the zero set $\iota_{\mathcal{K}}(X) = |s_{\mathcal{K}}|^{-1}(0)$ is contained in $\pi_{\mathcal{K}}(\mathcal{V}) = \bigcup_{I\in\mathcal{I}_{\mathcal{K}}} \pi_{\mathcal{K}}(V_I)$.*

Given a reduction $\mathcal{V}$, we define the **reduced domain category $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}$** *and the* **reduced obstruction category $\mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$** *to be the full subcategories of $\mathbf{B}_{\mathcal{K}}$ and $\mathbf{E}_{\mathcal{K}}$ with objects $\bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} V_I$ resp. $\bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} V_I \times E_I$, and denote by $s|_{\mathcal{V}} : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ the section given by restriction of $s_{\mathcal{K}}$.*

It is again crucial in this context that the quotient map Obj $\mathbf{B}_{\mathcal{K}} \to$ Obj $\underline{\mathbf{B}}_{\mathcal{K}}$ is proper (cf.Lemma 3.1.5), so that the pullback V_I of a precompact subset $\underline{V}_I \sqsubset \underline{U}_I$ is still precompact in U_I. Because of this, we can establish the existence and uniqueness of reductions modulo cobordism by working in the intermediate category, hence proving the analog of [MW12, Proposition 7.1.11]. Further, the results on **nested (cobordism) reductions** can be interpreted at the intermediate level, and hence go through as before. Here, we say that two reductions $\mathcal{C}, \mathcal{V}$ are **nested** (written $\mathcal{C} \sqsubset \mathcal{V}$) if C_I is a precompact subset of V_I for all I. The only real change needed to the discussion in §2.4 above is that, to achieve transversality we should work with "multisections" rather than sections. In our categorical framework, these can be defined very easily.

Definition 3.2.9. *A* **reduced section** ν *of* $\mathcal{K}$ *is a smooth map*

$$\nu : \mathcal{V} = \bigsqcup_{I \in \mathcal{I}_K} V_I \longrightarrow \mathrm{Obj}_{\mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}}$$

between the spaces of objects in the reduced domain and obstruction categories of some reduction $\mathcal{V}$ *of* $\mathcal{K}$, *such that* $\mathrm{pr}_{\mathcal{K}} \circ \nu$ *is the identity. Further, we require that* $\nu = (\nu_I)_{I \in \mathcal{I}_{\mathcal{K}}}$ *is given by a family of smooth maps* $\nu_I : V_I \to E_I$ *that are compatible with coordinate changes in the sense that*

$$\nu_J\big|_{\widetilde{U}_{IJ} \cap V_J \cap \rho_{IJ}^{-1}(V_I)} = \widehat{\phi}_{IJ} \circ \nu_I \circ \rho_{IJ}\big|_{\widetilde{U}_{IJ} \cap V_J \cap \rho_{IJ}^{-1}(V_I)} \quad \forall I \subset J. \tag{3.2.5}$$

We say that a reduced section ν *is an* **admissible perturbation** *of* $s_{\mathcal{K}}|_{\mathcal{V}}$ *if*

$$\mathrm{d}_y \nu_J(\mathrm{T}_y V_J) \subset \operatorname{im} \widehat{\phi}_{IJ} \qquad \forall\, I \subsetneq J,\ y \in \widetilde{U}_{IJ} \cap V_J \cap \rho_{IJ}^{-1}(V_I).$$

The above compatibility condition implies that when $I \subset J$ the section ν_J is determined by ν_I on the part $\widetilde{U}_{IJ} \cap V_J \cap \rho_{IJ}^{-1}(V_I)$ of V_J that lies over V_I. In particular it takes values in $E_I \subset E_J$ and is invariant under the action of $\Gamma_{J \setminus I}$, and this means that ν is compatible with morphisms of type (b) on page 37. However, ν is *not* in general a functor $\mathbf{B}_{\mathcal{K}}\big|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}\big|_{\mathcal{V}}$ since it is not required to be equivariant under the group actions. Hence it induces a multivalued map on the realization $|\mathcal{V}|$. This can be written down most easily in terms of the **equivariant completion** of ν, which consists of the family of maps

$$\gamma\nu_I : V_I \to E_I, \quad x \mapsto \gamma\nu_I(x) \qquad \forall I \in \mathcal{I}_{\mathcal{K}}, \gamma \in \Gamma_I, \tag{3.2.6}$$

where we use the action $(\gamma, v) \mapsto \gamma v$ of Γ_I on E_I. The following notions will allow us to control the topology of the zero set. (The second part of the transversality requirement is needed in order to apply the results of [M07].)

Definition 3.2.10 (cf. [MW12] Definition 7.2.6)**.** *We say that a reduced section* $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ *is* **precompact** *if there is a nested reduction* $\mathcal{C} \sqsubset \mathcal{V}$ *such that*

$$\bigcup_{I \in \mathcal{I}_{\mathcal{K}}} \pi_{\mathcal{K}}\big((s_I|_{V_I} + \gamma\nu_I)^{-1}(0)\big) \subset \pi_{\mathcal{K}}(\mathcal{C}), \qquad \forall \gamma \in \Gamma_I, I \in \mathcal{I}_K.$$

We say it is **transverse to** 0 *if the following conditions hold for each* $I \in \mathcal{I}_{\mathcal{K}}, \gamma \in \Gamma_I$*:*

- $s_I|_{V_I} + \gamma\nu_I : V_I \to E_I$ *is transverse to* 0*;*
- *the intersection of the graph of* $s_I|_{V_I} + \gamma\nu_I$ *with the singular set* $\{(x, e) \in V_I \times E_I : |\mathrm{Stab}_{x,e}| > 1\}$ *has empty interior.*

For each I, there is a corresponding map $\underline{s_I|_{V_I} + \Gamma_I \nu_I}$ from the reduced intermediate domain $\underline{V}_I$ to the set $\mathrm{Fin.Set}\big(\underline{V_I \times E_I}\big)$ of finite subsets of $\underline{V_I \times E_I}$, namely

(3.2.7)
$$\underline{s_I|_{V_I} + \Gamma_I \nu_I} \,:\, \underline{V}_I \to \mathrm{Fin.Set}\big(\underline{V_I \times E_I}\big), \quad \underline{x} \mapsto \big\{ \underline{\big(x, (s_I|_{V_I} + \gamma \nu_I)(x)\big)} \,\big|\, \gamma \in \Gamma_I \big\}.$$

If each point $\underline{(x, e)} \in \underline{V_I \times E_I}$ is given the weight induced by the stabilizers of the action of Γ_I on $V_I \times E_I$,

$$m(\underline{(x, e)}) := {}^{\left|\mathrm{Stab}_{(x,e)}\right|}\!/_{|\Gamma_I|}, \tag{3.2.8}$$

then the sum of the weights in each set $\underline{s_I|_{V_I} + \Gamma_I \nu_I}(\underline{x})$ is equal to 1.

Orientations and Effectiveness. In order to apply the theory developed in [M07] concerning branched manifolds, we need to require that the action of Γ_I on $U_I \times E_I$ preserves an orientation, and also that the atlas is **effective** in the sense that for each $I \in \mathcal{I}_\mathcal{K}$ the restriction of the Γ_I action to any open subset of $U_I \times E_I$ is effective.[14] Note in particular if the action of Γ_I is noneffective on some V_I then one can never satisfy the second transversality condition above. However, if this orientation and effectiveness condition is satisfied, we can consider each family of sections $(\gamma \nu_I)_{\gamma \in \Gamma_I}$ to be a multisection in the sense of [M07, Definition 4.12]. Hence, by [M07, Definition 4.15] each local zero set of a transverse section is a weighted nonsingular branched (wnb) groupoid. Just as in the case with trivial isotropy, one can define an appropriate notion of orientation bundle on the atlas $|\mathcal{K}|$. One then shows that if ν is precompact these local zero sets fit together for the different $I \in \mathcal{I}_\mathcal{K}$ to form a compact wnb groupoid, that is oriented if $\mathcal{K}$ is. Hence, by the results in [M07, §3.4], it has a fundamental class. Thus the following analog of Proposition 2.4.7 holds.

Proposition 3.2.11. *Let $\mathcal{K}$ be an oriented, effective tame d-dimensional Kuranishi atlas with a reduction $\mathcal{V} \sqsubset \mathcal{K}$, and suppose that $\nu : \mathbf{B}_\mathcal{K}|_\mathcal{V} \to \mathbf{E}_\mathcal{K}|_\mathcal{V}$ is a precompact transverse perturbation. Then $|\mathbf{Z}_\nu| = |(s+\nu)^{-1}(0)|$ is a compact oriented weighted nonsingular branched d-dimensional manifold. Moreover, its quotient topology agrees with the subspace topology on $|(s+\nu)^{-1}(0)| \subset |\mathcal{K}|$.*

To complete the proof of Theorem B in the case with nontrivial isotropy it remains to construct suitable reductions and sections ν. Reductions (as well as the needed cobordism reductions) can be constructed on the level of the intermediate category, and hence exist by previous arguments. For the section, we just need to construct a single valued section $\nu : \mathcal{V} \to \mathrm{Obj}_{\mathbf{E}|_\mathcal{V}}$ as described in Definition 3.2.9 and then extend it by the group action. Although this section must be constructed on the space of objects $\bigsqcup_I U_I$ of $\mathbf{B}_\mathcal{K}$, the sets such as $V_J^{|J|}$ and $B^J_\delta(N^{|J|}_{JI})$ used in §2.4 to describe its inductive construction can all be pulled back from corresponding subsets of the domains of the intermediate category (which after all is where the metric lives). It follows that the construction goes through with essential change. The only point worthy of note is that

[14] One could probably dispense with these assumptions, but we will use them for simplicity.

at each stage one must take extra care in choosing the last small section $\nu_\pitchfork$ so as to satisfy the strengthened transversality condition in Definition 3.2.10.

Finally we note that, as before, a limiting procedure gives an element $[X]^{vir} \in \check{H}_d(X;\mathbb{R})$. Since all these constructions are unique up to cobordism, this class is independent of choices. This completes the sketch proof of Theorem B. For more details see [MW14].

Remark 3.2.12 (Relation to work of Fukaya et al in [FOOO, FOOO12])**.** The main difference between Kuranishi atlases and Kuranishi structures is that in the latter context one does not attempt to construct sum charts whose footprint is the full intersection F_I. When the isotropy is trivial this makes no real difference. However in the presence of isotropy, our approach gives more precise information about the isotropy groups, which makes it easier to control the construction of the perturbation section. If one uses a smooth gluing theorem and defines the invariant as the zero set of a perturbed multisection, then again it is not clear that this makes a decisive difference. However, in the de Rham context that Fukaya et al are currently developing, one needs auxiliary bundles that certainly would be easier to describe and understand in the language of atlases. In the world of [FOOO], Kuranishi atlases and good coordinate charts are somewhat different in nature, and in this new theory one needs to understand how to go back and forth between these notions; while for us a reduction is simply a subcategory of the Kuranishi category, and so is the same kind of structure. Further, if one uses a weak gluing theorem as in [MWss] and yet wants to construct a class of dimension $d > 1$, then our precise control of the isotropy group actions is essential. The isotropy action and coordinate changes are now not sufficiently smooth to preserve the notion of transversality automatically, i.e. they are not strongly SS, and hence one needs very precise information about the morphisms in the Kuranishi category; for more information see the discussion after Deefinition 3.3.4.

3.3. **Stratified smooth atlases.** As we will see in §5 in order to build a smooth Kuranishi atlas on a GW moduli space such as $X = \overline{\mathcal{M}}_{0,k}(M, J, A)$ we need a smooth version of the gluing theorem that builds a curve with smooth domain from one with nodal domain. Even if we ask that the structural maps in $\mathcal{K}$ are $\mathcal{C}^1$-smooth rather than $\mathcal{C}^\infty$-smooth, this is more than is provided by the simplest gluing theorems such as that in [MS]. On the other hand, in order to get a VFC we do not need the domains U_I of the Kuranishi charts to be smooth manifolds: since all we want is a homology class that we define as the zero set of transverse section ν, it is enough that U_I is stratified, with smooth top stratum and lower strata of codimension at least 2.[15] Here we briefly explain some elements of the approach in [MWss]. What we describe is enough to define $[X]^{vir}_{\mathcal{K}}$ if this is zero dimensional (therefore with one dimensional cobordisms), and hence enough to calculate all numerical GW invariants; cf. §5.2 [b]. We begin with some basic definitions.

[15]Pardon [P13] uses a homological way to define $[X]^{vir}$ and hence only needs the U_I to be topological manifolds. Thus a gluing theorem such as that in [MS] suffices.

Definition 3.3.1. *A pair* $(X, \mathcal{T})$ *consisting of a topological space* X *together with a finite partially ordered set* $(\mathcal{T}, \leq)$ *is called a* **stratified space with strata** $(X_T)_{T\in\mathcal{T}}$ *if:*

(i) *X is the disjoint union of the strata, i.e.* $X = \cup_{T\in\mathcal{T}} X_T$*, where* $X_T \neq \emptyset$ *for all* $T \in \mathcal{T}$ *and* $X_S \cap X_T = \emptyset$ *for all* $S \neq T$*;*

(ii) *the closure of each stratum intersects only deeper strata, i.e.* $cl(X_T) \subset \cup_{S\leq T} X_S$.

We denote the induced strict order by $S < T$ *iff* $S \leq T, S \neq T$.

Definition 3.3.2. *A* **stratified continuous** *map* $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ *between stratified spaces is a continuous map* $f : X \to Y$ *that induces a map* $f_* : \mathcal{T} \to \mathcal{S}$ *which preserves strict order. More precisely,*

(i) *f_* preserves strict order in the sense that* $T < S \Rightarrow f_*T < f_*S$ *, and*

(ii) *f maps strata into strata in the sense that* $f(X_T) \subset Y_{f_*T}$ *for all* $T \in \mathcal{T}$.

A stratified continuous map $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ *is called a* **stratified homeomorphism***, if f is a homeomorphism and f_* is bijective. In this case the spaces* $(X, \mathcal{T}), (Y, \mathcal{S})$ *are called* **stratified homeomorphic**.

Example 3.3.3. Let M be a smooth k-dimensional manifold and let $n \in \mathbb{N}_0$. The **standard SS space** $M \times \mathbb{C}^{\underline{n}}$ is the topological space $M \times \mathbb{C}^n$ with the following extra structure:

- the stratification $M \times \mathbb{C}^{\underline{n}} = \cup_{T\in\mathcal{T}^n} (M \times \mathbb{C}^{\underline{n}})_T$, where $\mathcal{T}^n$ is the set of all (possibly empty) subsets $T \subset \{1, \dots, n\}$, partially ordered by the subset relation, and whose strata are given by

$$(M \times \mathbb{C}^{\underline{n}})_T := \big\{(x; \mathbf{a}) \in M \times \mathbb{C}^n \,\big|\, \mathbf{a} = (a_1, \dots, a_n) \text{ with } a_i \neq 0 \Leftrightarrow i \in T\big\};$$

- the smooth structure induced on each stratum by the embedding $(M \times \mathbb{C}^{\underline{n}})_T \hookrightarrow M \times \mathbb{C}^{|T|}$, $(x; \mathbf{a}) \mapsto (x; (a_i)_{i\in T})$.

Every subset $U \subset M \times \mathbb{C}^{\underline{n}}$ inherits a stratification $\big(U_T = U \cap (M \times \mathbb{C}^{\underline{n}})_T\big)_{T\in\mathcal{T}_U^n}$ in the sense of Definition 3.3.1, which is called the **SS stratification** on U. Here we denote by $\mathcal{T}_U^n$ or sometimes just $\mathcal{T}_U$ the subset of $T \in \mathcal{T}$ for which the stratum U_T is nonempty.

Thus $\mathbb{C}^{\underline{1}}$ is the space $\mathbb{C}$ equipped with the two smooth strata $\{0\} = (\mathbb{C}^{\underline{1}})_\emptyset$ and $\mathbb{C}\setminus\{0\} = (\mathbb{C}^{\underline{1}})_{\{1\}}$, while if $M = \mathbb{R}^k$ we have the **Euclidean SS space** $\mathbb{R}^k \times \mathbb{C}^{\underline{n}}$. Note that the (real) codimension of the stratum $(M \times \mathbb{C}^{\underline{n}})_T \subset M \times \mathbb{C}^{\underline{n}}$ is $2(n - |T|)$ and hence is always even. We think of the components a_j of $\mathbf{a} \in \mathbb{C}^n$ as **strata variables**, while the components x_i of a local coordinate system near $x \in M$ are called **smooth variables**. This is the natural given by complex gluing parameters at nodes. This is also convenient notationally since it allows us to distinguish between the two types of variables. However, maps between coordinate charts need not be in any sense holomorphic, and (unless defined on a neighbourhood of zero) need not preserve the distinction between these two types of variable.

Definition 3.3.4. *Let* $f : U \to Y \times \mathbb{C}^{\underline{m}}$ *be a stratified continuous map defined on an open subset* $U \subset M \times \mathbb{C}^{\underline{n}}$*. We call f* **weakly stratified smooth** *(abbreviated* **weakly**

SS) *if it restricts to a smooth map* $U_T = U \cap (M \times \mathbb{C}^{\underline{n}})_T \to (Y \times \mathbb{C}^{\underline{m}})_{f_* T}$ *on each stratum* $T \in \mathcal{T}_U^n$. *A* **weakly SS diffeomorphism** *is an injective, weakly SS map* $\phi : U \to \phi(U)$ *with open image and a weakly SS inverse.*

It is easy to check that the composite of two weakly SS maps is weakly SS. Hence it is possible to define the notion of a **weakly SS manifold**, namely a stratified topological manifold whose transition functions are weakly SS diffeomorphisms between open subsets of Euclidean SS spaces. Such manifolds (if closed and with oriented top stratum) do have a fundamental class, since the singular set has codimension ≥ 2. However, unfortunately this context is not sufficiently rich to support a good theory of transversality. The maps we are interested in are sections of the local bundles $U \times E$ given by functions of the form $f := s + \nu : U \to E = \mathbb{R}^c$. If f is weakly SS and $f(w) = 0$ for $w \in U_S$, we say that f **is transverse to** 0 if the derivative $\mathrm{d}_w^S f : \mathrm{T}_w U_S \to \mathbb{R}^c$ is surjective at w, where d_w^S is the differential of the smooth restriction of f to the stratum U_S. However, this is not an open condition. For example, the weakly SS function $f : \mathbb{R} \times \mathbb{C}^{\underline{1}} \to \mathbb{R}$ given by $(x; a) \mapsto x(1 + |a| \sin(\frac{1}{|a|}))$ is transverse to zero at $(0; 0)$ in the above sense, since its restriction to the stratum $a = 0$ is transverse. However its zero set is not a manifold near $(0, 0)$. Therefore we cannot hope to define the VFC as the zero set of such a function. To get a good transversality theory we need to consider (strongly) SS functions, rather than weakly SS functions. We will avoid this problem here by restricting consideration to atlases of dimension $d = 0$ with cobordisms of dimension 1. Then one can always build sections whose zero set is contained in the top stratum, where everything is smooth so that one can use standard results on transversality. The general case will be treated in [MWss].

Let $(X, \mathcal{T})$ be a stratified space. We say that $\mathbf{K}_I = (U_I, E_I, \Gamma_I, s_I, \psi_I)$ is a **weakly SS Kuranishi chart** on X if the conditions of Definition 3.1.6 hold in the category of weakly SS manifolds and weakly SS diffeomorphisms. Thus U_I is an open subset of a weakly SS manifold, all maps are weakly SS diffeomorphisms, and the footprint map $\psi : s_I^{-1}(0) \to F_I$ is stratified continuous. Similarly, $\mathcal{K} = (\mathbf{K}_I, \widehat{\Phi}_{IJ})_{I \subset J, I, J \in \mathcal{I}_{\mathcal{K}}}$ is a **weakly SS (weak) Kuranishi atlas** if all the conditions of Definition 3.1.15 hold in the weakly SS category.

The arguments outlined above prove the following.

Proposition 3.3.5. *Let* $\mathcal{K}$ *be an oriented,* 0*-dimensional, weak, effective, weakly SS Kuranishi atlas on a compact metrizable stratified space* X. *Then* $\mathcal{K}$ *determines a rational number* $[X]^{vir}_{\mathcal{K}}$ *that depends only on the oriented cobordism class of* $\mathcal{K}$.

More formally, $[X]^{vir}_{\mathcal{K}}$, which is represented by a finite union of oriented, weighted points in $|\mathcal{K}|$, may be considered as an element in the Čech homology group $\check{H}_0(X; \mathbb{Q})$, which is canonically identified with $\mathbb{Q}$.

4. Additivity and products

We now generalize the definition of an atlas to allow its sum charts to have more general index sets, obstruction spaces and groups. This involves slightly changing the definition of the coordinate change $\widehat{\Phi}_{IJ}$ since the groups Γ_I, Γ_J are no longer products of groups indexed by the elements of I, J. However, their main characteristic remains unchanged, namely that they are defined by group covering maps $(\rho_{IJ}, \rho^\Gamma_{IJ}) : (\widetilde{U}_{IJ}, \Gamma_J) \to (U_{IJ}, \Gamma_I)$ where the kernel of $\rho^\Gamma_{IJ} : \Gamma_J \to \Gamma_I$ acts freely on $\widetilde{U}_{IJ}$. Our main aim is to adapt the construction so as to be compatible with products, and with structures relevant to the case when X has boundary; cf Example 4.1.3 (ii), (iii) and Proposition 4.1.11. Section §4.1 discusses the general theory, while Section 4.2 explains the relation of the semiadditive theory to our earlier results.

4.1. **Indexing sets.** We will require the index set $\mathcal{I}$ for the atlas charts to be a finite poset[16] with **enough minimal elements** in the following sense:

- every subset $\{I_1, \dots, I_k\}$ of $\mathcal{I}$ with an upper bound J has a unique least upper bound $\mathrm{lub}(I_1, \dots, I_k)$;
- if $m(\mathcal{I})$ denotes the set of minimal elements in $\mathcal{I}$, then each $I \in \mathcal{I}$ is the least upper bound of the set $m(I) = \{H \in m(\mathcal{I}) \mid H \le I\}$ of minimal elements it dominates;
- the set $\{m(I) : I \in \mathcal{I}\}$ of subsets of $m(\mathcal{I})$ is closed under taking nonempty intersection.

The second condition implies that each element $I \in \mathcal{I}$ is determined by $m(I)$, so that $\mathcal{I}$ injects into the poset $\mathcal{P}^*\big(m(\mathcal{I})\big)$ of nonempty subsets of the finite set $m(\mathcal{I})$, while the third condition implies that any two elements $I, J \in \mathcal{I}$ with $m(I) \cap m(J) \ne \emptyset$ have a greatest lower bound that we can think of as their intersection and is given by

$$I \cap J := \mathrm{glb}\big(m(I) \cap m(J)\big). \tag{4.1.1}$$

In particular, the image of $\mathcal{I}$ in $\mathcal{P}^*\big(m(\mathcal{I})\big)$ is closed under nonempty intersection. The charts indexed by elements $i \in m(\mathcal{I})$ play the role of the basic charts, while the others will be thought of as sum charts.

Example 4.1.1. (i) Take a finite collection $\mathcal{F} := (F_i)_{i=1,\dots,N}$ of nonempty subsets of some set X and then define $\mathcal{I}_\mathcal{F}$ to be the set of all elements $I \in \mathcal{P}^*(\{1, \dots, N\})$ such that $F_I := \bigcap_{i \in I} F_i \ne \emptyset$, ordered by inclusion. Then $m(\mathcal{I}_\mathcal{F}) = \{1, \dots, N\}$ and $\mathcal{I}_\mathcal{F}$ has enough minimal elements. Further, there is an order reversing map $I \mapsto F_I$. This is often injective, but as the next example shows, need not be.

(ii) (Products) Given two collections $\mathcal{F}, \mathcal{G}$ of subsets of X, Y respectively, the intersections of the sets in $\mathcal{F} \times \mathcal{G} := (F_i \times G_j)_{F_i \in \mathcal{F}, G_j \in \mathcal{G}}$ are labelled by the elements $(I, J) \in \mathcal{I}_\mathcal{F} \times \mathcal{I}_\mathcal{G}$ with order $(I, J) \le (I', J')$ iff $I \le I', J \le J'$. On the other hand, the elements of $\mathcal{I}_{\mathcal{F}\times\mathcal{G}}$ are collections $\big((i_k, j_k)\big)_{1 \le k \le \ell}$ such that $I = \{i_k : 1 \le k \le \ell\} \in \mathcal{I}_\mathcal{F}$ and $J = \{j_k : 1 \le k \le \ell\} \in \mathcal{I}_\mathcal{G}$. Both $\mathcal{I}_\mathcal{F} \times \mathcal{I}_\mathcal{G}$ and $\mathcal{I}_{\mathcal{F}\times\mathcal{G}}$ are posets with enough minimal elements. Further, in both cases the set of minimal elements can be identified with the set of pairs $(F_i, G_j) \in \mathcal{F} \times \mathcal{G}$. However, the map $\mathcal{I}_{\mathcal{F}\times\mathcal{G}} \to \mathcal{I}_\mathcal{F} \times \mathcal{I}_\mathcal{G}$ is not injective. For example, the element $(\{i_1, i_2\}, \{j_1, j_2\}) \in \mathcal{I}_\mathcal{F} \times \mathcal{I}_\mathcal{G}$ has seven preimages in $\mathcal{I}_{\mathcal{F}\times\mathcal{G}}$, including $\{(i_1, j_1), (i_2, j_2)\}$ and $\{(i_1, j_2), (i_2, j_1)\}$.

[16] a poset is a partially ordered set, i.e. it has a reflexive, transitive and antisymmetric relation.

Definition 4.1.2. *Consider a family $\mathcal{K} := (\mathbf{K}_I, \widehat{\Phi}_{IJ})_{I\le J, I,J\in\mathcal{I}}$ of Kuranishi charts and coordinate changes on X whose charts are indexed by a set $\mathcal{I}$ with enough minimal elements as above. We will denote minimal elements of $\mathcal{I}$ by $i \in m(\mathcal{I})$, and assume for all $I \in \mathcal{I}$ that*

$$\mathbf{K}_I = (U_I, E_I, \Gamma_I, s_I, \psi_I)$$

has footprint $F_I := \bigcap_{i\in m(I)} F_i \subset X$. We say that $\mathcal{K}$ is **semi-additive** *if the following conditions hold on the charts and coordinate changes.*

- *There is a surjection $\tau_{E\Gamma} : \mathcal{A}_E \to \mathcal{A}_\Gamma$ between two finite sets, and tuples $(E_\alpha)_{\alpha\in\mathcal{A}_E}$, $(\Gamma_\alpha)_{\alpha\in\mathcal{A}_\Gamma}$ where E_α is a finite dimensional vector space and Γ_α is a finite group that acts on $E_{\alpha'}$ whenever $\tau_{E\Gamma}(\alpha') = \alpha$. For each subset $A \subset \mathcal{A}_E$ (resp. $A \subset \mathcal{A}_\Gamma$) we define $E_A := \prod_{\alpha\in A} E_\alpha$, (resp. $\Gamma_A := \prod_{\alpha\in A}\Gamma_\alpha$). Then each Γ_α acts on $E_{\tau_{E\Gamma}^{-1}(\alpha)}$ by the diagonal action, so that there is a well defined product action of $\Gamma_{\tau_{E\Gamma}(A)}$ on E_A.*
- *There is an injective map $\tau_E : \mathcal{I} \to \mathcal{P}^*(\mathcal{A}_E)$ satisfying*

$$\tau_E(I) = \bigcup_{i\in m(I)} \tau_E(i), \tag{4.1.2}$$

  such that, with $\tau_\Gamma := (\tau_{E\Gamma})_* \circ \tau_E : \mathcal{I} \to \mathcal{P}^*(\mathcal{A}_\Gamma)$, the following holds for the groups and obstruction spaces of $\mathbf{K}_I$.
  - *$\Gamma_I := \Gamma_{\tau_\Gamma(I)}$ for each I, and the surjections $\rho^\Gamma_{IJ} : \Gamma_J \to \Gamma_I$ in the coordinate changes are given by the projections $\prod_{\alpha\in\tau_\Gamma(J)}\Gamma_\alpha \to \prod_{\alpha\in\tau_\Gamma(I)}\Gamma_\alpha$ and hence have kernel $\prod_{\alpha\in\tau_\Gamma(J)\setminus\tau_\Gamma(I)}\Gamma_\alpha$; in particular, by definition of coordinate change each group $\prod_{\alpha\in\tau_\Gamma(J)\setminus\tau_\Gamma(I)}\Gamma_\alpha$ acts freely on the set $\widetilde{U}_{IJ}$ with quotient Γ_I-equivariantly isomorphic to (U_{IJ}, Γ_I).*
  - *E_I is compatibly isomorphic to $\prod_{\alpha\in\tau_E(I)} E_\alpha$ for each I. In other words, there are Γ_I-equivariant isomorphisms $\sigma_I : \prod_{\alpha\in\tau(I)} E_\alpha \to E_I$ such that the following diagrams commute for all $I \le J$:*

$$\begin{array}{ccc} \prod_{\alpha\in\tau(I)} E_\alpha & \xrightarrow{\sigma_I} & E_I \\ \downarrow \iota_{IJ} & & \downarrow \widehat{\phi}_{IJ} \\ \prod_{\alpha\in\tau(J)} E_\alpha & \xrightarrow{\sigma_J} & E_J, \end{array}$$

    where ι_{IJ} is the natural inclusion and $\widehat{\phi}_{IJ} : E_I \to E_J$ are the inclusions occurring in the coordinate changes.

We say that $\mathcal{K}$ is a (weak) **semi-additive atlas** *if in addition the tangent bundle condition and (weak) cocycle condition hold. For short we will denote a semi-additive atlas by the tuple $(\mathbf{K}_I, \widehat{\Phi}_{IJ})_{\mathcal{I},\mathcal{A},\tau}$.*

Example 4.1.3. If $\mathcal{K}$ is an atlas with $\mathcal{A}_E = \mathcal{A}_\Gamma = m(\mathcal{I})$ and $\tau_E : m(\mathcal{I}) \to \mathcal{P}^*(\mathcal{A}_E)$ is induced by the identity map $m(\mathcal{I}) \to \mathcal{A}_E = m(\mathcal{I})$ then the above notion of semi-additivity reduces to the notion of additivity in [MW12]. In particular, the basic charts are those indexed by the elements in $m(\mathcal{I}) =: \{1, \dots, N\}$. In this case we write $\mathcal{I} = \mathcal{I}_\mathcal{K}$ and say that $\mathcal{K}$ is **standard**. Further we say that $\mathcal{K}$ is **additive** if it is semi-additive with $\mathcal{A}_E = m(\mathcal{I})$. See Remark 4.1.6 below for an explanation of why we make no similar requirement on $\mathcal{A}_\Gamma$.

Our main example is that of products. For simplicity we will consider their properties only in the case of standard atlases with $m(\mathcal{I}_i) =: \mathcal{A}_i$ as above. Let $\mathcal{A}_E = \mathcal{A}_\Gamma = \mathcal{A}_1 \sqcup \mathcal{A}_2$ and suppose that the poset $\mathcal{I}$ is a subset of the product $\mathcal{P}^*(\mathcal{A}_1) \times \mathcal{P}^*(\mathcal{A}_2)$ that contains all pairs $(\alpha_1, \alpha_2) \in \mathcal{A}_1 \times \mathcal{A}_2$. Then we can identify $m(\mathcal{I})$ with the product $\mathcal{A}_1 \times \mathcal{A}_2$, and define

$$\tau_E : m(\mathcal{I}) \to \mathcal{P}^*(\mathcal{A}_1 \sqcup \mathcal{A}_2), \qquad \tau_E(\alpha_1, \alpha_2) = \{\alpha_1, \alpha_2\}.$$

The elements of $\mathcal{I}$ are pairs (I_1, I_2) of nonempty subsets $I_j \in \mathcal{P}^*(\mathcal{A}_j)$, and $\tau_E = \tau_\Gamma$ is the map that considers $I_1 \cup I_2$ as an element of $\mathcal{P}^*(\mathcal{A}_1 \sqcup \mathcal{A}_2)$. With these choices, we assume given vector spaces $E^k_\alpha, \alpha \in \mathcal{A}_k$ and define $E_{(I_1,I_2)} :\cong E^1_{I_1} \times E^2_{I_2}$. Similarly, we have $\Gamma_{(I_1,I_2)} := \Gamma^1_{I_1} \times \Gamma^2_{I_2}$. In other words, we get the indexing structure of the product of two additive atlases as in the following definition.

Definition 4.1.4. *Let $(\mathcal{K}_i)_{i=1,2}$ be standard atlases $\big((\mathbf{K}_I, \widehat{\phi}_{IJ})_{I \subset J \in \mathcal{I}_i}\big)_{i=1,2}$ with basic charts indexed by $\mathcal{A}_i := \{1, \dots, N_i\}$. The product atlas $\mathcal{K}_1 \times \mathcal{K}_2$ is an atlas on $X_1 \times X_2$ with indexing set $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2$, charts equal to the products*

$$\mathbf{K}_{I_1,I_2} := \big(U_{I_1} \times U_{I_2}, E_{I_1} \times E_{I_2}, \Gamma_{I_1} \times \Gamma_{I_2}, s_{I_1} \times s_{I_2}, \psi_{I_1} \times \psi_{I_2}\big), \quad I_i \in \mathcal{I}_i,$$

and coordinate changes also given by product maps.

One should check that $\mathcal{K}_1 \times \mathcal{K}_2$ is a semiadditive atlas.

Example 4.1.5 (Atlases when X has boundary)**.** If the space X has a codimension 1 boundary $\partial X = Y_1 \times Y_2$ that is a product (or more generally a fiber product), it is natural to build an atlas on X whose indexing set $\mathcal{I}$ is a hybrid of standard and product types. To be consistent with the approach to cobordism taken in [MW12], we will assume that X has collared boundary, i.e. that a neighborhood of its boundary is identified with $\partial X \times [0, 2\varepsilon)$. The basic charts are of two kinds, either product charts $\mathbf{K}^\partial_{\alpha_1,\alpha_2}$ indexed by $(\alpha_1, \alpha_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ as in (ii) whose footprint has the form $(F^\partial_{\alpha_i} \times F^\partial_{\alpha_2}) \times [0, 2\varepsilon)$ where $F^\partial_{\alpha_i} \subset Y_i$ is the footprint of $\mathbf{K}_{\alpha_i}$, or an interior chart $\mathbf{K}_i$ with footprint $F_i \subset X \setminus \partial X \times [0, \varepsilon]$ indexed by $i \in m(\mathcal{I}_0)$. (See [MW12, §6.4] for precise definitions.) Then the set of basic charts in the whole atlas is indexed by $m(\mathcal{I}) = (\mathcal{A}_1 \times \mathcal{A}_2) \sqcup m(\mathcal{I}_0)$, while the sum charts, together with their obstruction spaces and groups, are indexed by triples $(I_1, I_2, J) \in \mathcal{P}^*(\mathcal{A}_1 \sqcup \mathcal{A}_2) \times \mathcal{I}_0$, where $\mathbf{K}_{(I_1,I_2,J)}$ has footprint $F_J \cap (F^\partial_{I_i} \times F^\partial_{I_2}) \times [0, 2\varepsilon)$ and we allow $J = \emptyset$ or $I_1 \cup I_2 = \emptyset$. It follows as in the proof of Proposition 5.2.3 that such charts can be built in the Gromov–Witten context.

Remark 4.1.6. (i) Definition 4.1.2 requires τ_E to be injective, though τ_Γ need not be. Thus the set of semi-additive families is *not* closed under the operation of refinement, in which for example a single basic chart $\mathbf{K}_i = (U_i, E_i, \Gamma_i, s_i, \psi_i)$ is replaced by a family of charts $\big((W_{ij}, E_i, \Gamma_i, s_i, \psi_i)\big)_{j=1,\dots,k_i}$, where the $(W_{ij})_j$ form an open cover of U_i. We restrict to the case of injective τ_E so that the existence proof for tame shrinkings carries through with minor changes; cf. Proposition 4.1.9. One natural setting in which to consider refinements is when the chart domains are general étale groupoids rather than group quotients, and the bundles are general orbibundles rather than trivialized bundles with a diagonal group action. Allowing charts to have these features would take us very far from our original idea. We explain in §6.2 an alternative approach to dealing with these issues.

(ii) In contrast, the precise indexing set chosen for the groups is largely irrelevant to the abstract theory. Additivity conditions are used to get tameness. But the existence proof for tameness is carried out on the level of the intermediate category, i.e. *after* we have quotiented out by the isotropy groups. Therefore it makes no difference if many sum charts have the same isotropy group. We assumed in Definition 4.1.2 that each group Γ_I is a product and that the group homomorphisms $\rho^\Gamma_{IJ} : \Gamma_J \to \Gamma_I$ are projections so that their kernel is naturally identified with $\Gamma_{J \setminus I}$. However, it is possible to consider more general families provided that they satisfy the coherence conditions formulated in Remark 3.1.18.

(iii) As we will see in §4.2, every semi-additive atlas has a cobordant extension whose obstruction bundles are additive. For abstract atlases we have not worked out a way to enlarge the groups to make them have the product form of a standard atlas. However, it is possible that one might be able to do this by exploiting the morphisms in $\mathbf{B}_{\mathcal{K}}$ as in Proposition 6.1.3. In the Gromov–Witten setting one can always do this, provided that each factor Γ_i is associated with a slicing manifold; cf. §5.2.

As with the notion of additivity, the semi-additive condition implies that

$$\widehat{\phi}_{IJ}(E_I) \;\cap\; \widehat{\phi}_{HJ}(E_H) \;=\; \widehat{\phi}_{(I\cap H)J}(E_{I\cap H}), \quad \forall H, I, J \in \mathcal{I}_{\mathcal{K}} \text{ with } H, I \subset J. \tag{4.1.3}$$

This holds because each obstruction space E_J is the direct product of the E_α over the index set $\alpha \in \tau(J)$ and, by equation (4.1.2), $\tau(I) \cap \tau(H) = \tau(I \cap H)$ for all H, I, where $I \cap H$ is defined in (4.1.1). Hence

$$s_J^{-1}(\widehat{\phi}_{IJ}(E_I)) \;\cap\; s_J^{-1}(\widehat{\phi}_{HJ}(E_H)) \;=\; s_J^{-1}(\widehat{\phi}_{(I\cap H)J}(E_{I\cap H})).$$

Note that with $E_\emptyset := \{0\}$, this equation holds when $I \cap H = \emptyset$, which in the current context means that $m(I) \cap m(H) = \emptyset$. A similar identity holds on the level of the intermediate charts, except that now we must replace the map s_I by the section $\underline{S}_I : \underline{U}_I \to \underline{U_I \times E_I}$ and understand $\underline{\widehat{\phi}_{IJ}}$ to be the orbibundle embedding $\underline{U_{IJ} \times E_I} \to \underline{U_J \times E_J}$ induced by $\phi_{IJ} \times \widehat{\phi}_{IJ} : U_I \times E_I \to U_J \times E_J$:

$$\underline{S}_J^{-1}(\operatorname{im}(\underline{\widehat{\phi}_{IJ}})) \;\cap\; \underline{S}_J^{-1}(\operatorname{im}(\underline{\widehat{\phi}_{HJ}})) \;=\; \underline{S}_J^{-1}(\operatorname{im}(\underline{\widehat{\phi}_{(I\cap H)J}})). \tag{4.1.4}$$

Notice also that, because the map $\tau : \mathcal{I} \to \mathcal{P}^*(\mathcal{A})$ in (4.1.2) is injective, the map $I \mapsto E_I$ is also injective unless $E_i = \{0\}$ for some i. In particular, if $I \neq H$ above $\dim(E_{I\cap H}) \leq \min(\dim(E_I), \dim(E_H))$ with strict inequality unless $E_i = \{0\}$ for all elements of $H \setminus I \cap H$ or $I \setminus I \cap H$.

Definition 4.1.7. *A semi-additive atlas is called* **tame** *if it satisfies the taming conditions*

$$U_{IJ} \cap U_{IK} = U_{I(J \cup K)} \qquad \forall I \subset J, K; \tag{4.1.5}$$

$$\phi_{IJ}(U_{IK}) = U_{JK} \cap s_J^{-1}\big(\widehat{\phi}_{IJ}(E_I)\big) \qquad \forall I \subset J \subset K. \tag{4.1.6}$$

Definition 4.1.8. *A semi-additive atlas is* **good** *if*

- *its realization $|\mathcal{K}| = \bigcup_I U_I/\sim$ is Hausdorff in the quotient topology;*
- *each projection $\pi_{\mathcal{K}} : U_I \to |\mathcal{K}|$ is a homeomorphism onto its image.*

Further, we say that a good atlas $\mathcal{K}$ is metrizable if there is a metric d on $|\mathcal{K}|$ such that its pull back to each U_I induces its standard topology.

Proposition 4.1.9. *Every weak semi-additive atlas $\mathcal{K}$ has a tame shrinking $\mathcal{K}'$. Moreover every such tame shrinking $\mathcal{K}'$ is good.*

Proof. This is proved in the additive case in [MW12] Propositions 6.2.3 and 6.3.4. No essential changes are needed in order for this argument to apply in the current situation, since, as explained in §2.2 above, it is based on the taming equations (2.2.3) and (2.2.4) together with the additivity condition (2.2.7), all of which are unchanged in the current context; cf. equations (4.1.5), (4.1.6) and (4.1.4). □

The notions of a Kuranishi cobordism and orientation bundle defined in [MW12, MW14] extend immediately to the semi-additive case. As before we say that $\mathcal{K}$ is effective if for each chart the group Γ_I acts effectively on all open subsets of the product $U_I \times E_I$.[17] We can now proceed to construct representatives for the fundamental class $[X]^{vir}_{\mathcal{K}}$ as before. Briefly, the idea is as follows. Starting from a good atlas, we first construct a **reduction** $\mathcal{V}$ of $\mathcal{K}$, i.e. a collection of open sets $V_I \sqsubset U_I$ such that

$$\bigcup_{I \in \mathcal{I}_{\mathcal{K}}} \pi_{\mathcal{K}}(V_I) \cap s_I^{-1}(0)) \supset \iota(X); \quad \pi_{\mathcal{K}}(\overline{V_I}) \cap \pi_{\mathcal{K}}(\overline{V_J}) \neq \emptyset \Longrightarrow I \subset J \text{ or } J \subset I.$$

Next we construct a coherent family of sections $\nu := \big(\nu_I : V_I \to E_I\big)_{I \in \mathcal{I}}$, such that $s|_{V_I} + \gamma\sigma_I$ is transverse to 0 for all $\gamma \in \Gamma_I$, and so that the

$$\mathbf{Z}_\nu := \textstyle\bigcup_{I \in \mathcal{I}, \gamma \in \Gamma_I} \big|(s_{V_I} + \nu)^{-1}(0)\big| \subset |\mathcal{K}|$$

is a compact oriented d-dimensional manifold without boundary (that is weighted and branched if there is isotropy).[18] Thus $\mathbf{Z}_\nu$ has a fundamental class that is represented in the singular homology of a small neighbourhood of the zero set $\iota_{\mathcal{K}}(X)$ in $|\mathcal{K}|$. Taking a sequence ν_k of admissible sections with norm converging to 0, one obtains an element in the Čech homology of X. It is unique because it is possible to join any two sections by a cobordism. Therefore, finally we obtain the followng result.

[17] As explained in more detail in [MW14, M14] we assume effectiveness for convenience. It is probably not necessary.

[18] Here d is the dimension of $\mathcal{K}$, i.e. $d = \dim(U_I) - \dim(E_I)$, a number that is independent of I because of the tangent bundle condition. For the precise conditions required of ν see [M14] or [MW12, MW14].

Corollary 4.1.10. *Every oriented, effective, weak, semi-additive atlas $\mathcal{K}$ on X of dimension d determines a Čech homology class $[X]^{vir}_{\mathcal{K}} \in \check{H}_d(X;\mathbb{Q})$ that depends only on the cobordism class of $\mathcal{K}$.*

We illustrate this construction by proving the following.

Proposition 4.1.11. *If $\mathcal{K}_i$ is a weak, oriented, effective, semi-additive Kuranishi atlas on X_i of dimension d_i for $i = 1, 2$, then the fundamental class $[X_1 \times X_2]^{vir}_{\mathcal{K}_1 \times \mathcal{K}_2}$ of the product atlas $\mathcal{K}_1 \times \mathcal{K}_2$ on $X = X_1 \times X_2$ is the product*

$$[X_1]^{vir}_{\mathcal{K}_1} \times [X_2]^{vir}_{\mathcal{K}_2} \in \check{H}_{d_1+d_2}(X_1 \times X_2;\mathbb{Q}).$$

This is not completely obvious because the product of two reductions does not have the right intersection pattern to be a reduction. This is clear even on the level of the footprint covering. If the subsets $(Z_I)_{I\in\mathcal{I}}$ of X satisfy

$$Z_I \cap Z_J \neq \emptyset \implies I \subset J \text{ or } J \subset I,$$

then the product covering on $X \times X$ has nonempty intersections $(Z_I \times Z_J) \cap (Z_J \times Z_I)$ for non-comparable pairs $(I, J), (J, I)$. We now show how to modify the product of two reductions to obtain a reduction of a product atlas.

First we prove a general result about coverings.

Lemma 4.1.12. *Suppose given a finite open cover of a compact metrizable Hausdorff space $X = \bigcup_{I\in\mathcal{I}} P_I$ such that*

$$P_I \cap P_J \subset P_{I\cup J} \quad \forall I, J \in \mathcal{I}.$$

Then there exists a **cover reduction** $\big(Z_I\big)_{I\in\mathcal{I}}$ *with the following properties: The $Z_I \subset X$ are (possibly empty) open subsets satisfying*

(i) *$Z_I \sqsubset P_I$ for all I;*
(ii) *if $\overline{Z_I} \cap \overline{Z_J} \neq \emptyset$ then $I \subset J$ or $J \subset I$;*
(iii) *$X = \bigcup_I Z_I$.*

Proof. Since X is compact Hausdorff, we may choose precompact open subsets $Q'_I \sqsubset P_I$ that still cover X. We claim we may enlarge these sets to Q_I with $Q'_I \subset Q_I \sqsubset P_I$ so that $Q_I \cap Q_J \subset Q_{I\cup J}$ for all I, J. For this, we define $Q_I = Q'_I$ if $|I| = 1$ and then define Q_I by induction over $|I|$ by setting

$$Q_I = Q'_I \cup \textstyle\bigcup_{(I_j)\in\mathcal{S}(I)} \bigcap_j Q_{I_j},$$

where $\mathcal{S}(I)$ is the set of all collections $(I_j)_{j\in H}$ such that $\cup_{j\in H} I_j = I$ and $I_j \neq I$. Note that for each such collection $(I_j)_{j\in H}$, the induction hypothesis implies that

$$\textstyle\bigcap_{j\in H} Q_{I_j} \quad \sqsubset \quad \bigcap_{j\in H} P_{I_j} \subset P_I.$$

Therefore $Q_I \sqsubset P_I$ since it is a finite union of precompact subsets of P_I.

Repeating this procedure $2M$ times where $M = \max_{I\in\mathcal{I}} |I|$, we obtain families of nested sets

$$Q^0_I \sqsubset P^1_I \sqsubset Q^1_I \sqsubset P^2_I \sqsubset \ldots \sqsubset Q^M_I := P_I, \tag{4.1.7}$$

such that $P_I^k \cap P_J^k \subset P_{I\cup J}^k$, and $Q_I^k \cap Q_J^k \subset Q_{I\cup J}^k$ for all k. Now define

$$Z_I \;:=\; P_I^{|I|} \smallsetminus \bigcup_{|J|>|I|} \overline{Q_J^{|I|}}. \tag{4.1.8}$$

These sets are open since they are the complement of a finite union of closed sets in the open set $P_I^{|I|}$. The precompact inclusion $Z_I \sqsubset P_I$ in (i) holds since $P_I^{|I|} \sqsubset P_I$.

To prove the covering in (iii) let $x \in X$ be given. Then we claim that $x \in Z_{I_x}$ for

$$I_x := \bigcup_{x\in P_I^{|I|}} I \quad \subset \quad \{1,\dots,N\}.$$

Indeed, $i \in I_x$ implies $x \in P_I^{|I|}$ for some $i \in I \subset I_x$. Therefore we can write $I_x = \cup_{j\in H} I_j$ where $x \in P_{I_j}^{|I_j|}$ for all $j \in H$. Hence

$$x \in \bigcap_{j\in H} P_{I_j}^{|I_j|} \subset \bigcap_{j\in H} P_{I_j}^{|I_x|} \subset P_{I_x}^{|I_x|},$$

where the last step holds by assumption on the covering $(P_I^k)_{I\in\mathcal{I}}$. On the other hand, if $x \in \overline{Q_J^{|I_x|}}$ for some J with $|J| > |I_x|$, then $x \in \overline{Q_J^{|I|}} \subset P_J^{|J|}$, which contradicts the definition of I_x. Hence $x \in Z_{I_x}$ as claimed.

To prove the intersection property (ii), suppose to the contrary that $x \in \overline{Z_I} \cap \overline{Z_J}$ where $|I| \le |J|$ but $I \smallsetminus J \ne \emptyset$. Then $I\cup J \supsetneq J$ and we have $x \in \overline{Z_I} \cap \overline{Z_J} \subset Q_I^{|I|} \cap Q_J^{|J|} \subset Q_{J\cup I}^{|J|}$, which is impossible because $Q_{J\cup I}^{|J|}$ has been removed from $\overline{Z_J}$. Thus the sets Z_I form a cover reduction. □

Corollary 4.1.13. *Suppose that $\mathcal{K}$ is a good atlas with footprint cover $(F_I)_{I\in\mathcal{I}}$, and let $P_I \subset F_I$ be any sets such that $P_I \cap P_J \subset P_{I\cup J}$. Suppose further that $W_I \sqsubset U_I$ are Γ_I-invariant sets such that $W_I \cap s_I^{-1}(0) = \psi_I^{-1}(P_I)$. Then $\mathcal{K}$ has a reduction $\mathcal{V} := (V_I)_{I\in\mathcal{I}}$ such that $V_I \subset W_I$ for all I. It is unique up to cobordism.*

Proof. Choose a cover reduction $(Z_I \sqsubset P_I)_{I\in\mathcal{I}}$ of X as in the previous lemma, and then for each I choose an open set $W_I' \sqsubset W_I$ such that $W_I' \cap s_I^{-1}(0) = \psi_I^{-1}(Z_I)$. For each I, let $\mathcal{C}(I) = \{J \in \mathcal{I} : I \subset J,\text{ or } J \subset I\}$ and then define $Y_I := \bigcup_{J\notin\mathcal{C}(I)} \overline{W_I'} \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(\overline{W_J'}))$. Then Y_I is closed because $\pi_{\mathcal{K}} : U_J \to |\mathcal{K}|$ is homeomorphism for each J by goodness. Further $Y_I \cap s_I^{-1}(0) \subset \psi_I^{-1}(F_I \cap F_J) = \emptyset$ by construction. Hence we may choose an open neighbourhood $\mathcal{N}(Y_I)$ of Y_I in $U_I \smallsetminus s_I^{-1}(0)$, and then set $V_I := W_I' \smallsetminus \mathcal{N}(Y_I)$. The statement about cobordism follows as in [MW12]. □

Proof of Proposition 4.1.11. By Proposition 4.1.9 we may suppose that each $\mathcal{K}_i$ is good, and then choose reductions $\mathcal{V}_i$ of $\mathcal{K}_i$ and admissible sections $\nu_i : \mathbf{B}_{\mathcal{K}_i}\big|_{\mathcal{V}_i} \to \mathbf{E}_{\mathcal{K}_i}\big|_{\mathcal{V}_i}$. For each chart $\mathbf{K}_{I_i}\times\mathbf{K}_{I_2}$ of the product atlas, define $P_{I_1,I_2} = V_{I_1}\times V_{I_2} \subset U_{I_1}\times U_{I_2}$. Since $P_{I_1,I_2} \cap P_{J_1,J_2} = P_{I_1\cup J_1, I_2\cup J_2}$, we can apply Corollary 4.1.13 to construct a reduction $\mathcal{V} = (V_{I_1,I_2})_{(I_1,I_2)\in\mathcal{I}_{\mathcal{K}_1}\times\mathcal{I}_{\mathcal{K}_2}}$ with $V_{I_1,I_2} \subset V_{I_1}\times V_{I_2}$. Now define

$$\nu : \mathbf{B}_{\mathcal{K}_1\times\mathcal{K}_2}\big|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}_1\times\mathcal{K}_2}\big|_{\mathcal{V}},$$

by setting

$$\nu_{I_1,I_2} : V_{I_1,I_2} \to E_{I_1}\times E_{I_2}, \quad (u_1,u_2) = \big(\nu_1(u_1), \nu_2(u_2)\big).$$

It is easy to check that ν satisfies the necessary conditions. In particular, $s+\nu$ is transverse to 0 on each V_{I_1,I_2} with zero set equal to the product of the zero sets of ν_1 and ν_2. It follows that the realization $\mathbf{Z}_\nu$ of the zero set is the product $\mathbf{Z}_{\nu_1}\times\mathbf{Z}_{\nu_2}$. Hence the resulting homology class $[X_1\times X_2]^{vir}_{\mathcal{K}_1\times\mathcal{K}_2}$ in $\check{H}_{d_1+d_2}(X_1\times X_2)$ is the product $[X_1]^{vir}_{\mathcal{K}_1}\times[X_2]^{vir}_{\mathcal{K}_2}$. □

4.2. **Additive extensions.** The more flexible definition of Kuranishi atlas introduced above allows us to construct two atlases that have the same basic charts but different indexing sets for the obstruction spaces and groups. In the Gromov–Witten setting there is a very powerful sum construction that allows us to prove that atlases constructed in different ways from the same basic charts are always cobordant. For example, we show in Proposition 5.2.3 that the product of two GW atlases $\mathcal{K}_1\times\mathcal{K}_2$ with indexing set $\mathcal{I}_{\mathcal{K}_1}\times\mathcal{I}_{\mathcal{K}_2}$ as in Definition 4.1.4 is cobordant to the standard GW atlas with the same set of basic charts $\mathbf{K}_{i_1}\times\mathbf{K}_{i_2}$ but indexing set $\mathcal{I}_{\mathcal{K}_1\times\mathcal{K}_2}$ as defined in Example 4.1.1 (ii). It follows that these atlases define the same virtual class.

However, in the abstract it is not clear how to construct such cobordisms; in fact it is not even clear how to build a (semi)-additive atlas from a set of basic charts since there is no abstract sum construction. The next proposition shows that we can promote every semi-additive atlas $\mathcal{K}$ to an additive atlas $\mathcal{K}'$ whose indexing set $\mathcal{I}_{\mathcal{K}'}$ is determined by the intersection pattern of the footprints. Here $\mathcal{K}'$ is additive in the sense of Example 4.1.3, i.e. the groups $(\Gamma_I)_{I\in\mathcal{I}_\mathcal{K}}$ will in general not be products $\prod_{i\in I}\Gamma_i$ of the groups Γ_i of the basic charts, but rather will be the same as those in $\mathcal{K}$.

Proposition 4.2.1. (i) *Every semi-additive weak atlas $\mathcal{K}=(\mathbf{K}_I,\widehat{\Phi}_{IJ})_{\mathcal{I},\mathcal{A},\tau}$ has a canonical extension to an additive weak atlas $\mathcal{K}'$ with the same basic charts as $\mathcal{I}$.*

(ii) *Moreover, if $\mathcal{K}$ is an atlas, so is $\mathcal{K}'$ and there is a functor $f:\mathbf{B}_{\mathcal{K}'}\to\mathbf{B}_\mathcal{K}$ such that the induced map $|\mathcal{K}'|\to|\mathcal{K}|$ is surjective with contractible fibers.*

(iii) *The two (weak) atlases $\mathcal{K}'$ and $\mathcal{K}$ are semi-additively cobordant.*

Proof. Let $m(\mathcal{I})$ be the set of minimal elements in $\mathcal{I}$ and denote

$$\mathcal{I}_{\mathcal{K}'}:=\{J\subset\mathcal{P}^*(m(\mathcal{I}))\;\big|\;F_J:=\textstyle\bigcup_{j\in J}F_j\neq\emptyset\}.$$

By assumption on $\mathcal{I}$, the least upper bound function

$$\ell:\mathcal{P}^*(m(\mathcal{I}))\to\mathcal{I},\quad J\mapsto\ell(J):=\text{l.u.b.}(m(J))$$

defines a map $\ell:\mathcal{I}_{\mathcal{K}'}\to\mathcal{I}$ such that $F_J=F_{\ell(J)}$ for all $J\in\mathcal{I}_{\mathcal{K}'}$. We define the weak atlas $\mathcal{K}'$ to have charts indexed by $J\in\mathcal{I}_{\mathcal{K}'}$. We take $\mathcal{A}'_E=m(\mathcal{I})$ with

$$\tau'_E:\mathcal{P}^*(m(\mathcal{I}))\to\mathcal{P}^*(\mathcal{A}'_E) \tag{4.2.1}$$

induced by the identity, and take $\mathcal{A}'_\Gamma=\mathcal{A}_\Gamma$ where

$$\tau'_\Gamma:\mathcal{I}_{\mathcal{K}'}=\mathcal{P}^*\big(m(\mathcal{I})\big)\to\mathcal{P}^*\big(\mathcal{A}_\Gamma\big)\text{ is induced by }\tau_\Gamma:m(\mathcal{I})\to\mathcal{A}_\Gamma.$$

For $i\in m(\mathcal{I})$ define $E'_i:=E_i=\prod_{\alpha\in\tau_E(i)}$. Then $E'_I=\prod_{i\in I}E'_i$ may also be written as the product $\prod_{\alpha\in\tau_E(I)}E_\alpha^{m_{\alpha,I}}$, where the multiplicities $m_{\alpha,I}\geq1$ are defined as follows:

$$m_{\alpha,I}=\big|\{i\in I\mid\alpha\in\tau_E(i)\}\big|.$$

Therefore we can write the elements e'_I of E'_I as tuples $\big(\vec{e}_\alpha\big)_{\alpha\in\tau_E(I)}$ where $\vec{e}_\alpha = (e^k_\alpha)_{1\le k\le m_{\alpha,I}}$ is an $m_{\alpha,I}$-tuple of vectors in E_α. With this notation the map $\widehat{\phi}'_{IJ}: E'_I \to E'_J$ is the obvious inclusion with image equal to

$$\widehat{\phi}'_{IJ}(E'_I) = \Big\{ \big(\vec{e}_\alpha\big)_{\alpha\in\tau_E(J)} \mid \vec{e}_\alpha = \vec{0}\ \forall \alpha\in \tau_E(J){\smallsetminus}\tau_E(I) \Big\} \subset E'_J. \tag{4.2.2}$$

Since we chose $\mathcal{A}'_\Gamma = \mathcal{A}_\Gamma$, the group Γ'_I can be identified with $\Gamma_{\ell(I)} = \prod_{\alpha\in\tau_\Gamma(\ell(I))}\Gamma_\alpha$. Hence it acts on E'_I via the diagonal action of each Γ_α on the elements $\vec{e}_{\alpha'} = (e^k_{\alpha'})_k$ of $E_{\alpha'}$, where $\tau_{E\Gamma}(\alpha') = \alpha$. Further, because $E_{\ell(I)} = \prod_{\alpha\in\tau_E(\ell(I))} E_\alpha$ there is a projection:

$$\sigma_I : E'_I \to E_{\ell(I)}, \quad \big(\vec{e}_\alpha\big)_{\alpha\in\tau(\ell(I))} \mapsto \big(\sigma_\alpha(\vec{e}_\alpha)\big)_{\alpha\in\tau(\ell(I))}, \tag{4.2.3}$$

where we define $\sigma_\alpha(\vec{e}_\alpha) := \sum_{k=1}^{m_\alpha} e^k_\alpha \in E_\alpha$. This map is $(\Gamma'_I, \Gamma_{\ell(I)})$ equivariant, and also satisfies the compatibility condition:

$$\sigma_J\circ\widehat{\phi}'_{IJ} = \widehat{\phi}_{IJ}\circ\sigma_I, \quad E'_I\to E_J.$$

Now define the domains U'_I of the charts of $\mathcal{K}'$ and the section s'_I by setting:

$$U'_I := \Big\{ (e'_I, u)\in E'_I\times U_{\ell(I)} \mid s_{\ell(I)}(u) = \sigma_I(e'_I)\Big\}, \quad s'_I(e'_I,u) := e'_I. \tag{4.2.4}$$

Thus, since $\ell(i) = i$, the basic chart $\mathbf{K}_i = (U_i, E'_i = E_i, \Gamma'_i = \Gamma_i, s'_i, \psi'_i)$ has domain U'_i consisting of all pairs $(e_i, u)\in E_i\times U_i$ with $s_i(u) = e_i\in E_i$. Thus we can identify U'_i with U_i, and take $\phi'_i = \psi_i$ so that $\mathbf{K}'_i\cong\mathbf{K}_i$. On the other hand, if $|I|>1$ there is a fibration

$$f_I : U'_I \to U_{\ell(I)}, \quad (e'_I, u)\mapsto (\sigma'_I(e'_I), u)\mapsto u\in U_{\ell(I)},$$

where the second map is a diffeomorphism since $\sigma'_I(e'_I)\in E_{\ell(I)} = s_{\ell(I)}(u)$ is uniquely determined by u. Note also that f_I restricts to a diffeomorphism from $(s'_I)^{-1}(0) = \{(e'_I,u)\in U'_I \mid e'_I = 0\}$ to $s^{-1}_{\ell(I)}(0)$, which implies that the footprint map $\psi'_I := \psi_{\ell(I)}\circ f_I : (s'_I)^{-1}(0)\to F_I$ can be identified with $\psi_{\ell(I)} : (s_{\ell(I)})^{-1}(0)\to F_{\ell(I)} = F_I$ and hence induces a homeomorphism ${(s'_I)^{-1}(0)}/{\Gamma_I}\to F_I$. Therefore the chart

$$\mathbf{K}'_I := \big(U'_I, \Gamma'_I = \Gamma_{\ell(I)}, E'_I, s'_I, \psi'_I\big),$$

has the footprint $F_I = F_{\ell(I)}$.

If $I\subset J$ define

$$\widetilde{U}'_{IJ} := \big\{(e'_J, u)\in U'_J \mid u\in \widetilde{U}_{\ell(I)\ell(J)},\ e'_J\in \operatorname{im}\widehat{\phi}_{IJ}(E'_I)\big\}.$$

By (4.2.2), the elements $e'_J\in\operatorname{im}\widehat{\phi}_{IJ}(E'_I)$ can be identified with a unique element $\rho^E_{IJ}(e'_J)\in E'_I$. Hence the natural map

$$\rho'_{IJ} : \widetilde{U}'_{IJ}\to U_{\ell(I)}, \quad (e'_J, u)\mapsto \big(\rho^E_{IJ}(e'_J), \rho_{IJ}(u)\big)$$

is injective on the first component (on which $\Gamma'_{J\smallsetminus I}$ acts trivially), and hence quotients out by the action of $\Gamma'_{J\smallsetminus I}\cong\Gamma_{\ell(J)\smallsetminus\ell(I)}$ as required.

This completes the construction of the weak atlas $\mathcal{K}'$. It is additive by construction. Further, because all the domains $U'_J, \widetilde{U}'_{IJ}$ in $\mathcal{K}'$ are products with a suitable vector space of the corresponding domains in $\mathcal{K}$, the weak atlas $\mathcal{K}'$ satisfies the cocycle condition precisely if $\mathcal{K}$ does. Thus (i) and the first part of (ii) hold. The functor $f : \mathcal{K}' \to \mathcal{K}$ is induced by the projections $f_i : U'_I \to U_{\ell(I)}, (e'_I, u) \to u$. Each fiber of f_I is a vector space isomorphic to the product of the kernels of the maps $\big(\sigma_\alpha : E_\alpha^{\times m_\alpha} \to E_\alpha\big)_{\alpha \in \tau_E(\ell(I))}$ in (4.2.3). Therefore the fiber over a point $|(\ell(I), x)| \in |\mathcal{K}|$ is isomorphic to the quotient of a vector space by the action of the finite group $\mathrm{Stab}(\ell(I), x)$, which is contractible (though it need not be a vector space). This proves (ii).

To prove (iii) we must construct a cobordism between $\mathcal{K}$ and $\mathcal{K}'$, i.e. a atlas over $X \times [0,1]$ with product form near the boundary, that restricts to $\mathcal{K}$ over $X \times \{0\}$ and to $\mathcal{K}'$ over $X \times \{1\}$. This cobordism will contain the product charts[19] $(\mathbf{K}_I \times [0, \frac{2}{3}))_{I \in \mathcal{I}}$ and $(\mathbf{K}'_J \times (\frac{1}{3}, 1])_{J \in \mathcal{I}_{\mathcal{K}'}}$ as well as additional sum charts $\mathcal{K}^{01}_{(I,J)}$ that are indexed by the pairs $(I, J) \in \mathcal{I} \times \mathcal{I}_{\mathcal{K}'}$ for which the intersection $\big(F_I \cap F_{\ell(J)}\big)$ is nonempty. We define $\mathcal{K}^{01}_{(I,J)} = (U_{I,J}, E_I \times E'_J, \Gamma_{I \cup \ell(J)}, s_{I,J}, \psi_{I,J})$ where

$$\begin{aligned} U_{I,J} = \ & \big\{(e_I, e'_J, u, t) \in E_I \times E'_J \times U_{I \cup \ell(J)} \times (\tfrac{1}{3}, \tfrac{2}{3}) : \\ & \qquad s_{I \cup \ell(J)} u = \widehat{\phi}_{I(I \cup \ell(J))} e_I + \widehat{\phi}_{\ell(J)(I \cup \ell(J))} \sigma_J(e'_J)\big\} \end{aligned}$$

and $s_{I,J}(e_I, e'_J, u) = (e_I, e'_J)$. The group $\Gamma_{I \cup \ell(J)}$ acts on $U_{I \cup \ell(J)}$ by definition, and also on $E_I \times E'_J$ because it is a product of factors Γ_α each of which acts diagonally on the set of factors E_α occurring in $E_I \times E'_J$. It is straightforward to check that $\mathcal{K}^{01}_{(I,J)}$ is a chart with footprint $(F_I \cap F_{\ell(J)}) \times (\frac{1}{3}, \frac{2}{3})$.

To finish the definition of an atlas we need to describe the coordinate changes. These are indexed by pairs $(I, J), (H, L)$ where $I \subset H, J \subset L$ and are determined by the choice of subset $\widetilde{U}_{(H,J),(I,L)}$ of $U_{(I,L)}$. We take

$$\widetilde{U}_{(H,J),(I,L)} = \big\{(e_I, e'_L, u, t) \in U_{I,L} \mid e_I \in \widehat{\phi}_{HI}(E_I), e'_L \in \widehat{\phi}_{KL}(E'_K), u \in \widetilde{U}_{(H \cup \ell(J),(I \cup \ell(L)}\big\},$$

with projection to $\widetilde{U}_{(H,J)}$ given by

$$\rho_{(H,J),(I,L)} : (e_I, e'_L, u, t) \mapsto \big(\widehat{\phi}^{-1}_{HI}(e_I), \widehat{\phi}^{-1}_{KL}(e'_J), \rho_{(H \cup \ell(J))(I \cup \ell(L))}(u), t\big).$$

Just as before, the kernel $\Gamma_{(I \cup \ell(L)) \smallsetminus (H \cup \ell(J))}$ of the projection $\Gamma_{I,L} \to \Gamma_{(H,J)}$ acts freely on the u-component of the elements in $\widetilde{U}_{(H,J),(I,L)}$, and its orbits may be identified with points $U_{H \cup \ell(J)}$. Since all required compatibility conditions are satisfied, this completes the construction. □

5. Gromov–Witten atlases

We begin by discussing the proof of Theorem A. Consider a closed $2n$-dimensional symplectic manifold (M, ω) with ω-tame almost complex structure J, and let $X = \overline{\mathcal{M}}_{0,k}(M, A, J)$, the space of equivalence classes of genus zero, k-marked stable maps to

[19] if $\mathcal{K}$ is a chart (U, E, Γ, s, ψ) on X and $A \subset [0, 1]$ is an interval, then the product chart $\mathcal{K} \times A$ on $X \times [0, 1]$ is $(U \times A, E, \Gamma, s \circ pr_U, \psi \times \mathrm{id})$, where $\mathrm{pr}_U : U \times A \to U$ is the projection. For a detailed discussion of cobordisms see [MW12, §6.4] and [MW14]. There are further comments in Remark 6.1.4.

M in class $A \in H_2(M;\mathbb{Z})$. Section 5.1 explains how to construct a d-dimensional weak Kuranishi atlas on X, where

$$d := \operatorname{ind}(A) = 2n + 2c_1(A) + 2k - 6. \tag{5.0.5}$$

This atlas is either weakly SS or ($\mathcal{C}^1$)-smooth, depending on the gluing theorem that we use, see (VII) below, and is unique up to cobordism as required by Theorem A. Though we describe the construction in some detail, we do not carry out the necessary analysis; for this, see [C].

Section §5.2 explains variants of the basic method.

5.1. **Construction of charts in the genus zero GW setting.** We begin by explaining how to build a basic chart $\mathbf{K} = (U, E, \Gamma, s, \psi)$ near a point $[\Sigma_0, \mathbf{z}_0, f_0] \in X$. We denote by $[\Sigma_0, \mathbf{z}_0, f_0]$ the equivalence class of the stable map $(\Sigma_0, \mathbf{z}_0, f_0)$. The domain Σ_0 is a connected finite union of standard spheres $(S^2)_{\alpha\in T}$ joined at nodal pairs $(S^2)_\alpha \ni n^0_{\alpha\beta} = n^0_{\beta\alpha} \in (S^2)_\beta$, with the marked point z^i_0 lying on the component $(S^2)_{\alpha_i}$. Thus $f_0 : \Sigma_0 = \bigcup_{\alpha\in T}(S^2)_\alpha \to M$ satisfies $f(n^0_{\alpha\beta}) = f(n^0_{\beta\alpha})$. Because Σ_0 has genus zero, T is a tree whose directed edges determine a symmetric relation E on T such that $(S^2)_\alpha$ is joined to $(S^2)_\beta$ at $n^0_{\alpha\beta} = n^0_{\beta\alpha}$ exactly if $\alpha E \beta$ and $\beta E \alpha$. For short, the nodal points are denoted $\mathbf{n}_0 := (n^0_{\alpha\beta})_{\alpha E\beta}$.

Quite a few choices are involved in constructing the chart; we list the main ones here.

(I): The added marked points. The chart is determined by the choice of a **slicing manifold** Q, a codimension 2 (open, possibly disconnected) submanifold of M that is transversal to $\operatorname{im} f_0$ and meets it in enough points $f_0^{-1}(Q) = \{w^1_0, \dots, w^L_0\} =: \mathbf{w}_0$ to stabilize its domain, i.e. so that there are at least three special points (nodal or marked) on each component. We assume the points w^ℓ_0 are disjoint from $\mathbf{z}_0 \cup \mathbf{n}_0$. (If $[\Sigma_0, \mathbf{z}_0]$ is already stable there is no need to add these points. In this case we allow $\mathbf{w}_0$ to be the empty tuple.) Since $(\Sigma_0, \mathbf{w}_0, \mathbf{z}_0)$ is stable, it is described up to biholomorphism by its tuple of special points. Thus we may write $\delta_0 \in \overline{\mathcal{M}}_{0,k+L}$ either as $[\mathbf{n}_0, \mathbf{w}_0, \mathbf{z}_0]$ or as $[\Sigma_0, \mathbf{w}_0, \mathbf{z}_0]$, and will work over a suitable neighbourhood Δ of δ_0 in $\overline{\mathcal{M}}_{0,k+L}$.

(II): The group. We take Γ to be the stabilizer subgroup of $[\Sigma_0, \mathbf{z}_0, f_0]$, so that each $\gamma \in \Gamma$ acts on Σ_0 by a biholomorphism $\phi_\gamma : \Sigma_0 \to \Sigma_0$, permuting the points in $\mathbf{w}_0$ (and hence also T and $\mathbf{n}_0$) while fixing those in $\mathbf{z}$ and leaving f_0 unchanged: $f_0 = f_0 \circ \phi_\gamma$. We therefore consider Γ to be a subgroup of S_L, the symmetric group on L letters, acting via[20]

$$\mathbf{w}_0 \mapsto \gamma \cdot \mathbf{w}_0 := (w^{\gamma(\ell)})_{1\le\ell\le L}. \tag{5.1.1}$$

The induced action $\alpha \mapsto \gamma(\alpha)$ on T has the property that

$$w^\ell_0 \in (S^2)_\alpha \implies (\gamma \cdot \mathbf{w}_0)^\ell = w^{\gamma(\ell)} \in (S^2)_{\gamma(\alpha)}.$$

[20]We could take Γ to be any subgroup of S_L that contains the isotropy group, but this complicates the description of the action given in (5.1.9) below; cf. [MW14, Remark XX].

Correspondingly we define a Γ action on $\mathbf{n}$ by $(\gamma\cdot\mathbf{n})_{\alpha\beta} := n_{\gamma(\alpha)\gamma(\beta)}$. We choose small disjoint neighbourhoods $(D^\ell_0)_{\ell=1,\dots,L} \subset \Sigma_0\smallsetminus(\mathbf{z}_0\cup \mathit{nodes})$ of the (w^ℓ_0), averaging them over the Γ-action, so that Γ acts on them by permutation. Later we will use these discs to control the added marked points, specially those in a sum chart.

(III): The normalization conditions and universal curve. The above description of Σ_0 in terms of its nodal points $\mathbf{n}_0$ is not unique since the Möbius group acts on each component. However, in order to describe the equation satisfied by the elements in the domain of the chart, it is important to fix a parametrization for Σ_0 and the nearby domains. To this end, we fix the positions of three of the special points $\mathbf{n}_0\cup\mathbf{w}_0\cup\mathbf{z}_0$ on each component to be $0,1,\infty$. Thus we choose an injective function

$$\mathbf{P}: T\times\{0,1,\infty\}\to\ \{(\alpha,\beta)\mid \alpha E\beta\})\cup\{1,\dots,L\}\cup\{1,\dots,k\} \tag{5.1.2}$$

that takes $\{\alpha\}\times\{0,1,\infty\}$ to three labels for points in $(S^2)_\alpha$. We denote the set of points with labels in $\operatorname{im}(\mathbf{P})$ by $(\mathbf{n}_0)_\mathbf{P}\cup(\mathbf{w}_0)_\mathbf{P}\cup(\mathbf{z}_0)_\mathbf{P}$, and write p_n, p_w, p_z for the number of points of each type. Thus $3|T| = p_n+p_w+p_z$. We then parametrize Σ_0 by identifying the collection of points with labels in $\mathbf{P}$ with the corresponding fixed positions on the standard sphere S^2, denoting this parametrization of Σ_0 by $\Sigma_{\mathbf{P},0}$. Note that this normalization $\mathbf{P}$ does not uniquely determine the domain Σ_0 up to biholomorphism since the positions of the nodal points in $\mathbf{n}_0\smallsetminus(\mathbf{n}_0)_\mathbf{P}$ must still be specified. If we want, we can reduce this indeterminacy by putting nodal labels into $\operatorname{im}(\mathbf{P})$ wherever possible, but we cannot always eliminate it; cf. Figure 5.1.1. Thus as a stable curve, the tuple $(\Sigma_{\mathbf{P},0},(\mathbf{w}_0)_\mathbf{P},(\mathbf{z}_0)_\mathbf{P})$ represents the element $\delta_{\mathbf{P},0} := [\mathbf{n}_0,(\mathbf{w}_0)_\mathbf{P},(\mathbf{z}_0)_\mathbf{P}]\in\overline{\mathcal{M}}_{0,p}$, where $p := p_w+p_z$. We denote the nearly elements in $\overline{\mathcal{M}}_{0,p}$ by $\delta_\mathbf{P} := [\mathbf{n},\mathbf{w}_\mathbf{P},\mathbf{z}_\mathbf{P}]$, reserving the name δ to denote stable curves $[\mathbf{n},\mathbf{w},\mathbf{z}]\in\overline{\mathcal{M}}_{0,k+L}$.

We now discuss the structure of a neighbourhood $\Delta_\mathbf{P}$ of $\delta_{\mathbf{P},0}$ in the Deligne–Mumford space $\overline{\mathcal{M}}_{0,p}$. We denote the universal curve over $\Delta_\mathbf{P}$ by $\mathcal{C}|_{\Delta_\mathbf{P}}$ with fibers $\Sigma_{\delta_\mathbf{P}}, \delta_\mathbf{P}\in\Delta_\mathbf{P}$. A normalized representation of the surface $\Sigma_{\delta_\mathbf{P}}$ may be obtained from $\Sigma_{\mathbf{P},0}$ by varying the positions of the nodal points not in $\operatorname{im}(\mathbf{P})$ and then gluing. More precisely, if $\Sigma_{\mathbf{P},0}$ has K nodes, then there are $2K$ nodal points $\mathbf{n}_0$, $2K-p_n$ of which can move, and K small complex gluing parameters $\mathbf{a} := (a_1,\dots,a_K)$, one at each node, such that all nearby fibers $\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$ may be obtained from $\Sigma_{\mathbf{P},0}$ by first varying the $2K-p_n$ points in $\mathbf{n}_0\smallsetminus\mathbf{P}$ via complex parameters denoted $\mathbf{b} = (b_1,\dots,b_{2K-p_n})$ and then cutting out discs of radius $|a_i|<\varepsilon$ near the ith pair of nodal points, gluing the boundaries of these discs with the twist $\arg(a_i)$. We suppose $|a_i|,|b_j|<\varepsilon$, where $\varepsilon>0$ is chosen so that the union $\mathcal{N}^{2\varepsilon}_{nodes}$ of the 2ε-discs around the nodes of $\Sigma_{\mathbf{P},0}$ does not intersect the discs D^ℓ_0 or the marked points $\mathbf{z}_0$. Thus, for some small neighborhood B^{6K-2p_n} of 0 in $\mathbb{C}^{3K-p_n}$, we have a fiberwise embedding

$$\begin{aligned} \iota_\mathbf{P} &: \ \big(\Sigma_{\mathbf{P},0}\smallsetminus\mathcal{N}^{2\varepsilon}_{nodes}\big)\times B^{6K-2p_n}\to\mathcal{C}|_{\Delta_\mathbf{P}},\ \text{ where}\\ \iota_{\mathbf{P},\mathbf{a},\mathbf{b}} &: \ \big(\Sigma_{\mathbf{P},0}\smallsetminus\mathcal{N}^{2\varepsilon}_{nodes}\big)\times\{\mathbf{a},\mathbf{b}\}\ \mapsto\ \Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}\smallsetminus\mathcal{N}^{2\varepsilon}_{nodes}, \end{aligned} \tag{5.1.3}$$

Thus $\iota_{\mathbf{P},\mathbf{a},\mathbf{b}}$ takes the $p=p_w+p_z$ marked points in $\Sigma_{\mathbf{P},0}\smallsetminus\mathcal{N}^{2\varepsilon}_{nodes}$ to the marked points $\mathbf{w}_\mathbf{P},\mathbf{z}_\mathbf{P}$ in the fiber $\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$, and the discs $\bigcup_\ell D^\ell_0\subset\Sigma_{\mathbf{P},0}\smallsetminus\mathcal{N}^{\varepsilon}_{nodes}$ to corresponding discs

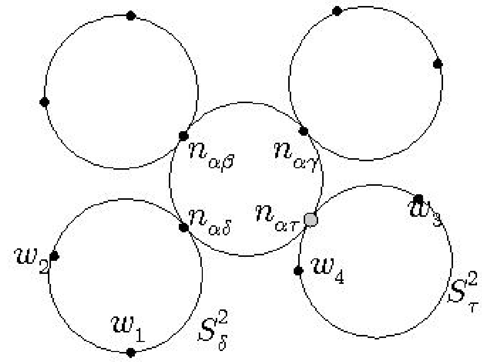

FIGURE 5.1.1. Here all the nodes and special points are in $\mathbf{P}$ except for $n_{\alpha\tau}$. When the node joining $(S^2)_\alpha$ to $(S^2)_\delta$ is resolved, two points of $\mathbf{P}$ are removed, namely the nodal pair, so that the new glued component contains 4 points of $\mathbf{P}$, one more than is needed stabilize it. This extra point records the gluing parameter. For example, if we fix the parametrization of the new glued sphere by $n_{\alpha\beta}, n_{\alpha\gamma}, w_1$, the gluing parameter is determined by the position of w_2, i.e. by the cross ratio $cr(n_{\alpha\beta}, n_{\alpha\gamma}, w_1, w_2)$. On the other hand, if we resolve at $n_{\alpha\tau}$, then we lose one point of $\mathbf{P}$, and we can take the cross ratio $cr(n_{\alpha\beta}, n_{\alpha\gamma}, n_{\alpha\delta}, w_3)$ to parametrize the position of the nodal point $n_{\alpha\tau}$, while $cr(n_{\alpha\beta}, n_{\alpha\gamma}, n_{\alpha\delta}, w_4)$ gives the gluing parameter. Similarly, if we resolve at a node with neither point in $\mathbf{P}$ then, after gluing, the three points in $\mathbf{P}$ not needed for stability parametrize the positions of the two nodal points and the gluing parameter.

in $\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$. For each $\mathbf{a}, \mathbf{b}$ the injection $\iota_{\mathbf{P},\mathbf{a},\mathbf{b}}$ is defined on the subset of $\Sigma_{\mathbf{P},0}$ that is not cut out by the gluing, i.e. on $\bigcup_\alpha \big((S^2)_\alpha \setminus \bigcup_\beta D_{n_\alpha\beta}(|a_{\alpha\beta}| + |b_{\alpha\beta}|\big)$, where $a_{\alpha\beta}, b_{\alpha\beta}$ are the relevant parameters $\mathbf{a}, \mathbf{b}$ at the nodal point $n_{\alpha\beta}$.

Remark 5.1.1. (i) These coordinates $(\mathbf{a}, \mathbf{b})$ for the neighbourhood $\Delta_{\mathbf{P}} \subset \overline{\mathcal{M}}_{0,p}$ are given by the positions of the free nodes (parametrized by $\mathbf{b}$) and the gluing parameters $\mathbf{a}$, and are the most convenient ones in which to write down the equation; cf (VI). In order to understand the group action it is helpful to note that one can read off the parameters $\mathbf{a}, \mathbf{b}$ from the (extended) cross ratios[21] of the points $\mathbf{w}_{\mathbf{P}}, \mathbf{z}_{\mathbf{P}}$ in the fiber $\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$; cf. Figure 5.1.1 and [MS, Appendix D]. Hence we can write down the group action in terms of the induced permutation of the special points as in (5.1.9) below.

[21] In this extension we allow at most pairs of points to coincide, so that cr may equal $0, 1, \infty$. The presence of such special values signals the existence of a node, and the resulting combinatorics gives the tree.

(ii) We will often denote the normalized domain of the stable curve $\delta_{\mathbf{P}} := [\mathbf{n}, \mathbf{w}_{\mathbf{P}}, \mathbf{z}_{\mathbf{P}}] = [\Sigma_{\delta_{\mathbf{P}}}, \mathbf{w}_{\mathbf{P}}, \mathbf{z}_{\mathbf{P}}]$ as $\Sigma_{\mathbf{P},\delta_{\mathbf{P}}}$ instead of $\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$. Thus

$$\Sigma_{\mathbf{P},\delta_{\mathbf{P}}} := \Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}.$$

(iii) As a check on dimensions, note that $\dim_{\mathbb{C}}(\overline{\mathcal{M}}_{0,p}) = p-3$, while there are $3K-p_n$ parameters $\mathbf{a}, \mathbf{b}$, and $p + p_n = 3K+3$ by definition of $\mathbf{P}$, so that the total number of parameters $\mathbf{a}, \mathbf{b}$ is $p-3$. Note also that the normalization $\mathbf{P}$ of the central fiber Σ_0 labels enough points to normalize the nearby fibers, since one needs three fewer points in $\mathbf{P}$ for each node that is glued. As illustrated in Figure 5.1.1, some points in $\mathbf{P}$ may be cut out by a gluing, but the extra elements in $\mathbf{P}$ can always be interpreted in terms of gluing parameters $\mathbf{a}$ and the parameters $\mathbf{b}$ pertaining to the nodes that have been glued. (This point is discussed more fully in (VIII)[b].)

Now consider the stable curves $\delta := [\Sigma_\delta, \mathbf{w}, \mathbf{z}]$ with the full set of marked points. If we parametrize the domain as $\Sigma_{\mathbf{P},\delta_{\mathbf{P}}}$, then the points in $\mathbf{w}_{\mathbf{P}}, \mathbf{z}_{\mathbf{P}}$ have fixed positions while the other marked points (as well as the nodes not in $\mathbf{n}_{\mathbf{P}}$) can move. The map $\iota_{\mathbf{P}}$ in (5.1.3) therefore extends to a parametrization of the universal curve $\mathcal{C}|_\Delta$ away from the nodes:

$$\iota_{\mathbf{P}} : \big(\Sigma_{\mathbf{P},0} \setminus \mathcal{N}^{\varepsilon}_{nodes}\big) \times B^{6K-2p_n} \times B^{2(k+L-p)} \;\to\; \mathcal{C}|_\Delta : \tag{5.1.4}$$

where the small parameters $\omega^\ell, \zeta^j \in B^{2(k+L-p)} \subset \mathbb{C}^{k+L-p}$ describe the positions of the points in $\iota^{-1}_{\mathbf{P},\mathbf{a},\mathbf{b}}(\mathbf{w} \cup \mathbf{z}) \setminus (\mathbf{w}_{\mathbf{P}} \cup \mathbf{z}_{\mathbf{P}})$, taking the value 0 at $\mathbf{w}_0, \mathbf{z}_0$. The map $\overline{\mathcal{M}}_{0,k+L} \to \overline{\mathcal{M}}_{0,p}$ that forgets the points in $(\mathbf{w} \cup \mathbf{z}) \setminus (\mathbf{w}_{\mathbf{P}} \cup \mathbf{z}_{\mathbf{P}})$ lifts to a forgetful map *forget* : $\mathcal{C}|_\Delta \to \mathcal{C}_{\Delta_{\mathbf{P}}}$ that fits into the following commutative diagram

$$\begin{tikzcd}
\big(\Sigma_{\mathbf{P},0} \setminus \mathcal{N}^{\varepsilon}_{nodes}\big) \times B^{6K-2p_n} \times B^{2(k+L-p)} \arrow[r, "\iota_{\mathbf{P}}"] \arrow[d, "\mathrm{proj}"'] & \mathcal{C}|_\Delta \arrow[d, "forget"'] \\
\big(\Sigma_{\mathbf{P},0} \setminus \mathcal{N}^{\varepsilon}_{nodes}\big) \times B^{6K-2p_n} \arrow[r, "\iota_{\mathbf{P}}"] & \mathcal{C}|_{\Delta_{\mathbf{P}}}.
\end{tikzcd} \tag{5.1.5}$$

We will denote the element $\delta \in \Delta$ as $\delta := [\Sigma_\delta, \mathbf{w}, \mathbf{z}] = [\mathbf{n}, \mathbf{w}, \mathbf{z}]$, with chosen representative denoted either $(\Sigma_{\mathbf{P},\delta}, \mathbf{w}, \mathbf{z})$ or $(\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}, \mathbf{w}, \mathbf{z})$. Here $\mathbf{w}, \mathbf{z}$ are tuples of points in the curve $\Sigma_{\mathbf{P},\delta} = \Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$; their pullbacks by $\iota_{\mathbf{P},\mathbf{a},\mathbf{b}}$ to the fixed fiber $\Sigma_{\mathbf{P},0}$ are given by the complex parameters $\vec{\omega}, \vec{\zeta}$, that we assume to vanish at $\mathbf{w}_0, \mathbf{z}_0$ and have length $< \varepsilon$ so that

$$\omega^\ell := \iota^{-1}_{\mathbf{P},\mathbf{a},\mathbf{b}}(w^\ell) \in D^\ell_0. \tag{5.1.6}$$

(IV): The group action. Since Γ is the stabilizer of $[\Sigma_0, \mathbf{z}_0, f_0]$ and acts on the added marked points $\mathbf{w}_0$ by permutation, with an associated action on the nodes, this action extends to a neighbourhood of $[\Sigma_0, \mathbf{w}_0, \mathbf{z}_0]$. Hence we may assume that Δ is invariant under this action $\delta \mapsto \gamma^*(\delta)$ of Γ, where $\gamma^*(\delta) = [\gamma \cdot \mathbf{n}, \gamma \cdot \mathbf{w}, \mathbf{z}] =: [\mathbf{n}', \mathbf{w}', \mathbf{z}]$ as in (5.1.1) ff. Correspondingly there is an action $[\mathbf{n}, \mathbf{w}, \mathbf{z}, f] \mapsto [\gamma \cdot \mathbf{n}, \gamma \cdot \mathbf{w}, \mathbf{z}, f] = [\mathbf{n}', \mathbf{w}', \mathbf{z}, f]$ on the space of stable maps. To obtain an explicit formula for this action, we normalize the domains via the labelling $\mathbf{P}$. We may assume that f is defined on the normalized domain $\Sigma_{\mathbf{P},\delta}$. However, $\Sigma_{\mathbf{P},\delta} \neq \Sigma_{\mathbf{P},\gamma^*(\delta)}$ since in $\Sigma_{\mathbf{P},\gamma^*(\delta)}$ the points whose *new* labels are in $\mathbf{P}$ are put in standard position. Therefore the normalized action may be written as

$$(5.1.7)\qquad \big(\Sigma_{\mathbf{P},\delta}, \mathbf{w}, \mathbf{z}, f\big) \mapsto \big(\Sigma_{\mathbf{P},\gamma^*(\delta)}, \phi^{-1}_{\gamma,\delta}(\gamma\cdot\mathbf{w}), \phi^{-1}_{\gamma,\delta}(\mathbf{z}), f\circ\phi_{\gamma,\delta}\big)$$

where

$$(5.1.8)\qquad \phi_{\gamma,\delta}: \Sigma_{\mathbf{P},\gamma^*(\delta)} \to \Sigma_{\mathbf{P},\delta}$$

is defined to be the unique biholomorphic map that takes the special points $\mathbf{n}', \mathbf{w}', \mathbf{z}'$ in $\Sigma_{\mathbf{P},\gamma^*(\delta)}$ with labels in $\operatorname{im}(\mathbf{P})$ (that are in standard position) to the corresponding points in Σ_δ, i.e. the map $\phi_{\gamma,\delta}$ takes

$$(5.1.9)\qquad n'_{\alpha\beta} \mapsto n_{\gamma(\alpha)\gamma(\beta)},\ \ (w')^\ell \mapsto w^{\gamma(\ell)},\ \ (z')^i \mapsto z^i \quad \text{if } (\alpha,\beta),\ \ell,\ i \in \operatorname{im}(\mathbf{P}).$$

The positions of the other special points in $\mathbf{n}', \mathbf{w}', \mathbf{z}'$ in $\Sigma_{\mathbf{P},\gamma^*(\delta)}$ are then determined by (5.1.7); in particular, $\mathbf{w}' = \phi^{-1}_{\gamma,\delta}(\gamma\cdot\mathbf{w})$ and $\mathbf{z}' = \phi^{-1}_{\gamma,\delta}(\mathbf{z})$ as claimed in (5.1.7).

One can also pull these maps $\phi_{\gamma,\delta}$ back to partially defined maps $\phi_{\mathbf{P},\gamma,\delta}$ on the fixed surface $\Sigma_{\mathbf{P},0}\smallsetminus\mathcal{N}^{2\varepsilon}_{nodes}$, as follows:

$$(5.1.10)\qquad \phi_{\mathbf{P},\gamma,\delta} := \\ \Sigma_{\mathbf{P},0}\smallsetminus\mathcal{N}^{2\varepsilon}_{nodes} \xrightarrow{\iota_{\mathbf{P}}(\cdot,\mathbf{a}',\mathbf{b}')} \Sigma_{\mathbf{P},\mathbf{a}',\mathbf{b}'} = \Sigma_{\mathbf{P},\gamma^*(\delta)} \xrightarrow{\phi_{\gamma,\delta}} \Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}} = \Sigma_{\mathbf{P},\delta} \xrightarrow{\iota_{\mathbf{P}}(\cdot,\mathbf{a},\mathbf{b})^{-1}} \Sigma_{\mathbf{P},0}\smallsetminus\mathcal{N}^{\varepsilon}_{nodes}.$$

Then $\phi_{\mathbf{P},\gamma,\delta}$ is almost equal to $\gamma: \Sigma_{\mathbf{P},0} \to \Sigma_{\mathbf{P},0}$, because the inverse image by $\iota_{\mathbf{P}}(\cdot,\mathbf{a},\mathbf{b})$ of $w^{\gamma(\ell)} \in \Sigma_{\mathbf{P},\delta}$ is close to $w_0^{\gamma(\ell)}$ while $\iota_{\mathbf{P}}(\cdot,\mathbf{a}',\mathbf{b}')^{-1}\circ\phi^{-1}_{\gamma,\delta}(w^{\gamma(\ell)})$ is close to w_0^ℓ.

Because the permutation action $\mathbf{w}\mapsto\gamma\cdot\mathbf{w}$ satisfies $(\alpha\gamma)\cdot\mathbf{w} = \gamma\cdot(\alpha\cdot w)$, we have $(\alpha\gamma)^*(\delta) = [(\alpha\gamma)\cdot\mathbf{w},\mathbf{z}] = \gamma^*(\alpha^*(\delta))$. Hence the composite $\phi_{\alpha,\delta}\circ\phi_{\gamma,\alpha^*\delta}$ is defined and maps from $\Sigma_{\mathbf{P},\gamma^*(\alpha^*(\delta))}$ through $\Sigma_{\mathbf{P},\alpha^*(\delta)}$ to $\Sigma_{\mathbf{P},\delta}$. It follows easily that

$$(5.1.11)\qquad \phi_{\alpha\gamma,\delta} = \phi_{\alpha,\delta}\circ\phi_{\gamma,\alpha^*\delta}: \Sigma_{\mathbf{P},(\alpha\gamma)^*(\delta)} \to \Sigma_{\mathbf{P},\delta}.$$

Figure 5.1.2 explains this action in a case in which there is a trivial induced action of Γ on the set of components of Σ_0 and hence on the nodes.

Using the map $\iota_{\mathbf{P}}$ in (5.1.4), we can push the discs $(D_0^\ell)_\ell$ in (I) forward to subsets of $\Sigma_{\mathbf{P},\delta}$, and then average them for each δ to obtain discs $(D_\delta^\ell)_\ell \subset \Sigma_{\mathbf{P},\delta}$ whose union is invariant under this action of Γ. Thus the set of discs has $|\Gamma|$ allowed labelings that form an orbit under the Γ action. All these formulas and constructions are explained in more detail in [MW14].

(V): The obstruction space. Consider the bundle $\operatorname{Hom}^{0,1}_J(\mathcal{C}|_\Delta\times M)$ whose fiber at (z,x) is $\operatorname{Hom}^{0,1}_J(T_z\Sigma_{\mathbf{P},\delta}, T_xM)$ in normalized coordinates. Choose a vector space E_0 and a (not necessarily injective) linear map $\lambda: E_0 \to \mathcal{C}^\infty\big(\operatorname{Hom}^{0,1}_J(\mathcal{C}|_\Delta\times M)\big)$ whose image consists of sections that vanish near the nodal points of the fibers. More precisely, the sections should be supported in the image of the embedding $\iota_{\mathbf{P}}$ of (5.1.4). Define $E := \prod_{\gamma\in\Gamma} E_0$, the product of $|\Gamma|$ copies of E_0 with elements $\vec{e} := (e^\gamma)_{\gamma\in\Gamma}$, on which Γ

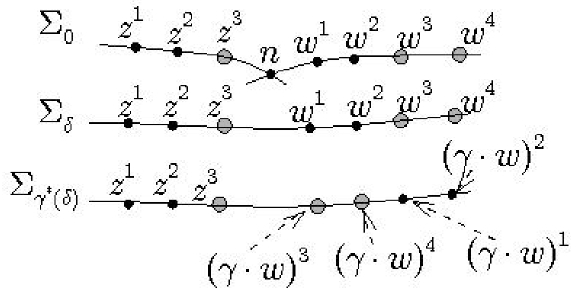

FIGURE 5.1.2. The points in $\mathbf{P}$ are the two nodal points plus z^1, z^2, w^1, w^2 with solid dots. These are shown on Σ_0 and Σ_δ. The group $\Gamma = \mathbb{Z}/2\mathbb{Z}$ interchanges w^1, w^3 and w^2, w^4, so that $\Sigma_{\gamma^*(\delta)}$ has the same marked points as Σ_δ, but with different labels. Hence it is normalized differently, using the points labelled w^3, w^4 on Σ_δ instead of w^1, w^2. Usually $cr(z^1, z^2, w^1, w^2) \neq cr(z^1, z^2, (\gamma \cdot w)^1, (\gamma \cdot w)^2)$, so that the normalized domains $\Sigma_{\mathbf{P},\delta}$ and $\Sigma_{\mathbf{P},\gamma^*(\delta)}$ are obtained from Σ_0 by gluing with different parameters. We have drawn the figure so that the map $\phi_{\gamma,\mathbf{P},\delta} : \Sigma_{\mathbf{P},\gamma^*(\delta)} \to \Sigma_{\mathbf{P},\delta}$ identifies points vertically, taking $(\gamma \cdot w)^1$ to w^3 and so on.

acts by permutation so that $\big(\alpha \cdot \vec{e}\big)^\gamma = e^{\alpha\gamma}$ for $\alpha \in \Gamma$. Then extend λ equivariantly to a linear map

$$(5.1.12) \qquad \lambda : E \to \mathcal{C}^\infty\big(\mathrm{Hom}^{0,1}_J(\mathcal{C}|_\Delta \times M)\big), \qquad \vec{e} := (e^\gamma)_{\gamma\in\Gamma} \mapsto \textstyle\sum_{\gamma\in\Gamma} \gamma^*(\lambda(e^\gamma)).$$

Here we use the fact that the isotropy group Γ acts fiberwise on $\mathcal{C}|_\Delta$ as explained in (IV), taking the fiber $\Sigma_{\mathbf{P},\delta}$ (with relabelled marked points $\mathbf{w}$) to the fiber $\Sigma_{\mathbf{P},\gamma^*(\delta)}$ by a map that in normalized coordinates is $(\phi_{\gamma,\mathbf{P},\delta})^{-1}$; cf. (5.1.7), (5.1.9). The induced action of Γ on a section $\nu \in \mathcal{C}^\infty\big(\mathrm{Hom}^{0,1}_J(\mathcal{C}|_\Delta \times M)\big)$ is by pullback as follows: for $z \in \Sigma_{\mathbf{P},\delta}$ we have

$$(5.1.13) \quad \gamma^*(\nu)(z,x) := (\phi^{-1}_{\gamma,\delta})^*(\nu)(z,x) = \nu(\phi^{-1}_{\gamma,\delta}(z), x) \circ \mathrm{d}\phi^{-1}_{\gamma,\delta}(z) : \mathrm{T}_z\Sigma_{\mathbf{P},\delta} \to \mathrm{T}_x M.$$

It follows from (5.1.11) that $(\gamma\alpha)^*(\nu) = \gamma^*(\alpha^*(\nu))$.

There is quite a bit of choice for the space E_0. For example, we could ask that it is the pullback via (5.1.5) of a space of sections of $\mathrm{Hom}^{0,1}_J(\mathcal{C}|_{\Delta_\mathbf{P}} \times M)$. However, we do need E to consist of sections over Δ in order for it to support a Γ-action. Later we will require that E_0 is chosen so that the linearized Cauchy–Riemann operator is surjective; more precisely that condition (*) in (VI) holds.

(VI): The equation. The elements of the domain U of a basic chart near the point $[\Sigma_0, \mathbf{z}_0, f_0] \in X$ have the form $(\vec{e}, \mathbf{a}, \mathbf{b}, \vec{\omega}, \vec{\zeta}, f)$, where:

(i) $\vec{e} \in E$, for E chosen sufficiently large as specified below;

(ii) the parameters $\mathbf{a}, \mathbf{b}$ determine the normalized domain $\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$ and the parameters $\vec{\omega}, \vec{\zeta}$ describe the positions in $\Sigma_{\mathbf{P},0} \smallsetminus \mathcal{N}^{\varepsilon}_{nodes}$ of the points $\iota^{-1}_{\mathbf{P},\mathbf{a},\mathbf{b}}\big((\mathbf{w}\smallsetminus\mathbf{w_P}) \cup (\mathbf{z}\smallsetminus\mathbf{z_P})\big)$ as in the discussion after (5.1.4) and (5.1.6); in particular, $\omega^\ell \in D^\ell_0$ and the tuple $\mathbf{a}, \mathbf{b}, \vec{\omega}, \vec{\zeta}$ determines a unique fiber $\delta := [\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}, \mathbf{w}, \mathbf{z}]$ in $\mathcal{C}|_\Delta$ whose underlying surface we call either $\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$ or $\Sigma_{\mathbf{P},\delta}$;

(iii) the map $f : \Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}} \to M$ represents the class $A \in H_2(M)$ and is a solution of the equation

$$\overline{\partial}_J(f) = \lambda(\vec{e})|_{\mathrm{graph} f} := \textstyle\sum_{\gamma\in\Gamma} \gamma^*\big(\lambda(e^\gamma)\big)|_{\mathrm{graph} f}. \tag{5.1.14}$$

where $\gamma^*(\lambda)$ is defined as in (5.1.13),with δ as in (ii).

The solution set of this equation is the zero set of the section

$$\begin{aligned} F : E \times B^{2(L+k-p)} \times W^{1,p}(\mathcal{C}|_\Delta, M) &\to L^p\big(\mathrm{Hom}^{0,1}_J(\mathcal{C}|_\Delta \times M)\big), \\ (\vec{e}, \mathbf{a}, \mathbf{b}, \vec{\omega}, \vec{\zeta}, f) &\mapsto \overline{\partial}_J f - \lambda(\vec{e})|_{\mathrm{graph} f} \in L^p\big(\mathrm{Hom}^{0,1}_J(\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}} \times M)\big), \end{aligned} \tag{5.1.15}$$

where the domain $W^{1,p}(\mathcal{C}|_\Delta, M)$ is the Sobolev space of $(1,p)$ maps from the fibers of $\mathcal{C}|_\Delta$ to M, and the range consists of L^p sections of the bundle considered in (V). If we fix $\mathbf{a}, \mathbf{b}$ so that the domain $\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$ of f is fixed, then the operator F is $\mathcal{C}^1$ because (5.1.13) shows that $\lambda(\vec{e})$ is a sum of terms

$$z \mapsto \gamma^*\big(\lambda(e^\gamma)\big)|_{(z,f(z))} = \lambda(e^\gamma)(\phi^{-1}_{\gamma,\delta}(z), f(z)) \circ \mathrm{d}_z \phi^{-1}_{\gamma,\delta} \tag{5.1.16}$$

where $\phi_{\gamma,\delta}$ does *not* depend explicitly on f but just on the parameters $\mathbf{a}, \mathbf{b}$ (which we have fixed) and on $\vec{\omega}, \vec{\zeta}$. (We allow the points in $(\mathbf{w} \cup \mathbf{z}) \smallsetminus (\mathbf{w_P} \cup \mathbf{z_P})$ to vary freely until we have solved the equation.) Hence[22] the operator has a linearization $\mathrm{d}F$. Consider the restriction F_0 of F to a neighbourhood of $\vec{0} \times f_0$ in the space $E \times W^{1,p}(\Sigma_{\mathbf{P},0})$ of tuples with the fixed domain $\Sigma_{\mathbf{P},0}$. Then $F_0(\vec{e}, f) = \overline{\partial}_J f - \lambda(\vec{e})|_{\mathrm{graph} f}$. It follows that

$$\mathrm{d}_{(\vec{0},f_0)} F_0(\xi, \vec{e}) = \mathrm{d}_{f_0}(\overline{\partial}_J)(\xi) - \lambda(\vec{e})|_{\mathrm{graph} f_0},$$

where

$$\mathrm{d}_{f_0}(\overline{\partial}_J) : \mathcal{D}_0 \to \prod_{\alpha\in T} L^p(\mathrm{Hom}^{0,1}_J((S^2)_\alpha, f^*_{0,\alpha}(\mathrm{T}M)) \tag{5.1.17}$$

has domain[23]

$$\mathcal{D}_0 := \big\{\xi_\alpha \in \textstyle\prod_{\alpha\in T} W^{1,p}((S^2)_\alpha, f^*_{0,\alpha}(\mathrm{T}M)) \;\big|\; \xi_\alpha(n_{\alpha\beta}) = \xi_\beta(n_{\beta\alpha}) \;\forall \alpha E \beta\big\}. \tag{5.1.18}$$

Therefore, the requirement on the obstruction space $E := \prod_{\gamma\in\Gamma} E_0$ is as follows:

[22] See [MW14] for the analytic details.

[23] Here we assume that the domain Σ_0 is connected, i.e. we identify the different components at the nodal points, so that tangent vectors must satisfy $\xi_\alpha(n_{\alpha\beta}) = \xi_\beta(n_{\beta\alpha})$. Equivalently, one could set up the equation on the disjoint union of spheres $\bigsqcup_\alpha (S^2)_\alpha$ and require that the evaluation map ev_{node} at the nodes is transverse to the corresponding diagonal $\big\{(x_{\alpha\beta}) : \alpha E \beta \Rightarrow x_{\alpha\beta} = x_{\mathbf{a}\alpha}\big\} \subset M^{2K}$, where K is the number of nodes, and hence the number of edges in the tree. For variety, we took this second approach in the discussion of condition $(*_c)$ below; cf. (5.3.1).

(∗) the elements in the image of $\lambda : E \to \mathbb{C}^\infty\big(\mathrm{Hom}^{0,1}_J(\mathcal{C}|_W \times M)\big)$ restrict on graphf_0 to a subspace of $\prod_\alpha L^p(\mathrm{Hom}^{0,1}_J((S^2)_{\alpha\in T}, f^*_{0,\alpha}(\mathrm{T}M))$ that covers the cokernel of $\mathrm{d}_{f_0}(\overline{\partial}_J)$.

Since the regularity condition is open, if we allow the nodal parameters $\mathbf{b}$ and also $\vec{\omega}, \vec{\zeta}$ to vary (fixing the gluing parameters $\mathbf{a} = 0$) we get transversality for each nearby domain in the given stratum of $\overline{\mathcal{M}}_{0,k+L}$. However, in general we need a gluing theorem in order to claim that condition (*) implies that the linearization $\mathrm{d}F$ is surjective for all sufficiently close tuples $(\vec{e}, \mathbf{a}, \mathbf{b}, \vec{\omega}, \vec{\zeta}, f)$, and that the space of solutions near the center point $(\vec{0}, 0, 0, 0, 0, f_0)$ (where $\vec{e}, \mathbf{a}, \mathbf{b}, \vec{\omega}, \vec{\zeta}$ all vanish) is the product of the space of solutions at $\mathbf{a} = \mathbf{b} = 0$ with a small neighborhood of 0 in the parameter space $(\mathbf{a}, \mathbf{b})$. The gluing theorem in [MS] suffices for this purpose, but it does not show that the resulting set of solutions is a smooth manifold: although the solution depends smoothly on $\mathbf{b}$, and on $\mathbf{a}$ as long as no component goes to zero, it does not establish any differentiability with the respect to the gluing parameters a_i as these converge to 0. Thus the solution space has a weakly SS structure as in Definition 3.3.4. Therefore one must either work in a stratified smooth situation or prove a more powerful gluing theorem.

We will assume here that we have a more powerful gluing theorem that gives at least $\mathcal{C}^1$ smoothness with respect to $\mathbf{a}$. (See [MWss] for the general case.) We then define $\widehat{U}$ to be a small $\mathcal{C}^1$-open neighbourhood of $(\vec{0}, 0, 0, 0, 0, f_0)$ in $F^{-1}(0)$ where F is as in (5.1.15). Condition (*) on E implies that $\widehat{U}$ is a manifold of dimension $\dim \widehat{U} = \dim E + 2L + \mathrm{ind}\,(A)$, where $\mathrm{ind}\,(A) = 2n + 2c_1(A) + 2k - 6$ as in (5.0.5).

We now show that $\alpha \in \Gamma$ acts on the solutions of (5.1.14) by

$$\alpha^*(\vec{e}, \mathbf{a}, \mathbf{b}, \vec{\omega}, \vec{\zeta}, f) = \big(\alpha \cdot \vec{e}, \mathbf{a}', \mathbf{b}', \phi^{-1}_{\mathbf{P},\alpha,\delta}(\alpha \cdot \vec{\omega}), \phi^{-1}_{\mathbf{P},\alpha,\delta}(\vec{\zeta}), f \circ \phi_{\alpha,\delta}\big). \tag{5.1.19}$$

where $\Sigma_{\mathbf{P},\delta} := \Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$, $\Sigma_{\mathbf{P},\alpha^*(\delta)} = \Sigma_{\mathbf{P},\mathbf{a}',\mathbf{b}'}$, $\phi_{\alpha,\delta} : \Sigma_{\mathbf{P},\mathbf{a}',\mathbf{b}'} \to \Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}}$ is as in (5.1.8), and $\phi_{\mathbf{P},\alpha,\delta} : \Sigma_0 \smallsetminus N^{2\varepsilon}_{nodes}$ is its normalization defined in (5.1.10), with the obvious induced action on the parameters $\vec{\omega}, \vec{\zeta}$. To simplify the calculation we consider a point $v \in \Sigma_{\mathbf{P},\alpha^*(\delta)}$, and write $z := \phi_{\alpha,\delta}(v)$. If $(\vec{e}, \mathbf{a}, \mathbf{b}, \vec{\omega}, \vec{\zeta}, f)$ is a solution, then by (5.1.16) we have for fixed $\alpha \in \Gamma$ that

$$\begin{aligned}
&\lambda\big(\alpha \cdot \vec{e}, \mathbf{a}', \mathbf{b}', \phi^{-1}_{\mathbf{P},\alpha,\delta}(\alpha \cdot \vec{\omega}), \phi^{-1}_{\mathbf{P},\alpha,\delta}(\vec{\zeta}), f \circ \phi_{\alpha,\delta})(v) \\
&\qquad = \textstyle\sum_{\gamma\in\Gamma} \lambda(e^{\alpha\gamma})\big(\phi^{-1}_{\gamma,\alpha^*(\delta)}(v), f \circ \phi_{\gamma,\delta}(v)\big) \circ \mathrm{d}_v \phi^{-1}_{\gamma,\alpha^*(\delta)} \\
&\qquad = \textstyle\sum_{\gamma\in\Gamma} \lambda(e^{\alpha\gamma})\big(\phi^{-1}_{\gamma,\alpha^*(\delta)}(\phi^{-1}_{\alpha,\delta}(z)), f(z)\big) \circ \mathrm{d}_v \phi^{-1}_{\gamma,\alpha^*(\delta)} \\
&\qquad = \textstyle\sum_{\alpha\gamma\in\Gamma} \lambda\big(e^{\alpha\gamma})(\phi^{-1}_{\alpha\gamma,\delta}(z), f(z)\big) \circ \mathrm{d}_z \phi^{-1}_{\alpha\gamma,\delta} \circ \mathrm{d}_v \phi_{\alpha,\delta} \\
&\qquad = \overline{\partial}_J(f)(z) \circ \mathrm{d}_v \phi_{\alpha,\delta} \\
&\qquad = \overline{\partial}_J(f \circ \phi_{\alpha,\delta})(v).
\end{aligned}$$

where the third equality uses (5.1.11) twice and the next one uses (5.1.16). Hence, because Γ fixes the element $(\vec{0}, 0, 0, 0, 0, f_0) \in \widehat{U}$, we may assume that $\widehat{U}$ is Γ-invariant.

(Replace $\widehat{U}$ by $\bigcap_{\gamma\in\Gamma}\gamma^*(\widehat{U})$.) Notice also that although the Γ-action of (5.1.19) looks quite complicated in normalized coordinates, the induced action on the equivalence classes $\big(\vec{e},[\mathbf{n},\mathbf{w},\mathbf{z},f]\big)$ of the elements in $\widehat{U}$ modulo biholomorphic reparametrizations (where we now describe the marked points by their images in the fiber $\Sigma_\delta=[\mathbf{n},\mathbf{w},\mathbf{z}]$) may be written in the notation of (II) as

$$\gamma^*\big(\vec{e},[\mathbf{n},\mathbf{w},\mathbf{z},f]\big)=\big(\gamma\cdot\vec{e},[\gamma\cdot\mathbf{n},\gamma\cdot\mathbf{w},\mathbf{z},f]\big). \tag{5.1.20}$$

(VII) The basic chart $\mathbf{K}:=(U,E,\Gamma,s,\psi)$:

To obtain a chart from the solution space $\widehat{U}$ we impose **slicing conditions** on the tuples $(\vec{e},\mathbf{a},\mathbf{b},\vec{\omega},\vec{\zeta},f)$ in $\widehat{U}$. Because $\operatorname{im} f_0|_{D_0^\ell}$ meets Q transversally in a unique point for each $\ell=1,\dots,L$ and $\Gamma=\operatorname{Stab}[\Sigma_0,\mathbf{z},f_0]$, we may choose the small Γ-invariant $\mathcal{C}^1$-open neighbourhood $\widehat{U}$ of (VI) so that it satisfies the following condition:

(iv) for all $(\vec{e},\mathbf{a},\mathbf{b},\vec{\omega},\vec{\zeta},f)\in\widehat{U}$ and $1\le\ell\le L$ the image $\operatorname{im} f\circ\iota_{\mathbf{P},\mathbf{a},\mathbf{b}}|_{D_0^\ell}$ meets Q transversally in a single point; moreover, $(f\circ\iota_{\mathbf{P},\mathbf{a},\mathbf{b}})^{-1}(Q)\subset\bigcup_{1\le\ell\le L}D_0^\ell$.

Now consider the following set U',

$$U':=\big\{(\vec{e},\mathbf{a},\mathbf{b},\vec{\omega},\vec{\zeta},f)\in\widehat{U}\;\big|\;f\circ\iota_{\mathbf{P},\mathbf{a},\mathbf{b}}(\omega^\ell)\in Q\;\;\forall 1\le\ell\le L\big\}.$$

Note that the Γ action of (5.1.19)

$$\alpha^*(\vec{e},\mathbf{a},\mathbf{b},\vec{\omega},\vec{\zeta},f)=\big(\alpha\cdot\vec{e},\mathbf{a}',\mathbf{b}',\phi^{-1}_{\mathbf{P},\alpha,\delta}(\alpha\cdot\vec{\omega}),\phi^{-1}_{\mathbf{P},\alpha,\delta}(\vec{\zeta}),f\circ\phi_{\alpha,\delta}).$$

preserves the slicing conditions because, by (5.1.10), $\phi^{-1}_{\mathbf{P},\alpha,\delta}$ is the pullback of $\phi_{\alpha,\delta}$ to the fixed domain $\Sigma_{\mathbf{P},0}$. Further, one can check that the slicing conditions are transverse (cf. [MW12]) so that the dimension of U' is $\dim E+\operatorname{ind}(A)$ as required. Define

$$s(\vec{e},\mathbf{a},\mathbf{b},\vec{\omega},\vec{\zeta},f):=\vec{e}\in E,\quad \psi(\vec{0},\mathbf{a},\mathbf{b},\vec{\omega},\vec{\zeta},f)=[\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}},\mathbf{z},f]\in X. \tag{5.1.21}$$

This tuple (U',E,Γ,s,ψ) satisfies all the requirements for a Kuranishi chart, except possibly the footprint condition: we need $\psi:s^{-1}(0)\to X$ to induce a homeomorphism from the quotient ${}^{s^{-1}(0)}/_{\Gamma}$ onto an open subset of X. The forgetful map $\psi:s^{-1}(0)\to X$ factors through the quotient ${}^{s^{-1}(0)}/_{\Gamma}$. Further, if $\psi(\vec{0},\mathbf{a},\mathbf{b},\vec{\omega},\vec{\zeta},f)=\psi(\vec{0},\mathbf{a}',\mathbf{b}',\vec{\omega}',\vec{\zeta}',f')$ there are biholomorphisms

$$\phi:\Sigma_{\mathbf{P},\mathbf{a}',\mathbf{b}'}\to\Sigma_{\mathbf{P},\mathbf{a},\mathbf{b}},\quad \phi_{\mathbf{P}}:=\iota_{\mathbf{P},\mathbf{a},\mathbf{b}}\circ\phi\circ\iota_{\mathbf{P},\mathbf{a}',\mathbf{b}'}:\Sigma_{\mathbf{P},0}\to\Sigma_{\mathbf{P},0}, \tag{5.1.22}$$

such that $f\circ\phi=f',\phi_{\mathbf{P}}^{-1}(\vec{\zeta})=\vec{\zeta}'$ and, by condition (iv) above, a permutation $\pi:\{1,\dots,L\}\to\{1,\dots,L\}$ such that $\phi_{\mathbf{P}}^{-1}(\omega^{\pi(\ell)})=(\omega')^\ell$. We need to see that $\pi\in\Gamma$. Without further conditions on U' this may not hold. However, since $\Gamma=\operatorname{Stab}([\Sigma_0,\mathbf{z}_0,f_0])$, we can choose U' so that it holds at $(\vec{0},0,0,0,0,f_0)$ itself and hence also on a sufficiently small neighbourhood of $(\vec{0},0,0,0,0,f_0)$ by continuity. Hence we may put a final condition on the domain U.

(v) for all $(\vec{e},\mathbf{a},\mathbf{b},\vec{\omega},\vec{\zeta},f)\in U$ and permutations $\pi:\{1,\dots,L\}\to\{1,\dots,L\}$, there is a tuple $(\vec{e},\mathbf{a}',\mathbf{b}',\vec{\omega}',\vec{\zeta}',f')\in U$ and maps $\phi,\phi_{\mathbf{P}}$ as in (5.1.22) such that

$$f\circ\phi=f',\quad \phi_{\mathbf{P}}^{-1}(\vec{\zeta})=\vec{\zeta}',\quad \phi_{\mathbf{P}}^{-1}(\omega^{\pi(\ell)})=(\omega')^\ell,1\le\ell\le L,$$

if and only if $\pi \in \Gamma$.

With this condition the footprint map is injective. It requires somewhat more work to show that its image F is open in X. The proof in the non-nodal situation may be found in [MW14]. In general, this is a consequence of the gluing theorem.

Definition 5.1.2. *We define the* **chart** $\mathbf{K} := (U, E, \Gamma, s, \psi)$ *with* $\Gamma = \mathrm{Stab}([\Sigma_0, \mathbf{z}_0, f_0]$ *and E as in (IV) by requiring that U, satisfying (v), be constructed as above from a set $\widehat{U}$ that satisfies (iv), and then defining s, ψ as in* (5.1.21).

This construction depends on the following choices:

- a **center** $\tau := [\Sigma_0, \mathbf{z}_0, f_0]$ used to fix the parametrization;
- a **slicing manifold** Q that is transverse to $\mathrm{im}\, f_0$, disjoint from $f_0(\mathbf{z}_0)$, and chosen so that the k points $\mathbf{z}_0$ together with the L points in $f_0^{-1}(Q)$ stabilize the domain of f_0;
- a **normalization** $\mathbf{P}$ for Σ_0 as in (5.1.2) which fixes the parametrization of $\Sigma_{\mathbf{P},0}$;
- a **disc structure** $\bigsqcup_{1\le \ell\le L} D^\ell \subset \Sigma_0 \smallsetminus \mathcal{N}^\varepsilon_{nodes}$ consisting of small disjoint neighbourhoods $D_0^\ell \ni w_0^\ell$ of the L points in $f_0^{-1}(Q)$ that are averaged over Γ so that the Γ-action permutes them; and
- an **obstruction space** E and Γ-invariant map $\lambda : E \to \mathcal{C}^\infty\big(\mathrm{Hom}^{0,1}_J(\mathcal{C}|_\Delta \times M)\big)$ as in (V), where Δ is a small neighbourhood of $[\Sigma_0, \mathbf{w}_0, \mathbf{z}_0]$ in $\overline{\mathcal{M}}_{0,k+L}$.

Notice that if $\tau \in X_S$, then the footprint is contained in $X_{\ge S}$, where S labels strata in the fine stratification introduced at the beginning of §5, since the elements of the chart U have domains obtained by resolving nodes of Σ_0. The disc structure $(D^\ell)_{1\le\ell\le L}$ will be important in the construction of sum charts; cf. the definition of $\mathcal{W}_{12,\mathbf{P}_2}$ in (5.1.26) below

(VIII): Change of coordinates. Before discussing sum charts, we consider the effect on a single chart of changing the normalization, center and slicing conditions.

[a] Change of normalization:

When defining a chart, the center and slicing manifold are needed to set up the framework, i.e. to specify the added marked points $\mathbf{w}$ and hence the neighborhood Δ of the stabilized domain $[\Sigma_0, \mathbf{w}_0, \mathbf{z}_0]$ in $\overline{\mathcal{M}}_{0,k+L}$. The normalization is then used in order to write down the equation (5.1.14) in coordinates so that one can understand its analytic properties. However, the equation itself makes sense as a section of a bundle over the space $\mathrm{Map}^\infty(\mathcal{C}|_\Delta; M)$ of C^∞ maps from the fibers of the universal curve to M. Therefore the following holds.

- *If we fix τ and Q and consider two possible normalizations $\mathbf{P}_1, \mathbf{P}_2$, then any chart $U_{\mathbf{P}_1}$ constructed using $\mathbf{P}_1$ is isomorphic to some chart $U_{\mathbf{P}_2}$ constructed using $\mathbf{P}_2$. In particular its footprint will not change.*

To see this, let $\phi_{\mathbf{P}_2,\mathbf{P}_1} : \Sigma_{\mathbf{P}_2,0} \to \Sigma_{\mathbf{P}_1,0}$ be the unique biholomorphism that takes the points $\mathbf{n}_0, \mathbf{w}_0, \mathbf{z}_0$ in $\Sigma_{\mathbf{P}_2,0}$ with labels in $\mathrm{im}\,(\mathbf{P}_1)$ to their standard positions in $\Sigma_{\mathbf{P}_1,0}$. Then for each $\delta \in \Delta$ the fiber Σ_δ has two normalizations $\Sigma_{\mathbf{P}_i,\delta} = \Sigma_{\mathbf{P}_i,\mathbf{a}_i,\mathbf{b}_i}, i = 1, 2,$

where $\mathbf{a}_i, \mathbf{b}_i$ are determined by appropriate cross ratios of the marked points $\mathbf{w}, \mathbf{z}$ in Σ_δ; cf Remark 5.1.1 (i). The change of normalization is given by a biholomorphism $\phi_{\mathbf{P}_2,\mathbf{P}_1,\delta} : \Sigma_{\mathbf{P}_2,\mathbf{a}_2,\mathbf{b}_2} \to \Sigma_{\mathbf{P}_1,\mathbf{a}_1,\mathbf{b}_1}$ that satisfies the formula

$$\phi_{\mathbf{P}_2,\mathbf{P}_1,\delta} := \iota_{\mathbf{P}_1,\mathbf{a}_1,\mathbf{b}_1} \circ \phi_{\mathbf{P}_2,\mathbf{P}_1} \circ \iota^{-1}_{\mathbf{P}_2,\mathbf{a}_2,\mathbf{b}_2} : \ \Sigma_{\mathbf{P}_2,\mathbf{a}_2,\mathbf{b}_2} \to \Sigma_{\mathbf{P}_1,\mathbf{a}_1,\mathbf{b}_1},$$

wherever the RHS is defined, and in particular, at the points $\mathbf{w}$. Hence there is an induced map $U_{\mathbf{P}_1} \to U_{\mathbf{P}_2}$ of the form

$$\begin{aligned} U_{\mathbf{P}_1} \ni (\vec{e}, \mathbf{a}_1, \mathbf{b}_1, \vec{\omega}, \vec{\zeta}, f) &\mapsto \phi^*(\vec{e}, \mathbf{a}_1, \mathbf{b}_1, \vec{\omega}, \vec{\zeta}, f) \\ &= \big(\vec{e}, \mathbf{a}_2, \mathbf{b}_2, \phi^{-1}(\vec{\omega}), \phi^{-1}(\vec{\zeta}), f \circ \phi_\delta\big) \in U_{\mathbf{P}_2}, \end{aligned} \tag{5.1.23}$$

where $\phi := \phi_{\mathbf{P}_2,\mathbf{P}_1}$ and $\phi_\delta = \phi_{\mathbf{P}_2,\mathbf{P}_1,\delta}$. Note that $\phi_{\mathbf{P}_2,\mathbf{P}_1,\delta}$ varies as $\delta = [\mathbf{n}, \mathbf{w}, \mathbf{z}]$ varies, and that the elements in $\vec{e}$ are not affected by the action.

[b] Change of center:

Now suppose given a chart $U_{\mathbf{P}_1}$ constructed using $\tau_1, Q, \mathbf{P}_1$ and with footprint F, and that we change the center from $\tau_1 := [\Sigma_{01}, \mathbf{z}_{01}, f_{01}]$ to $\tau_2 := [\Sigma_{02}, \mathbf{z}_{02}, f_{02}] \in F$, but keep the same slicing manifold and the same normalization (as far as possible). Thus we suppose that there is a lift $(\vec{0}, \mathbf{a}_{02}, \mathbf{b}_{02}, \mathbf{w}_{02}, \mathbf{z}_{02}, f_{02})$ of $[\Sigma_2, \mathbf{z}_2, f_2]$ to $U_{\mathbf{P}_1}$. We take a normalization $\mathbf{P}_2$ at $\tau_2 := [\Sigma_2, \mathbf{w}_2, \mathbf{z}_2]$ that includes all the nodal points in $\mathbf{P}_1$ that have not been glued (suppose there are $p_n - s$ of these), together with an appropriate subset of the points in $\mathbf{w}_{\mathbf{P}_1}, \mathbf{z}_{\mathbf{P}_1}$, if necessary assigned to different points $0, 1, \infty$ in S^2; cf. the example in Figure 5.1.1. If the stratum X_{S_2} containing τ_2 is strictly larger than X_{S_1} (i.e. τ_2 has fewer nodes than τ_1), then we cannot hope to represent the whole footprint F in the coordinates based at τ_2. However we claim:

- *there is a Γ-invariant neighbourhood $U_{\mathbf{P}_1}|_{\Delta_2}$ of $\psi^{-1}(F \cap X_{\ge S_2})$ in $U_{\mathbf{P}_1}$ that can be represented in terms of the normalization $\mathbf{P}_2$.*

To prove the claim, let us suppose that m of the gluing parameters $\mathbf{a}_{02}$ are nonzero, say $a_{02}^{K-m+1}, \dots, a_{02}^K$, so that Σ_2 has $K - m$ nodes. Then the "extra" marked points in $\mathbf{P}_1$ (namely those in $(\mathbf{w}_{\mathbf{P}_1} \cup \mathbf{z}_{\mathbf{P}_1}) \setminus (\mathbf{w}_{\mathbf{P}_2} \cup \mathbf{z}_{\mathbf{P}_2})$ can now move freely; cf. Figure 5.1.1 Consider the parametrization

$$\iota_{\mathbf{P}_1} : \big(\Sigma_{(\mathbf{P}_1,\delta_1),0} \setminus \mathcal{N}^\varepsilon_{nodes}\big) \times B^{6K-2p_n} \times B^{2(k+L-p)} \ \to \ \mathcal{C}|_\Delta$$

of (5.1.4) near $\delta_{01} := [\Sigma_{01}, \mathbf{w}_{01}, \mathbf{z}_{01}]$, and let $\Delta_2 \subset \Delta_1$ be a neighbourhood of τ_2 that contains the domains of the elements in $\psi^{-1}(F \cap X_{\ge S_2})$. There is a similar parametrization

$$\iota_{\mathbf{P}_2} : \big(\Sigma_{(\mathbf{P}_2,\delta_2),0} \setminus \mathcal{N}^\varepsilon_{nodes}\big) \times B^{6K-2(p_n-s)} \times B^{2(k+L-p+p^e)} \ \to \ \mathcal{C}|_{\Delta_2},$$

and the composite $\iota^{-1}_{\mathbf{P}_2} \circ \iota_{\mathbf{P}_1}|_{\Delta_2}$ has the form

$$\iota^{-1}_{\mathbf{P}_2} \circ \iota_{\mathbf{P}_1}|_{\Delta_2} : (\mathbf{a}_1, \mathbf{b}_1, \vec{\omega}_1, \vec{\zeta}_1) \mapsto (\mathbf{a}_2, \mathbf{b}_2, \vec{\omega}_2, \vec{\zeta}_2). \tag{5.1.24}$$

This is well defined over $\psi^{-1}(F \cap X_{\ge S})$ because the map $\iota_{\mathbf{P}_i,\mathbf{a}_i,\mathbf{b}_i}$ for $i = 1, 2$ takes the points with coordinates $\vec{\omega}_i, \vec{\zeta}_i$ to the same marked points $\mathbf{w}, \mathbf{z}$ in the fiber $\Sigma_{\mathbf{P}_1,\mathbf{a}_1,\mathbf{b}_1} = \Sigma_{\mathbf{P}_2,\mathbf{a}_2,\mathbf{b}_2}$. Hence this map is well defined over Δ_2 for sufficiently small Δ_2.

Define

$$U_{\mathbf{P}_1}|_{\Delta_2} := \big\{(\vec{e}, \mathbf{a}_1, \mathbf{b}_1, \vec{\omega}, \vec{\zeta}, f) \in U \mid [\Sigma_{\mathbf{P}_1,\mathbf{a}_1,\mathbf{b}_1}, \mathbf{w}, \mathbf{z}] \in \Delta_2\big\},$$

and denote by $\iota_{\mathbf{P}_1}|_{\Delta_2}$ the restriction of $\iota_{\mathbf{P}_1}$ to the domains occuring in $U_{\mathbf{P}_1}|_{\Delta_2}$. Given a formula such as (5.1.24) for the coordinate change on the parametrization of domains, we can derive a formula analogous to (5.1.23) for the effect on the elements of $U_{\mathbf{P}_1}|_{\Delta_2}$ of this change of center, namely

$$U_{\mathbf{P}_1}|_{\Delta_2} \ni (\vec{e}, \mathbf{a}_1, \mathbf{b}_1, \vec{\omega}_1, \vec{\zeta}_1, f) \;\mapsto \big(\vec{e}, \mathbf{a}_2, \mathbf{b}_2, \vec{\omega}_2, \vec{\zeta}_2, f \circ \phi_\delta\big) \in U_{\mathbf{P}_2}, \tag{5.1.25}$$

where $\mathbf{a}_2, \mathbf{b}_2, \vec{\omega}_2, \vec{\zeta}_2$, are as in (5.1.24), and where $\phi_\delta : \Sigma_{(\mathbf{P}_2,\delta_{02});\delta} \to \Sigma_{(\mathbf{P}_1,\delta_{01});\delta}$ is the biholomorphic map that equals $\iota_{\mathbf{P}_1,\mathbf{a}_1,\mathbf{b}_1} \circ \big(\iota_{\mathbf{P}_2}^{-1} \circ \iota_{\mathbf{P}_1}|_{\Delta_2}\big)^{-1} \circ \iota^{-1}_{\mathbf{P}_2,\mathbf{a}_2,\mathbf{b}_2}$ wherever this is defined.

Note the following:

- The map in (5.1.25) has the same form as that in (5.1.23) but with different ϕ, ϕ_δ. Hence a map that changes both the center and normalization also has this form.
- The resulting chart with domain $U_{\mathbf{P}_2}$ may not be not minimal in the sense of Definition 3.1.6 since Γ may now be larger than $\mathrm{Stab}(\tau_2)$.
- The composite of two such maps that change the center first from δ_0 to δ_1 and then from δ_1 to δ_2 equals the direct coordinate change from δ_0 to δ_2.

[c] **Change of slicing manifold:** Let us return to considering the chart U with center $\delta_0 = [\Sigma_0, \mathbf{w}_0, \mathbf{z}_0]$ as in defined in (VII), and suppose that we change the slicing manifold from Q_1 to Q_2. Let is first consider the case in which Q_2 is so close to Q_1 that the new set of slicing points $\mathbf{w}_2$ lies in the same set of discs $(D^\ell)_\ell$ as $\mathbf{w}_1$. Then there is a natural correspondence between the (ordered) tuples $\mathbf{w}_1$ and $\mathbf{w}_2$ so that we can use the same normalization $\mathbf{P}$ for both $\delta_1 := [\mathbf{n}, \mathbf{w}_1, \mathbf{z}]$ and $\delta_2 := [\mathbf{n}, \mathbf{w}_2, \mathbf{z}]$. Then if δ_1 is sufficiently close to the center δ_0 the element δ_2 will also lie in Δ. Hence the same obstruction space E can be used for both charts, and the corresponding change of coordinates $U_1 \to U_2$ is given by replacing the map $\phi_{\mathbf{P}_2,\mathbf{P}_1}$ in the above formulas by the map $\Sigma_{\mathbf{P}} \to \Sigma_{\mathbf{P}}$ that fixes the points in $\mathbf{n}_{\mathbf{P}}, \mathbf{z}_{\mathbf{P}}$ (that are in standard positions) and takes the points in $\mathbf{w}_2 \subset \Sigma_{\mathbf{P}}$ with labels in $\mathrm{im}\,(\mathbf{P})$ to their standard positions.

However, if the new slicing manifold Q_2 is sufficiently different from Q_1, there need be no obvious relation between the tuples $\mathbf{w}_1$ and $\mathbf{w}_2$. For example, suppose that the chart is centered on $[\Sigma_0, \mathbf{z}_0, f_0]$ where $\Sigma_0 = S^2$, $\mathbf{z}$ is the single point ∞ and $f_0 : S^2 \to M$ is a double cover that factors through the map $z \mapsto z^2$. Then the isotropy group is $\Gamma = \mathbb{Z}/2\mathbb{Z}$, and we need to add two points to stabilize the domain. We might choose Q_1 to have two components, one transverse to $\mathrm{im}\,(f_0)$ at $f_0(1) = f_0(-1)$ and the other transverse at $f_0(2) = f_0(-2)$ so that $\mathbf{w}_1 = (1, -1, 2, -2)$, while Q_2 might have a single component that is transverse to $\mathrm{im}\,(f_0)$ at $f_0(3) = f_0(-3)$, so that $\mathbf{w} = (3, -3)$. Since the obstruction bundle for U_1 might depend on all four entries in $\mathbf{w}_1$, while that for U_2 depends only on $\mathbf{w}_2$ there is no obvious relation between the obstruction spaces. Therefore there is no direct coordinate change from U_1 to U_2, and the easiest way to relate them is via sum charts.

(IX): Constructing the sum of two charts. Suppose that we are given two sets of data $\big(\tau_i := [\Sigma_i, \mathbf{z}_i, f_i], Q_i, (D_i^\ell)_{1\le\ell\le L_i}, \mathbf{P}_i, E_i, \lambda_i\big)_{i=1,2}$ that define charts $\mathbf{K}_i$ with overlapping footprints F_i. Then we aim to define a sum chart with

- footprint $F_{12} = F_1 \cap F_2$,
- obstruction space $E_{12} := E_1 \times E_2$,
- group $\Gamma_{12} := \Gamma_1 \times \Gamma_2$, and
- domain U_{12} of dimension $\dim(U_{12}) - \dim(E_{12}) = \dim U_i - \dim E_i = \mathrm{ind}\,(A)$, so that $\dim U_{12} = \dim U_1 + \dim E_2 = \dim U_2 + \dim E_1$.

In this paragraph we consider the case when the center of one chart is contained in the footprint of the other: say $\tau_2 \in F_1$ which implies $\tau_2 \in F_{12}$. Then we set up the sum chart using the coordinates provided by τ_2 and $\mathbf{P}_2$. As in (VIII) we may change coordinates on a neighbourhood of $\psi_1^{-1}(F_{12})$ in $U_1 := U_{1,\tau_1,\mathbf{P}_1}$ to obtain a chart $U_{1,\tau_2,\mathbf{P}_1}$ with data E_1, Γ_1, Q_1 and footprint F_{12}, but center and normalization $\tau_2, \mathbf{P}_1$. In particular the central fiber $\Sigma_{2,\mathbf{P}_2}$ contains two sets of discs, the standard discs $(D_2^\ell)_{1\le\ell\le L_2}$ for the chart U_2 as well as the image $(D_1^\ell)_{1\le\ell\le L_1}$ of the standard discs for the chart U_1.

With $L := L_1 + L_2$, $E_{12} := E_1 \times E_2$, and $p_{2,n}$ equal to the number of nodal points in $\mathbf{P}_2$, we set up an equation as in (VI) on tuples of the form

$$
\begin{aligned}
\mathcal{W}_{12,\mathbf{P}_2} :=& \big\{(\vec e_1, \vec e_2, \mathbf{a}, \mathbf{b}, \vec\omega_1, \vec\omega_2, \vec\zeta, f) \in E_{12} \times B^{6K-2p_{2,n}} \times B^{2(k+L)} \times W^{1,p}(\Sigma_{\mathbf{P}_2,\mathbf{a},\mathbf{b}}, M)\big\} \\
& \text{where} \quad \vec e_i \in E_i,\ \mathbf{a},\mathbf{b} \in B^{6K-2p_n}, \qquad f: \Sigma_{\mathbf{P}_2,\mathbf{a},\mathbf{b}} \to M, \\
& \exists \gamma \in \Gamma_1 : \omega_1^{\gamma(\ell)} \in D_1^\ell,\ 1 \le \ell \le L_1, \quad \omega_2^\ell \in D_2^\ell, 1 \le \ell \le L_2.
\end{aligned} \tag{5.1.26}
$$

Somewhat hidden in this notation is the fact that the tuple $\vec\omega_1$ contains L_1 elements since all the points $\mathbf{w}_1$ can vary, while the number of nonzero elements in the tuples $\vec\omega_2$ and $\vec\zeta$ is $\#(\mathbf{w}_2 \smallsetminus \mathbf{w}_{2,\mathbf{P}_2})$ and $\#(\mathbf{z} \smallsetminus \mathbf{z}_{\mathbf{P}_2})$. Also notice the different conditions on the tuples $\vec\omega_1, \vec\omega_2$ with respect to the discs.

In this notation, the thickened domain $\widehat U_{12,\mathbf{P}_2}$ is a suitable open subset of the following solution space:

$$
\begin{aligned}
\widehat U_{12,\mathbf{P}_2} \subset \Big\{ & (\vec e_1, \vec e_2, \mathbf{a}, \mathbf{b}, \vec\omega_1, \vec\omega_2, \vec\zeta, f) \in \mathcal{W}_{12,\mathbf{P}_2} \;\big|\; \vec e_i \in E_i, \\
& \overline\partial_J(\vec e_1, \vec e_2, \mathbf{a}, \mathbf{b}, \vec\omega_1, \vec\omega_2, \vec\zeta, f) = \textstyle\sum_{i=1,2} \sum_{\gamma\in\Gamma_i} \gamma^*\big(\lambda_i(e_i^\gamma)\big)|_{\mathrm{graph} f} \Big\},
\end{aligned}
$$

with

$$
\gamma^*\big(\lambda_i(e_i^\gamma)\big)|_{(z,f(z))} := \lambda_i(e_i^\gamma)(\phi_{i,\gamma}^{-1}(z), f(z)) \circ \mathrm{d}_z \phi_{i,\gamma}^{-1}, \quad \text{for } \gamma \in \Gamma_i,
$$

where $\phi_{i,\gamma} := \phi_{\gamma,\delta_i}$ as in as in (5.1.16) and (5.1.8), for

$$
\delta_i := [\Sigma_{\mathbf{P}_2,\mathbf{a},\mathbf{b}}, \mathbf{w}_i, \mathbf{z}_i], \quad i = 1,2,
$$

with (as usual) $\mathbf{w}_i = \iota_{\mathbf{P}_2,\mathbf{a},\mathbf{b}}(\vec\omega_i)$, and $\mathbf{z} = \iota_{\mathbf{P}_2,\mathbf{a},\mathbf{b}}(\vec\zeta_i)$. This equation has the same form as (5.1.14). Therefore because E_1, E_2 and hence $E_1 \times E_2$ satisfy (*) for all lifts of elements in the footprint F_{12} to $\widehat{\mathcal{W}}_{12,\mathbf{P}_2}$, we can choose the open set $\widehat U_{12,\mathbf{P}_2}$ so that it is a smooth manifold that contains all such lifts. We can also choose $\widehat U_{12,\mathbf{P}_2}$ to be invariant under the action of the group $\Gamma_{12} := \Gamma_1 \times \Gamma_2$. Here, since we normalize with respect to the chart $\mathbf{K}_2$, the action of $\gamma_1 \in \Gamma_1$ is simply by permutation:

$$\gamma_1^*(\vec{e}_1, \vec{e}_2, \mathbf{a}, \mathbf{b}, \vec{\omega}_1, \vec{\omega}_2, \vec{\zeta}, f) = (\gamma_1 \cdot \vec{e}_1, \vec{e}_2, \mathbf{a}, \mathbf{b}, \gamma_1 \cdot \vec{\omega}_1, \vec{\omega}_2, \vec{\zeta}, f). \tag{5.1.27}$$

However the elements of Γ_2 act by permutation plus renormalization:

$$\gamma_2^*(\vec{e}_1, \vec{e}_2, \mathbf{a}, \mathbf{b}, \vec{\omega}_1, \vec{\omega}_2, \vec{\zeta}, f) = (\vec{e}_1, \gamma_2 \cdot \vec{e}_2, \mathbf{a}, \mathbf{b}, \phi_{\gamma_2}^{-1}(\vec{\omega}_1), \phi_{\gamma_2}^{-1}(\gamma_2 \cdot \omega_2), \phi_{\gamma_2}^{-1}\vec{\zeta}, f \circ \phi_{\gamma_2,\delta_2}), \tag{5.1.28}$$

where $\phi_{\gamma_2} := \phi_{\mathbf{P}_2,\gamma_2,\delta_2}$ as in (5.1.10). This difference in action is compatible with the different conditions on $\vec{\omega}_1, \vec{\omega}_2$ in the definition of $\mathcal{W}_{12,\mathbf{P}_2}$.

We now choose $U_{12,\mathbf{P}_2}$ to be a suitable open subset of $\widehat{U}_{12,\mathbf{P}_2}$ on which the slicing conditions are satisfied. Thus

$$U_{12,\mathbf{P}_2} \subset \big\{(\vec{e}_1, \vec{e}_2, \mathbf{a}, \mathbf{b}, \vec{\omega}_1, \vec{\omega}_2, \vec{\zeta}, f) \in \widehat{U}_{12,\mathbf{P}_2} \,|\iota_{\mathbf{P},\mathbf{a},\mathbf{b}}(\vec{\omega}_i) \in f^{-1}(Q_i),\ i = 1, 2\big\}. \tag{5.1.29}$$

We choose $U_{12;\mathbf{P}_2}$ to be Γ_{12}-invariant (which is possible because the slicing conditions are preserved by this action), and so that the zero set of $s_{12} : (\vec{e}_1, \vec{e}_2, \mathbf{a}, \mathbf{b}, \vec{\omega}_1, \vec{\omega}_2, \mathbf{z}, f) \mapsto (\vec{e}_1, \vec{e}_2)$ is taken by the forgetful map $\psi : (\vec{0}, \vec{0}, \mathbf{a}, \mathbf{b}, \vec{\omega}_1, \vec{\omega}_2, \mathbf{z}, f) \mapsto [\Sigma_{\mathbf{P}_2,\mathbf{a},\mathbf{b}}, \mathbf{z}, f]$ onto F_{12}. We claim that $\mathbf{K}_{12} := \big(U_{12;\mathbf{P}_2}, E_{12}, \Gamma_{12}, s_{12}, \psi_{12}\big)$ is the required sum chart. This is immediate from the construction, except possibly for the fact that the footprint map ψ induces an injection ${s_{12}^{-1}(0)}/{\Gamma_{12}} \to F_{12}$. However this holds because the forgetful map

$$\rho_{2,12} : U_{12,\mathbf{P}_2} \cap s_{12}^{-1}(E_1) \to U_2 : (\vec{0}, \vec{e}_2, \mathbf{a}, \mathbf{b}, \vec{\omega}_1, \vec{\omega}_2, \vec{\zeta}, f) \mapsto (\vec{e}_2, \mathbf{a}, \mathbf{b}, \vec{\omega}_2, \vec{\zeta}, f) \in U_2.$$

induces an injection into U_2 from the quotient of $\widetilde{U}_{2,12} := U_{12,\mathbf{P}_2} \cap s_{12}^{-1}(E_1)$ by a free permutation action of Γ_1 on $\vec{\omega}_1$, and we have already checked that the footprint map ψ_2 induces a homeomorphism ${s_2^{-1}(0)}/{\Gamma_2} \to F_2$.

To complete the construction we must check that the required coordinate changes $\mathbf{K}_i \to \mathbf{K}_{12}$ exist. The coordinate change $\mathbf{K}_2 \to \mathbf{K}_{12}$ is induced by the above projection $\rho_{2,12}$. The coordinate change $\mathbf{K}_1 \to \mathbf{K}_{12}$ has domain $\widetilde{U}_{1,12} := U_{12,\mathbf{P}_2} \cap s_{12}^{-1}(E_2)$, and is given by first changing the normalization[24] from $\mathbf{P}_2$ to $\mathbf{P}_1$, and then forgetting the components of $\vec{\omega}_2$ to obtain a map $\rho_{1,12} : \widetilde{U}_{1,12} \to U_{1,\tau_2,\mathbf{P}_1}$. The reader can check that this change of normalization reverses the conditions on the tuples $\vec{\omega}_i$. In particular, afterwards $\vec{\omega}_2$ has L_2 potentially nonzero components $(\omega_2^\ell)_\ell$ with $\omega_2^\ell \in D_2^{\gamma(\ell)}$ for some $\gamma \in \Gamma_2$. Hence the forgetful map is the quotient by a free action of Γ_2 as required.

Remark 5.1.3. We constructed this sum chart under a restrictive condition on the footprints. If this condition is not satisfied we may not be able to find one set of coordinates that covers a neighbourhood of the full footprint F_{12}. The difficulty here is that the parametrization maps $\iota_{\mathbf{P}_1,\mathbf{a}_1,\mathbf{b}_1}$ in (5.1.4) are not defined near the nodes, so that their image may not contain all the points in the relevant inverse images $f^{-1}(Q_2)$. Therefore, one might not be able to pull all the points in the tuple $\mathbf{w}_2$ back to the center point τ_1, and similarly, the points in $\mathbf{w}_1$ might not all pull back to a center for

[24] If Q_1, Q_2 are disjoint we can simply apply (VIII) (a) with slicing manifold $Q_1 \cup Q_2$; the general case is similar.

the second chart. One could deal with this problem by requiring that if $F_{12} \neq \emptyset$, the corresponding slicing manifolds Q_1, Q_2 are not too different, but such conditions are hard to formulate precisely. Instead (as in [P13]) we dispense with the requirement that the chart have global coordinates. To prepare for the general definition given in (X) below, we now explain coordinate free version of the above construction.

We define U_{12} to be the image of $U_{12,\mathbf{P}_2}$ by the map

$$U_{12,\mathbf{P}_2} \ni \big(\vec{e}_1, \vec{e}_2, \mathbf{a}, \mathbf{b}, \vec{\omega}_1, \vec{\omega}_2, \vec{\zeta}, f\big) \mapsto \big(\underline{\vec{e}}, [\mathbf{n}, \mathbf{w}_1, \mathbf{w}_2, \mathbf{z}, f]\big)$$

where $[\mathbf{n}, \mathbf{w}_i, \mathbf{z}] \in \Delta_i$ is the domain stabilized via Q_i i.e. the stable curve $[\Sigma_{\mathbf{P}_2,\mathbf{a},\mathbf{b}}, \mathbf{w}_i, \mathbf{z}]$, and $\underline{\vec{e}} := (\vec{e}_1, \vec{e}_2) \in E_{12}$. Thus U_{12} is a subset of the following space

$$\left\{ \big(\underline{\vec{e}}, [\mathbf{n}, \mathbf{w}_1, \mathbf{w}_2, \mathbf{z}, f]\big) \;\middle|\; \begin{array}{ll} \underline{\vec{e}} \in E_{12}, & \delta_i := [\mathbf{n}, \mathbf{w}_i, \mathbf{z}] \in \Delta_i, \\ f(\mathbf{w}_i) \in Q_i, & \exists \gamma \in \Gamma_i, w_i^\ell \in D_{\delta_i}^{\gamma(\ell)} \\ \overline{\partial}_J f = \lambda(\underline{\vec{e}})|_{\mathrm{graph} f} & \end{array} \right\} \tag{5.1.30}$$

By choice of Q_i, the condition $f(\mathbf{w}_i) \in Q_i$ implies that there is exactly one element of $\mathbf{w}_i$ in each disc $D_{\delta_i}^\ell$. The labels of these discs are well defined modulo the action of Γ_i, and the condition $\exists \gamma \in \Gamma_i, w_i^\ell \in D_{\delta_i}^{\gamma(\ell)}$ implies that the tuple $\mathbf{w}_i$ has one of its admissible labellings; cf. the end of (IV). Hence in this formulation both groups $(\Gamma_i)_{i=1,2}$ act by permuting the elements in $\vec{e}_i, \mathbf{n}, \mathbf{w}_i$ as in (5.1.20). Further, the sum chart depends only on the footprint F_{12} and the choice of (Q_i, E_i, λ_i), the center τ_i, normalization $\mathbf{P}_i$ and discs $(D_i^\ell)_{1 \le \ell \le L}$ being irrelevant except insofar as they help guide the construction.

(X): Completion of the construction: Suppose given a collection $(\mathbf{K}_i)_{i \in I}$ of basic charts whose footprints $(F_i)_{1 \le i \le N}$ cover X. We aim to construct an atlas in the sense of Definition 2.1.9 in which the charts are indexed by $I \in \mathcal{I}_\mathcal{K}$ and have $E_I := \prod_{i \in I} E_i$, $\Gamma_I := \prod_{i \in I} \Gamma_i$. The easiest way to do this is in the coordinate free language introduced in Remark 5.1.3. To simplify notation we denote the elements of the obstruction space E_I by underlined tuples: $\underline{\vec{e}} := (\vec{e}_i)_{i \in I}$. Similarly $\underline{\mathbf{w}} := (\mathbf{w}_i)_{i \in I}$ are the sets of added marked points. We define U_I to be a Γ_I-invariant open subset of the following space:

$$U_I \subset \left\{ \big(\underline{\vec{e}}, [\mathbf{n}, \underline{\mathbf{w}}, \mathbf{z}, f]\big) \;\middle|\; \begin{array}{ll} \underline{\vec{e}} \in E_I, & \delta_i := [\mathbf{n}, \mathbf{w}_i, \mathbf{z}] \in \Delta_i, \forall i \in I, \\ f(\mathbf{w}_i) \in Q_i, & \exists \gamma \in \Gamma_i, w_i^\ell \in D_{\delta_i}^{\gamma(\ell)} \\ \overline{\partial}_J f = \lambda(\underline{\vec{e}})|_{\mathrm{graph} f} & \end{array} \right\}, \tag{5.1.31}$$

chosen so that the footprint is F_I. Since we take

$$s_I\big(\underline{\vec{e}}, [\mathbf{n}, \underline{\mathbf{w}}, \mathbf{z}, f]\big) = \underline{\vec{e}}, \quad \psi_I\big(\underline{\vec{0}}, [\mathbf{n}, \underline{\mathbf{w}}, \mathbf{z}, f]\big) = [\mathbf{n}, \mathbf{z}, f],$$

and Γ_I acts by permutation, this condition can always be satisfied. We claim that if U_I is a sufficiently small neighbourhood of $\psi_I^{-1}(F_I)$ then it is a smooth manifold. For this it suffices to check that each point τ of $\psi_I^{-1}(F_I)$ has such a neighbourhood, which one does by choosing a normalization $\mathbf{P}_i$ at τ for some $i \in I$, and then writing the definition of U_I in the corresponding local coordinates as in the explicit construction in (IX). Details are left to the reader.

In this coordinate free language, the atlas coordinate changes $\mathbf{K}_I \to \mathbf{K}_J$ are given by first choosing appropriate domains $\widetilde{U}_{IJ} \subset U_J$ and then simply forgetting the components $(\mathbf{w}_i)_{i\in(J\smallsetminus I)}$. To see these forgetful maps have the required properties, one should argue in coordinates as in the discussion after (5.1.29).

Remark 5.1.4. The smoothness of the charts, group actions and coordinate changes depends on the gluing theorem used. At the minimum (i.e. with the gluing theorem in [MS]) we get a weakly SS atlas. With more analytic input, we can get a $\mathcal{C}^1$-atlas or even a smooth atlas. However the sets U_I still have an underlying stratification (by the number of nodes in the domains Σ_δ of its elements) that is respected by all maps and coordinate changes. Hence, as explained in §3.3 the resulting zero set $|(s+\nu)^{-1}(0)|$ has a natural stratification that is sometimes useful.

(XI): Constructing cobordisms: To prove that the VFC is independent of choices we need to build cobordisms between any two atlases on X. We will not give the formal definition of cobordism here; cf. [MW12, §6.4] for a detailed treatment of cobordisms over the product $X \times [0,1]$. The following notion is also useful.

Definition 5.1.5. *Two atlases $\mathcal{K}, \mathcal{K}'$ on X are said to be* **directly commensurate** *if they are subatlases of a common atlas $\mathcal{K}''$. They are commensurate if there is a sequence of atlases $\mathcal{K} =: \mathcal{K}_1, \dots, \mathcal{K}_\ell := \mathcal{K}'$ such that any consecutive pair $\mathcal{K}_i, \mathcal{K}_{i+1}$ are directly commensurate.*

One useful result is that any two commensurate atlases are cobordant; cf. [MW12, Lemma 6.4.12] and the proof of Proposition 4.2.1 (iii) above. Note that because we can construct the sum of any number of charts provided only that their footprints have nonempty intersection, any two atlases constructed on X by the method described above with basic charts $(\mathbf{K}_i)_{1\le i\le N_1}$ and $(\mathbf{K}_i)_{N_1+1\le i\le N_2}$ are subatlases of a common atlas with basic charts $(\mathbf{K}_i)_{1\le i\le N_2}$. Thus they are **commensurate** and hence cobordant. (This result is mildly generalized in Proposition 5.2.3 below.)

A similar argument shows that the VFC is independent of the choice of almost complex structure J. More precisely, suppose that J_0, J_1 are two ω-tame almost complex structures on the symplectic manifold (M,ω), join them by a path $(J_t)_{t\in[0,1]}$ of ω-tame almost complex structures (where $t \mapsto J_t$ is constant for t near $0,1$), and define $X^{01} := \bigcup_{t\in[0,1]} \overline{\mathcal{M}}_{0,k}(M,A,J_t)$. In the same way that we build a cobordism atlas over $X \times [0,1]$, we can build a cobordism atlas $\mathcal{K}^{01}$ over X^{01}. Moreover, we can arrange that its restrictions $\mathcal{K}^\alpha := \mathcal{K}^{01}|_\alpha$ at the end points $\alpha = 0,1$ equal any given GW atlases $\mathcal{K}^\alpha$ for $X_\alpha := \overline{\mathcal{M}}_{0,k}(M,A,J_\alpha)$, and then prove that the two elements $\big([X_\alpha]^{vir}_{\mathcal{K}^\alpha}\big)_{\alpha=0,1}$ have the same image in $\check{H}_d(X^{01};\mathbb{Q})$. It follows that all GW invariants calculated using $[X]^{vir}_{\mathcal{K}}$ are independent of the choice of J. This argument is not yet written anywhere; however its details are very similar to those in [MW12, §7.5].

(XII): Proof of Theorem A: The above construction explains the proof of Theorem A. We set up the relevant equation in (VI), but the proof that it has the required properties assumes a gluing theorem that we did not even state precisely. The paper

[C] will complete the proof by providing the analytic details of a $\mathcal{C}^1$-gluing theorem, thus establishing a $\mathcal{C}^1$ version of Theorem A.

5.2. **Gromov–Witten atlases.** Let $X = \overline{\mathcal{M}}_{0,k}(M, A, J)$. Roughly speaking, a Gromov–Witten (GW for short) atlas on X is any Kuranishi atlas constructed by the procedure described above. We now aim to make this statement precise. In particular, we want to allow more general semi-additive indexing sets as in Definition 4.1.2. in order to show that the product of two GW atlases is also an atlas of this type.

In the construction given above each basic chart $\mathbf{K}_i$ depends on the choice of the following data $\big(\tau_i := [\Sigma_i, \mathbf{z}_i, f_i], Q_i, (D_i^\ell)_{1\le \ell\le L_i}, \mathbf{P}_i, E_i, \lambda_i\big)$, while the information recorded in the atlas is the tuple $(U_i, E_i, \Gamma_i, s_i, \psi_i)$. The center τ_i and normalization $\mathbf{P}_i$ are used to define coordinates over the domain U_i of the chart, and, though useful, are not really essential since one can define a coordinate free version of a chart. On the other hand, the slicing manifold Q_i and associated disc structure $(D_i^\ell)_{1\le\ell\le L_i}$ is essential to the construction, but appear in the chart only indirectly via the set $\mathbf{w}_i$ of added marked points that are permuted by the group Γ_i. In the semi-additive case we control the combinatorics of the obstruction spaces E_I and groups Γ_I via the indexing sets $\mathcal{A}_E, \mathcal{A}_\Gamma$ and choice of functions $\tau_E, \tau_{E\Gamma}$. In a semi-additive GW atlas we require that the slicing manifolds Q_i and hence the sets $\mathbf{w}_i$ are controlled in the same way as the groups. In §5.1, we defined the sets $\mathbf{w}_i$ to be the full inverse image $f^{-1}(Q_i)$, with order given by the disc neighbourhoods (D_i^ℓ), where we assumed that f is transverse to the manifold Q_i for all f in the footprint and hence all f in the domain. As we see from Example 5.2.2 it is convenient to consider slightly more general sets Q_α; the essential point is that they function in the same way as codimension 2 submanifolds, giving well defined tuples $\mathbf{w}_\alpha$ for each f. We leave this point a little vague in the definition since the only case we consider is that of products.

Definition 5.2.1. *Suppose given sets $\mathcal{I}, \mathcal{A}_E, \mathcal{A}_\Gamma$ and functions $\tau_E, \tau_{E\Gamma}$ that satisfy the conditions of Definition 4.1.2. Suppose further that $\mathcal{K}$ is a Kuranishi atlas on $X = \overline{\mathcal{M}}_{0,k}(M, A, J)$ whose charts $\mathbf{K}_I = (U_I, E_I, \Gamma_I, s_I, \psi_I)$ are indexed by $\mathcal{I}$, and are constructed as in §5.1 from data $\big(Q_i, (D_i^\ell)_{1\le\ell\le L_i}, E_i, \lambda_i\big)$ with elements $\big(\underline{\vec{e}}, [\mathbf{n}, \mathbf{w}_I, \mathbf{z}, f]\big)$ as in* (5.1.31). *We say that $\mathcal{K}$ is a* **semi-additive Gromov–Witten atlas** *if the following conditions hold:*

- *each E_I is defined by the tuples $(E_\alpha)_{\alpha\in A_E}$ via τ_E as in Definition 4.1.2;*
- *for each $\alpha \in \mathcal{A}_\Gamma$ there is a (generalized) slicing manifold Q_α such that each basic chart $\mathbf{K}_i$ is defined using the union $Q_i := \bigcup_{\alpha\in\tau(i)} Q_\alpha$ in the sense that*
  - *$\mathbf{w}_i = (\mathbf{w}_\alpha)_{\alpha\in\tau_\Gamma(i)}$, where $\mathbf{w}_\alpha = f^{-1}(Q_\alpha)$ (with appropriate order), and*
  - *$\Gamma_i = \prod_{\alpha\in\tau_\Gamma(i)}$ acts by permutation in each factor;*
- *more generally, $\mathbf{w}_I = (\mathbf{w}_\alpha)_{\alpha\in\tau(I)}$ with the product action of Γ_I.*

Example 5.2.2. If $\mathcal{K}^i$ is a given GW atlas on $X_i = \overline{\mathcal{M}}_{0,k}(M_i, A, J)$ for $i = 1, 2$ then the product atlas is also a GW atlas since we may take slicing "manifolds" $Q_{\alpha_1,\alpha_2} = Q^1_{\alpha_1} \times M_2 \cup M_1 \times Q^2_{\alpha_2}$. This of course is a slight liberty since Q_{α_1,α_2} is not a manifold. One could deal with this by removing the set $Q^1_{\alpha_1} \times Q^2_{\alpha_2}$, but then there might be awkward special cases. On the other hand, in context it makes sense since the maps in the product chart are pairs (f_1, f_2) and one can get transversality only when f_1 intersects $Q^1_{\alpha_1} \times M_2$ and f_2 intersects $M_1 \times Q^2_{\alpha_2}$.

Proposition 5.2.3. (i) *Let $X = \overline{\mathcal{M}}_{0,k}(M, J, A; Z_c)$. Any two GW atlases on X are directly commensurate and hence cobordant.*

(ii) *If $X_i = \overline{\mathcal{M}}_{0,k}(M_i, J, A_i; Z_{c_i})$ for $i = 1, 2$, every GW atlas on $X_1 \times X_2$ is cobordant to a product $\mathcal{K}_1 \times \mathcal{K}_2$ of GW atlases $\mathcal{K}_i$ on X_i.*

Sketch of proof. Suppose given two GW atlases $(\mathcal{K}^\beta)_{\beta=0,1}$ on X with basic charts $(\mathbf{K}^\beta_i)_{i\in m(\mathcal{I}^\beta)}$ built using the data

$$\mathcal{I}^\beta, \mathcal{A}^\beta_E, \mathcal{A}^\beta_\Gamma, E^\beta_\alpha, \Gamma^\beta_\alpha, Q^\beta_\alpha, \tau^\beta_E, \tau^\beta_{E\gamma}, \quad \beta = 0, 1,$$

According to Definition 5.1.5, we must show that one can include the union of these atlases into a GW atlas $\mathcal{K}$ on X with basic charts $\bigcup_{\beta=0,1}(\mathbf{K}^\beta_i)_{i\in m(\mathcal{I}^\beta)}$, and sum charts indexed by $\mathcal{I} \subset \mathcal{P}^*\big(m(\mathcal{I}^0) \cup m(\mathcal{I}^1)\big)$ where

$$I = \Big|\big\{I_0 \cup I_1 \in \mathcal{P}^*\big(m(\mathcal{I}^0) \cup m(\mathcal{I}^1)\big) \;\big|\; F_I := F_{I_0} \cap F_{I_1} \neq \emptyset\big\}.$$

The chart $\mathbf{K}_I$ has obstruction space $E^0_{I_0} \times E^1_{I_1}$, group $\Gamma^0_{I_0} \times \Gamma^1_{I_1}$, and added marked points $(\mathbf{w}_{I_\beta})_{\beta=0,1}$ defined by the slicing manifolds $(Q^\beta_\alpha)_{\alpha\in\tau^\beta_\Gamma(I_\alpha),\beta=0,1}$. One then builds a cobordism from $\mathcal{K}^0$ to $\mathcal{K}^1$ as in the proof of Proposition 4.2.1 (iii).

This proves (i). We saw in Example 5.2.2 that the product of two GW atlases is a GW atlas on the product space $X_1 \times X_2$. Hence (ii) follows from (i). □

Remark 5.2.4. For completeness one should define the notion of GW cobordism, and extend results such as Proposition 5.2.6 to cobordisms.

In order to show that abstract constructions such as those described in §4.2 or Remark 6.2.4 preserve the class of GW atlases, it is useful to make the following definition.

Definition 5.2.5. *Let $X = \overline{\mathcal{M}}_{0,k}(M, J, A)$. We say that an atlas $\mathcal{K}$ on X is* **isomorphic to a GW atlas** *if there is a GW atlas $\mathcal{K}'$ with the same indexing set and footprints, and collections of diffeomorphisms $\sigma_I : U_I \to U'_I$, linear isomorphisms $\widehat{\sigma}_I : E_I \to E'_I$ and group isomorphisms $\sigma^\Gamma_I : \Gamma_I \to \Gamma'_I$ that commute with all structural maps in the following sense:*

- *for each $I \in \mathcal{I}_\mathcal{K}$, $\widehat{\sigma}_I$ is the product $\prod_{i\in I} \widehat{\sigma}_i$, and σ^Γ_I is the product $\prod_{i\in I} \sigma^\Gamma_i$;*
- *for each I, $(\sigma_I, \sigma^\Gamma_I) : (U_I, \Gamma_I) \to (U'_I, \Gamma'_I)$ is equivariant and intertwines the sections s_I, s'_I and footprint maps:*

$$s'_I \circ \sigma_I = \widehat{\sigma}_I \circ s_I, \quad \psi'_I(\sigma_I(s_I^{-1}(0))) = \psi'_I(s_I^{-1}(0)) = F_I;$$

- $(\sigma_I, \sigma_I^\Gamma)$ *is compatible with coordinate changes in the sense that* $\sigma_J(\widetilde{U}_{IJ}) = \widetilde{U}'_{IJ}$ *and the following diagram commutes*

$$\begin{array}{ccc} (\widetilde{U}_{IJ}, \Gamma_J) & \xrightarrow{(\sigma_J, \sigma'_J)} & (\widetilde{U}'_{IJ}, \Gamma'_J) \\ \rho_{IJ}\Big\downarrow & & \Big\downarrow\rho'_{IJ} \\ (U_{IJ}, \Gamma_I) & \xrightarrow{(\sigma_I, \sigma'_I)} & (U'_{IJ}, \Gamma'_I). \end{array}$$

Proposition 5.2.6. *Let $\mathcal{K}$ be a semi-additive GW atlas. Then its additive extension $\mathcal{K}'$ defined as in Proposition 4.2.1 is isomorphic to a GW atlas.*

Proof. The charts in the additive extension $\mathcal{K}'$ are defined in (4.2.4). The domain U'_I has elements $(e'_I, u) \in E'_I \times U_{\ell(I)}$ where $s_{\ell(I)}(u) = \sigma_I(e'_I)$, and the section $s'_I : U'_I \to E'_I$ is the projection $(e'_I, u) \mapsto e'_I$. If $\mathcal{K}$ is a GW atlas constructed using the data $E_\alpha, \lambda_\alpha, \Gamma_\alpha$ and slicing manifolds Q_α, then u is a tuple of the form $(\vec{e}, [\mathbf{n}, \mathbf{w}_{\ell(I)}, \mathbf{z}, f])$ where $e = \sigma_I(e'_I)$, $s_{\ell(I)}(u) = \vec{e}$ and $\overline{\partial}_J f = \lambda(\vec{e})|_{\mathrm{graph\,f}}$. If we define $\lambda' := \lambda \circ s_{\ell(I)}$, then the elements $(e'_I, u) \in E'_I \times U_{\ell(I)}$ may be written as $(e'_I, [\mathbf{n}, \mathbf{w}_{\ell(I)}, \mathbf{z}, f])$ where $\overline{\partial}_J f = \lambda'(e'_I)|_{\mathrm{graph\,f}}$. Thus the charts of $\mathcal{K}'$ can be constructed using the same geometric data as $\mathcal{K}$ but with the new function τ'_E of (4.2.1). This completes the proof. □

5.3. **Variations on the construction.** There are two common variants of X: we can consider the subset of X formed by elements $[\Sigma, \mathbf{z}, f]$ where we constrain either the image of the evaluation map $\mathrm{ev}_k f := \big(f(z_1), \dots, f(z_k)\big) \in M^k$ or the topological type of the domain. In both cases, it is easy to modify the construction.

[a] **Adding homological constraints from M.**

Let $Z_c \subset M^k$ be a closed submanifold representing a homology class $c \in H_{\dim c}(M^k)$ and consider

$$X_{Z_c} := \overline{\mathcal{M}}_{0,k}(M, J, A; Z_c) := \big\{[\Sigma, \mathbf{z}, f] \in \overline{\mathcal{M}}_{0,k}(M, J, A) \;\big|\; ev_k(f) \in Z_c\big\}.$$

Then if $d := \mathrm{ind}\,(A)$ is the formal dimension $2n + 2c_1(A) + 2k - 6$ of $\overline{\mathcal{M}}_{0,k}(M, J, A)$, its subset X_c has formal dimension $d + \dim c - 2n = d - \operatorname{codim} c$. We can form a chart for X_c near $\tau_0 := [\Sigma_0, \mathbf{z}_0, f_0] \in X_c$ by modifying the requirement that E satisfy condition (*) as follows.

Choose subspaces $(V_i \subset \mathrm{T}_{f_0(z_i)} M)_{1 \le i \le k}$ whose complements $V_i^\perp$ span a complement to $\mathrm{T}_{\mathrm{ev}_k(f_0)} Z_c$. Then, if $z_i \in (S^2)_{\alpha(i)}$, consider the subspace

$$\mathcal{D}_c := \big\{(\xi_\alpha)_{\alpha \in T} \in \mathcal{D}_0 \;\big|\; \xi_{\alpha(i)}(z_i) \in V_i \ \forall i\big\},$$

where $\mathcal{D}_0$ is as in (5.1.18). Replace condition (*) by

($*_c$) the elements in the image of $\lambda : E \to \mathbb{C}^\infty\big(\mathrm{Hom}^{0,1}_J(\mathcal{C}|_W \times M)\big)$ restrict on graphf_0 to a subspace of $\prod_\alpha L^p(\mathrm{Hom}^{0,1}_J((S^2)_{\alpha \in T}, f^*_{0,\alpha}(\mathrm{T}M))$ that covers the cokernel of $\mathrm{d}_{f_0}(\overline{\partial}_J)|_{\mathcal{D}_c}$.

Now consider the set $\widehat{U}$ defined as in (VI). Condition $(*_c)$ on E implies that the linearization

$$\begin{aligned}\mathrm{d}_{f_0}(\mathrm{ev}_{node} \times \mathrm{ev}_k \times \overline{\partial}_J) : \prod_{\alpha\in T} W^{1,p}((S^2)_\alpha, f_{0,\alpha}^*(\mathrm{T}M))\\ \longrightarrow (\mathrm{T}M)^{2K+k} \times \prod_{\alpha\in T} L^p(\mathrm{Hom}_J^{0,1}((S^2)_\alpha, f_{0,\alpha}^*(\mathrm{T}M))\end{aligned} \tag{5.3.1}$$

is transverse to the product of the appropriate $2K$-dimensional diagonal with Z_c. Hence, there is an open neighbourhood of $(\vec{0}, \vec{w}_0, \mathbf{z}_0, f_0)$ in

$$\widehat{U}_c := \{(\vec{e}, \vec{w}, \mathbf{z}, f) \in \widehat{U} \mid \mathrm{ev}_k(f) \in Z_c\}$$

that is a manifold of dimension $\dim(\widehat{U}) - \mathrm{codim}\,(c)$. The rest of the construction goes through as before, giving a 0-dimensional atlas $\mathcal{K}$ on X_{Z_c}, whose virtual class $[X_{Z_c}]^{vir}_{\mathcal{K}}$ is a rational number.

If $c = c_1 \times c_k \in H_*(M^k)$, then this number is just the Gromov–Witten invariant $\langle c_1, \dots, c_k\rangle_{0,k,A} \in \mathbb{Q}$. If one has an appropriate gluing theorem, one can also form $[X]^{vir}_{\mathcal{K}}$, where $X := \overline{\mathcal{M}}_{0,k}(M, J, A)$. Because $[X]^{vir}_{\mathcal{K}} \in H_*(X, \mathbb{Q})$ it pushes forward by the evaluation map $\mathrm{ev} : X \to M^k$ to a homology class and one can also define this invariant using the intersection product in M^k:

$$\langle c_1, \dots, c_k\rangle_{0,k,A} := \mathrm{ev}_*([X]^{vir}_{\mathcal{K}}) \cdot Z_c.$$

It is not hard to check that these two definitions agree.

[b] **Restricting the domain of the stable maps.**

The easiest way to restrict the domain of a stable map is to specify a minimum number of nodes. For example, consider the space $X_{\le\mathcal{S}(p)}$ of elements in $\overline{\mathcal{M}}_{0,k}(M, J, A)$ whose (stabilized) domain has at least p nodes. In this case, the above construction builds an atlas that has all the required properties except that its domains may no longer be smooth manifolds. Rather they are stratified spaces with local models of the form

$$\mathbb{R}^k \times (\mathbb{C}^{\underline{n}})_s = \{(x; a_1, \dots, a_n) : \#\{i : a_i = 0\} \ge s\}.$$

(cf. Example 3.3.3.) This is a stratified space with smooth strata of even codimension that we label by the number of nodes.[25] If we require that the domains U_I of the Kuranishi charts are locally of this form, and that all group actions and coordinate changes respect this stratification, we can define an atlas on $X_{\le\mathcal{S}(p)}$ as before. Further, if we assume that our gluing theorem provides charts that are at least $\mathcal{C}^1$-smooth, we can construct $\mathcal{C}^1$-smooth perturbations ν, so that the stratawise transversality condition considered in §3.3 is open. The zero set of $s+\nu$ will no longer be a (branched) manifold, but rather a (branched) stratified space with strata of codimension at least 2. Such a space (if oriented) still carries a fundamental class. Hence all the arguments go through as before, and one again gets an analog of Theorem B.

[25] We forget the finer stratification $\mathcal{T}^n$ on $\mathbb{R}^k \times \mathbb{C}^{\underline{n}}$ since this does not extend in any natural way to X.

Another way to calculate an invariant involving $X_{\le \mathcal{S}(p)}$ is to build a Kuranishi atlas $\mathcal{K}$ for X in the standard way, together with a reduction $\mathcal{V}$ and transverse perturbation $\nu : \mathcal{B}_{\mathcal{K}}|_{\mathcal{V}}$, again using the transversality condition in §3.3 (which makes sense because all charts are stratified.) Then consider the part of the zero set $\big(s+\nu^{-1}(0)\big)|_{\le \mathcal{S}(p)}$ consisting of elements in strata at level at least p, i.e. the domains of the maps have at least p nodes. It is not hard to check that this represents a well defined homology class in $\check{H}_d(X_{\le \mathcal{S}(p)})$, that agrees with the one constructed earlier.

Remark 5.3.1. One could extend the definition of semiadditive GW atlas to include atlases over these more general spaces X. One could also amplify the discussion in Example 4.1.3 (iii) of atlases over manifolds with boundary. As explained there, the most natural indexing sets in this situation have hybrid type: over the interior they have the standard additive form, while charts that intersect the boundary have product form. Since we have no immediate applications in mind, we do not pursue these ideas further here.

6. Examples

In this section we give a few examples. We begin by showing that every compact smooth orbifold has an atlas. We then show how to use atlases to compute Gromov–Witten invariants in some very simple cases, for example if the moduli space is an orbifold with cokernels of constant rank. Finally, we revisit an argument in [M00] about the Seidel representation for general symplectic manifolds The "proof" given there assumed the existence of a construction for the VFC with slightly different properties from the one above, and does not work with the new definitions. However, it is not hard to give a proof using the current definitions.

6.1. **Orbifolds.** The aim of this subsection is to prove Proposition C stated in §1, i.e. to show that every compact orbifold Y has a Kuranishi atlas with trivial obstruction spaces. We will define orbifolds via the concept of **ep (étale proper) groupoid** $\mathcal{G}$. This is a category with smooth spaces of objects $\mathrm{Obj}_{\mathcal{G}}$ and morphisms $\mathrm{Mor}_{\mathcal{G}}$, such that

- all structural maps (i.e. source s, target t, identity, composition and inverse) are étale (i.e. local diffeomorphisms); and
- the map $s \times t : \mathrm{Mor}_{\mathcal{G}} \to \mathrm{Obj}_{\mathcal{G}} \times \mathrm{Obj}_{\mathcal{G}}$ given by taking a morphism to its source and target is proper.

The realization $|\mathcal{G}|$ of $\mathcal{G}$ is the quotient of the space of objects by the equivalence relation given by the morphisms: thus $x \sim y \;\Leftrightarrow\; \mathrm{Mor}_{\mathcal{G}}(x,y) \ne \emptyset$. The following definition is similar to that used by Moerdijk [Mo02]; also cf [M07].

def:orbstr

Definition 6.1.1. *An* **orbifold structure** *on a paracompact Hausdorff space Y is a pair $(\mathcal{G}, f)$ consisting of an ep (étale proper) groupoid $\mathcal{G}$ together with a map $f : \mathrm{Obj}_{\mathcal{G}} \to Y$ that factors through a homeomorphism $|f| : |\mathcal{G}| \to Y$. Two orbifold structures $(\mathcal{G}, f)$ and $(\mathcal{G}', f')$ are Morita equivalent if they have a common refinement, i.e. if* *there is a third structure* $(\mathcal{G}'', f'')$ *and functors* $F : \mathcal{G}'' \to \mathcal{G}, F' : \mathcal{G}'' \to \mathcal{G}'$ *such that* $f'' = f \circ F = f' \circ F'$.

An **orbifold** *is a second countable paracompact Hausdorff space* Y *equipped with an equivalence class of orbifold structures. We say that* Y *is oriented if the spaces of objects* $\mathrm{Obj}_{\mathcal{G}}$ *and morphisms* $\mathrm{Mor}_{\mathcal{G}}$ *have orientations that are preserved by all structure maps.*

Definition 6.1.2. *We say that an oriented orbifold* Y *has an* **orbifold atlas** $\mathcal{K}$ *if* Y *has an open covering* $Y = \bigcup_{i=1,\dots,N} F_i$ *such that the following conditions hold with*

$$\mathcal{I}_Y := \big\{ I \subset \{1, \dots, N\} : F_I := \bigcap_{i \in I} F_i \neq \emptyset \big\}.$$

- *For each* $I \in \mathcal{I}_Y$ *there is an oriented manifold* W_I *on which* $\Gamma_I := \prod_{i \in I} \Gamma_i$ *acts preserving orientation and a map* $\psi_I : W_I \to F_I$ *that induces a homeomorphism* $\underline{\psi}_I : {W_I}/{\Gamma_I} \to F_I$;
- *for all (nonempty) subsets* $I \subset J$ *the kernel* $\ker \rho^{\Gamma}_{IJ}$ *of the projection* $\Gamma_J \to \Gamma_I$ *acts freely on* W_J, *and the quotient map*
$$\rho_{IJ} : W_J \to W_{IJ} := (\psi_I)^{-1}(F_J)$$
is ρ^{Γ}_{IJ}*-equivariant, orientation preserving, and étale;*
- $\psi_I \circ \rho_{IJ} = \psi_J$, *and* $\rho_{IJ} \circ \rho_{JK} = \rho_{IK}$ *for all* $I \subset J \subset K$.

Thus the charts of this atlas $\mathcal{K}$ *are the tuples* $\big(\mathbf{K}_I := (W_I, \Gamma_I, \psi_I)\big)_{I \in \mathcal{I}_Y}$ *with footprints* $(F_I)_{I \in \mathcal{I}_Y}$, *and the coordinate changes are induced by the covering maps* ρ_{IJ}.

Such an atlas satisfies the strong cocycle condition, and is oriented. Further, the corresponding category $\mathbf{B}_{\mathcal{K}}$ has realization Y. Although it is not a groupoid since the nonidentity maps are not invertible, it has a **groupoid completion** $\mathcal{G}_{\mathcal{K}}$, obtained by adding in the relevant inverses and composites. In fact, for every (not necessarily nested) pair I, J with $F_I \cap F_J = F_{I \cup J} \neq \emptyset$ the subset of $\mathrm{Mor}_{\mathcal{G}_{\mathcal{K}}}$ consisting of morphisms from U_I to U_J can be identified with $U_{I \cup J} \times \Gamma_{I \cap J}$ with source and target maps given by

$$(s \times t)\big(z, \gamma\big) \;=\; \Big(\big(I, \gamma^{-1} \rho_{I(I \cup J)}(z)\big)\,,\, \big(J, \rho_{J(I \cup J)}(z)\big)\Big).$$

To prove this, recall that when (as here) the category $\mathbf{B}_{\mathcal{K}}$ is tame the equivalence relation on $\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$ generated by the morphisms in $\mathbf{B}_{\mathcal{K}}$ simplifies drastically. Indeed, applying Lemma 2.2.5 to the intermediate category, we find that if $(I, x) \sim (J, y)$ then there is an element $\underline{z} \in \underline{U}_{I \cup J}$ such that

$$(I, \underline{x}) \preceq (I \cup J, \underline{z}) \succeq (J, \underline{y}).$$

Therefore $(I, x) \sim (J, y)$ implies that there is a triple $(z, \gamma_I, \gamma_J) \in U_{I \cup J} \times \Gamma_I \times \Gamma_J$ such that $x = \gamma_I^{-1} \rho_{I(I \cup J)}(z), y = \gamma_J^{-1} \rho_{J(I \cup J)}(z)$. This triple is not unique since z is not uniquely determined by the morphism: for each $\delta_I \in \Gamma_{I \smallsetminus J}$ and $\delta_J \in \Gamma_{J \smallsetminus I}$ the triples (z, γ_I, γ_J) and $(\delta_I \delta_J(z), \delta_I \gamma_I, \delta_J \gamma_J)$ give the same morphism. (This makes sense because $\delta_I \delta_J = \delta_J \delta_I$ and $\rho_{IJ}(\delta_J(z)) = \rho_{IJ}(z)$.) Thus one can quotient the product

$U_{I\cup J}\times\Gamma_I\times\Gamma_J$ by $\Gamma_{I\smallsetminus J}\times\Gamma_{J\smallsetminus I}$ as well as one of the copies of $\Gamma_{I\cap J}$. It follows that the space of morphisms is $U_{I\cup J}\times\Gamma_{I\cap J}$, as claimed above.

Proposition 6.1.3. *Every compact orbifold Y has an orbifold atlas $\mathcal{K}$ with trivial obstruction spaces whose associated groupoid $\mathcal{G}_{\mathcal{K}}$ is an orbifold structure on Y. Moreover, there is a bijective correspondence between commensurability classes of such Kuranishi atlases and Morita equivalence classes of ep groupoids.*

Proof. Let $\mathcal{G}$ be an ep groupoid with footprint map $f:\mathcal{G}\to Y$. Our first aim is to construct an atlas $\mathcal{K}$ on Y together with a functor $\mathcal{F}:\mathcal{B}_{\mathcal{K}}\to\mathcal{G}$ that covers the identity map on Y and hence extends to an equivalence from the groupoid completion $\mathcal{G}_{\mathcal{K}}$ to $\mathcal{G}$.

By Moerdijk [Mo02], each point in Y is the image of a group quotient that embeds into $\mathcal{G}$. Therefore since Y is compact we can find a finite set of basic charts $\mathbf{K}_i := \big(W_i,\Gamma_i,\psi_i\big)_{1\le i\le N}$ on Y whose footprints $(F_i)_{1\le i\le N}$ cover Y, together with embeddings

$$\sigma:\bigcup_i W_i\hookrightarrow \mathrm{Obj}_{\mathcal{G}},\quad \widetilde{\sigma}:\bigcup_i W_i\times\Gamma_i\hookrightarrow \mathrm{Mor}_{\mathcal{G}}$$

that are compatible in the sense that the following diagrams commute:

$$\begin{tikzcd} W_i\times\Gamma_i \arrow[r, "\widetilde{\sigma}_i"] \arrow[d, "s\times t"'] & \mathrm{Mor}_{\mathcal{G}} \arrow[d, "s\times t"'] \\ W_i\times W_i \arrow[r, "\sigma_i"] & \mathrm{Obj}_{\mathcal{G}}\times\mathrm{Obj}_{\mathcal{G}}, \end{tikzcd} \qquad \begin{tikzcd} W_i \arrow[r, "\sigma_i"] \arrow[d, "\psi_i"'] & \mathrm{Obj}_{\mathcal{G}} \arrow[d, "f"'] \\ Y \arrow[r, "\mathrm{id}"] & Y, \end{tikzcd}$$

We claim that there is a Kuranishi atlas $\mathcal{K}$ with these basic charts whose footprint maps ψ_I extend $f\circ\sigma:\bigcup_i W_i\to Y$.[26] To see this, we first consider the sum of two charts. Given $I:=\{i_0,i_1\}$ with $F_I\neq\emptyset$, order its elements so that $i_0<i_1$ and consider the set

$$W_I := W_{\{i_1,i_0\}} := \mathrm{Mor}_{\mathcal{G}}(\sigma(W_{i_0}),\sigma(W_{i_1})) := (s\times t)^{-1}\big(\sigma(W_{i_0})\times\sigma(W_{i_1})\big)$$

of morphisms in $\mathcal{G}$ from $\sigma(W_{i_0})$ to $\sigma(W_{i_1})$. Then W_I is the inverse image of an open subset of $\mathrm{Obj}_{\mathcal{G}}\times\mathrm{Obj}_{\mathcal{G}}$, hence open in $\mathrm{Mor}_{\mathcal{G}}$, and thus a smooth manifold. Since the points in $f^{-1}(F_I)\cap\sigma(W_{i_0})$ are identified with points in $f^{-1}(F_I)\cap\sigma(W_{i_1})$ by morphisms in $\mathcal{G}$, the restrictions of s,t to W_I have images

$$s(W_I)=f^{-1}(F_I)\cap\sigma(W_{i_0}),\quad t(W_I)=f^{-1}(F_I)\cap\sigma(W_{i_1}).$$

Moreover, for any $x\in s(W_I)$ and $\alpha\in\mathrm{Mor}_{\mathcal{G}}(x,y)\in W_I$, we have

$$s^{-1}(x)\cap W_I\cong \mathrm{Mor}_{\mathcal{G}}\big(t(\alpha),\sigma(W_{i_1})\big)\cong\Gamma_{i_1},$$

where the second isomorphism holds because by assumption $f\circ\sigma$ is the footprint map $\psi_i: W_i\mapsto {}^{W_i}/_{\Gamma_i}\cong F_i$. Rephrasing this in terms of the action of the group $\Gamma_I:=\Gamma_{i_1}\times\Gamma_{i_0}$ on $\alpha\in W_I$ by

$$(\gamma_{i_1},\gamma_{i_0})\cdot\alpha=\widetilde{\sigma}(\gamma_{i_1})\circ\alpha\circ\widetilde{\sigma}(\gamma_{i_0}^{-1}),$$

[26]Our construction is reminiscent of the “resolution” of an orbifold in [M07]. However, the two constructions have different aims: here we want to build a model for $Y=|\mathcal{G}|$ with simple structure, while there we wanted to find a corresponding branched manifold, i.e. to make all stabilizers trivial.

one finds that Γ_{i_1} acts freely on W_I and that the source map $s : W_I \to \sigma(W_{i_0})$ induces a diffeomorphism ${}^{W_I}\!/_{\Gamma_{i_1}} \to \sigma(W_{i_0}) \cap f^{-1}(F_I)$. Similarly, Γ_{i_0} acts freely, and the target map $t : W_I \to \sigma(W_{i_1})$ induces a diffeomorphism ${}^{W_I}\!/_{\Gamma_{i_0}} \to \sigma(W_{i_1}) \cap f^{-1}(F_I)$. Since the footprint map for the chart W_i factors out by the action of Γ_i, the same is true for this sum chart: in other words the footprint map $\psi_I : W_I \to Y, \alpha \mapsto f\big(s(\alpha)\big) = f\big(t(\alpha)\big)$ induces an homeomorphism ${}^{W_I}\!/_{\Gamma_I} \xrightarrow{\cong} F_I$. Therefore W_I satisfies all the requirements of a sum of two charts.

To define a sum chart for general $I \in \mathcal{I}_Y$, enumerate its elements as $i_0 < i_1 \cdots < i_k$, where $k+1 := |I| \geq 2$ and define W_I to be the set of composable k-tuples of morphisms $(\alpha_{i_k}, \cdots, \alpha_{i_1})$, where $(s \times t)(\alpha_{i_\ell}) \in \big(\sigma(W_{i_{\ell-1}}), \sigma(W_{i_\ell}\big)$. If $H := (i_1, \cdots, i_k)$, then W_I is the fiber product $W_H\ {}_s\!\times_t W_{i_1 i_0}$. Since the target map $t : W_{i_1 i_0} \to W_{i_1}$ is étale and so locally submersive, it follows by induction on $|I|$ that W_I is a smooth manifold. Moreover, it supports an action of Γ_I given by

$$\gamma \cdot (\alpha_{i_k}, \cdots, \alpha_{i_1}) = (\alpha_{i_k}, \cdots, \alpha_{i_{\ell+1}} \widetilde{\sigma}(\gamma)^{-1}, \widetilde{\sigma}(\gamma)\alpha_{i_\ell}, \cdots, \alpha_{i_1}), \quad \gamma \in \Gamma_{i_\ell}.$$

For any $H \subsetneq I$ the subgroup $\Gamma_{I \smallsetminus H}$ acts freely, and the quotient can be identified with W_H by means of the appropriate partial compositions and forgetful maps. If $I = (i_0, \cdots, i_k) \supset H = (i_{n_0}, \cdots, i_{n_\ell})$ then

$$\rho_{HI}(\alpha_{i_k}, \cdots, \alpha_{i_1}) = \begin{cases} (\alpha_{i_{n_\ell}} \circ \cdots \circ \alpha_{i_{n_{\ell-1}+1}}, \cdots, \alpha_{i_{n_2}} \circ \cdots \circ \alpha_{i_{n_1+1}}), & \text{if } \ell \geq 1 \\ s(\alpha_{i_n+1}) = t(\alpha_{i_n}) & \text{if } \ell = 0 \end{cases}$$

For example if $H = \{1, 3, 6\} \subset I = \{0, 1, 2, 3, 4, 5, 6, 7\}$ then

$$\rho_{HI} : (\alpha_7, \cdots, \alpha_1) = (\alpha_6 \circ \alpha_5 \circ \alpha_4, \alpha_3 \circ \alpha_2), \quad \rho_{\{3\},I} : (\alpha_7, \cdots, \alpha_1) = s(\alpha_4) = t(\alpha_3).$$

It is clear from this description that $\rho_{HJ} = \rho_{HI} \circ \rho_{IJ}$ whenever $H \subset I \subset J$. Further the footprint map $\psi_I : W_I \to Y$ can be written as

$$\psi\big((\alpha_{i_k}, \cdots, \alpha_{i_1})\big) = f\big(s(\alpha_{i_p})\big) = f\big(t(\alpha_{i_p})\big), \quad \forall\ 1 \leq p \leq k.$$

This defines the atlas $\mathcal{K}$.

We define the functor $\mathcal{F} : \mathbf{B}_{\mathcal{K}} \to \mathcal{G}$ on objects by

$$W_I \to \mathrm{Obj}_{\mathcal{G}}, \quad \begin{cases} x \mapsto \sigma(x), & \text{if } I = \{i_0\}, x \in W_{i_0}, \\ (\alpha_{i_k}, \cdots, \alpha_{i_1}) \mapsto t(\alpha_{i_k}) \in \sigma(W_{i_k}) & \text{if } |I| > 1. \end{cases}$$

Recall from Lemma 3.2.1 that the morphisms in $\mathbf{B}_{\mathcal{K}}$ are given by $\bigcup_{I \subset J} W_J \times \Gamma_I$ where

$$(I, J, y, \gamma) : \big(I, \gamma^{-1}\rho_{IJ}(y)\big) \mapsto (J, y).$$

If $i_k = j_\ell$ then we define $F : W_J \times \Gamma_I \to \mathrm{Mor}_{\mathcal{G}}$ to be given by the initial inclusion $\widetilde{\sigma}$. More precisely, we define

$$F\big((\alpha_{j_\ell}, \cdots, \alpha_{j_1}), (\gamma_{j_\ell}, \cdots, \gamma_{i_0})\big) = \widetilde{\sigma}(t(\alpha_{j_\ell}), \gamma_{j_\ell}) \in \mathrm{Mor}_{\mathcal{G}}\big(\widetilde{\sigma}(\gamma_{j_\ell}^{-1})\ t(\alpha_{j_\ell}), t(\alpha_{j_\ell})\big).$$

Similarly, if $i_k = j_p < j_\ell$ define

$$F\big((\alpha_{j_\ell}, \cdots, \alpha_{j_1}), (\gamma_{i_k}, \cdots, \gamma_{i_0})\big) = (\alpha_{j_\ell} \circ \cdots \circ \alpha_{j_{p+1}}) \in \mathrm{Mor}_{\mathcal{G}}\big(t(\alpha_{j_p}), t(\alpha_{j_\ell})\big).$$

It is immediate that F is a functor that extends to an equivalence from the groupoid extension $\mathcal{G}_{\mathcal{K}}$ of $\mathbf{B}_{\mathcal{K}}$ to $\mathcal{G}$.

This shows that every orbifold has a Kuranishi atlas of the required type. Any two atlases constructed in this way from the same groupoid are directly commensurate. More generally, suppose given equivalent groupoid structures $(\mathcal{G}, f), (\mathcal{G}', f')$ on Y, and construct atlases $\mathcal{K}, \mathcal{K}'$ as above with functors $F_{\mathcal{K}} : \mathcal{G}_{\mathcal{K}} \to \mathcal{G}, F_{\mathcal{K}'} : \mathcal{G}_{\mathcal{K}'} \to \mathcal{G}'$. By hypothesis there is a common refinement $F : (\mathcal{G}'', f'') \to (\mathcal{G}, f), F' : (\mathcal{G}'', f'') \to (\mathcal{G}', f')$. Construct an atlas $\mathcal{K}''$ and functor $F_{\mathcal{K}''} : \mathcal{G}_{\mathcal{K}''} \to \mathcal{G}''$ as above. Since commensurability is an equivalence relation, it suffices to check that $\mathcal{K}''$ is commensurate to $\mathcal{K}$ and $\mathcal{K}'$. By definition, the category $\mathcal{G}''$ is the pullback of $\mathcal{G}$ by a local diffeomorphism $F : \mathrm{Obj}_{\mathcal{G}''} \to \mathrm{Obj}_{\mathcal{G}}$. If $\mathcal{K}$ has basic charts with domains $(U_i)_{1\le i\le N}$, define the groupoid $\mathcal{G}'''$ to be the pullback of $\mathcal{G}$ by the local diffeomorphism

$$F \cup \sigma : \mathrm{Obj}_{\mathcal{G}''} \cup \textstyle\bigsqcup_{1\le i\le N} U_i \to \mathrm{Obj}_{\mathcal{G}}.$$

Since by construction $\mathrm{Mor}_{\mathcal{G}}(\sigma(U_i), \sigma(U_i)) \cong U_i \times \Gamma_i$, the same is true for the set of morphisms $\mathrm{Mor}_{\mathcal{G}'''}(U_i, U_i))$ in the pullback $\mathcal{G}'$". Hence $\mathcal{K}$ is isomorphic to the atlas obtained from $\mathcal{G}'''$ with basic charts $(U_i)_{1\le i\le N}$. Similarly $\mathcal{K}''$ is isomorphic to the atlas obtained from $\mathcal{G}'''$. Hence $\mathcal{K}$ and $\mathcal{K}''$ are directly commensurate. Therefore each orbifold gives rise to a unique commensurability class of atlases.

Conversely, we must show that if $\mathcal{K}, \mathcal{K}'$ are commensurate, the groupoids $\mathcal{G}_{\mathcal{K}}$ and $\mathcal{G}_{\mathcal{K}'}$ are equivalent. It suffices to consider the case when $\mathcal{K}, \mathcal{K}'$ are directly commensurate. But then they are contained in a common atlas $\mathcal{K}''$ that defines a groupoid $\mathcal{G}_{\mathcal{K}''}$ such that there are equivalences $\mathcal{G}_{\mathcal{K}} \to \mathcal{G}_{\mathcal{K}''}$ and $\mathcal{G}_{\mathcal{K}'} \to \mathcal{G}_{\mathcal{K}''}$. This completes the proof. $\square$

Remark 6.1.4. The relation of cobordism between atlases $\mathcal{K}^0, \mathcal{K}^1$ on X requires there to be an atlas $\mathcal{K}^{01}$ over $X \times [0,1]$ with prescribed isomorphisms between the restrictions of $\mathcal{K}^{01}$ to the collars $X \times [0, \varepsilon)$ and $X \times (1-\varepsilon, 1]$ and the product atlases $\mathcal{K}^0 \times [0, \varepsilon), \mathcal{K}^1 \times (1-\varepsilon, 1]$. However, there is no requirement on the interior charts (i.e. those whose footprint does not intersect $X \times \{0,1\}$) that they are in any way compatible with the product structure, i.e. the local action of the stabilizer group of a point need not decompose as a product. Hence even if the obstruction bundles are trivial so that the footprint maps ψ_I are defined over the whole of the domains U_I, it is not immediately clear that the relation of commensurability for atlases on X with trivial obstruction spaces is the same as the notion of cobordism over the product $X \times [0,1]$, though it could well be true Since we are using the cobordism relation simply for convenience, we will not pursue this question further here.

6.2. **Nontrivial obstruction bundles.** When calculating Gromov–Witten invariants one often starts with moduli spaces X that have nice geometric structure, though they are not regular. For example, X might be a manifold (or more generally orbifold) of solutions to the Cauchy–Riemann equations, such that the cokernels form a bundle over X. In this case the VFC should be the Euler class of the (orbi)bundle. We now explain some simple examples of this type, both in the abstract and as applied in the GW setting.

There are two possible ways of incorporating nontrivial obstruction bundles into our framework. We can trivialize the bundle either by adding a complementary bundle or by using local trivializations. The first method is simpler, but may not adapt well to more complicated situations. The second method abstracts the procedure used to construct GW atlases.

Method 1: We explain this method in the case when the isotropy is trivial. It generalizes to cases when the obstruction bundle is a global quotient. With more complicated isotropy, one would need more charts and so should use Method 2.

Lemma 6.2.1. *Suppose that $\pi_X : E \to X$ is a nontrivial k-dimensional oriented dimensional bundle over a compact $d+k$ dimensional oriented manifold X. Then there is an oriented Kuranishi atlas $\mathcal{K}$ on X whose VMC equals the Euler class $\chi(E) \in H_d(X)$.*

Proof. Choose an oriented complementary bundle $E^\perp \to X$ such that $E \oplus E^\perp \cong X \times \mathbb{R}^m$. Denote by $\widehat{\iota} : E^\perp \to \mathbb{R}^m$ the composite of the inclusion $E^\perp \to X \times \mathbb{R}^m$ with the projection. Then define an atlas $\mathcal{K}$ with a single chart

$$\mathbf{K} = \bigl(U = E^\perp, E = \mathbb{R}^m, s = \widehat{\iota}, \psi\bigr),$$

where ψ identifies the zero section of $E^\perp$ with X. It has no nontrivial coordinate changes. Any section $\nu : X \to E$ that is transverse to the zero section gives rise to a section $s + \iota_E \circ \nu : U \to \mathbb{R}^m$, where $\iota_E : E \to \mathbb{R}^m$ is the inclusion. Then $s + \iota_E \circ \nu(u) = 0$ only if $s(u) = 0 \in E^\perp$ and $\nu(u) = 0 \in E$. Hence $s + \iota_E \circ \nu \pitchfork 0$ and its zero set equals that of ν. $\square$

Remark 6.2.2. Suppose that $X = \overline{\mathcal{M}}_{0,k}(M, A, J)$ is a manifold consisting of equivalence classes of stable maps with domains of constant topological type. If the cokernels of the linearized Cauchy–Riemann operator of (5.1.17) form a bundle over X of constant rank, then one might be able to carry out this construction in the GW setting since, at least locally, one can always find a suitable embedding λ of $E^\perp$. However there might be a problem with finding a global stabilization for the domains of the curves. The next method is more local, and hence more adaptable.

Method 2: We begin with the case of trivial isotropy. The first step is to build an oriented additive Kuranishi atlas that models the nontrivial bundle $E \to X$. To this end, choose a finite open cover $(F_i)_{i=1,\dots,N}$ of X together with trivializations $\tau_i : E_0 \times F_i \to E|_{F_i}$. We will define an atlas with indexing set $\mathcal{I}_\mathcal{K} = \{I \subset \{1, \dots, N\} : F_I \neq \emptyset\}$, and basic charts

$$\mathbf{K}_i := \bigl(F_i, E_i := E_0, s_i = 0, \psi_i = \mathrm{id}\bigr). \tag{6.2.1}$$

The sum charts are: $\mathbf{K}_I := \bigl(U_I, E_I := \prod_{i\in I} E_i, s_I, \psi_I\bigr)$ where

$$U_I = \Bigl\{(\vec{e}, x) \in E_I \times F_I \mid \sum_{i\in I} \tau_i(e_i, x) = 0\Bigr\}, \quad s_I(\vec{e}, x) = \vec{e}, \quad \psi_I(\vec{0}, x) = x \in F_I. \tag{6.2.2}$$

The coordinate changes have domains $U_{IJ} := E_I \times F_J$ and are induced by the obvious inclusions

$$\widehat{\phi}_{IJ} : E_I \to E_J, (e_i)_{i\in I} \mapsto \big((e_i)_{i\in I}, (0_j)_{j\in J\smallsetminus I}\big), \quad \phi_{IJ}(\vec{e}, x) = \big(\widehat{\phi}_{IJ}(\vec{e}), x\big).$$

Thus each chart has dimension $\dim X - \dim E_0$. The cocycle condition is immediate. Moreover the index condition holds because the inclusion $\widehat{\phi}_{IJ} : E_I|_0 \to E_J|_0$ and $\widehat{\phi}_{IJ} : E_I \to E_J$ have isomorphic cokernels, where $E|_0 = \{\vec{e} \in E : \sum \vec{e} = 0\}$. Therefore this set $\mathcal{K} = \big(\mathbf{K}_I, \widehat{\Phi}_{IJ})_{I\subset J, I, J\in\mathcal{I}_{\mathcal{K}}}$ of charts and coordinate changes defines an additive atlas, which is tame by construction. Note the commutative diagram

$$\begin{tikzcd} \mathbf{E}_{\mathcal{K}} \arrow[r, "\pi_{\mathcal{K}}"] \arrow[d, "\mathrm{pr}"'] & |\mathbf{E}_{\mathcal{K}}| \arrow[r, "\tau"] \arrow[d, "\mathrm{pr}"'] & E \arrow[d, "\pi_X"'] \\ \mathbf{B}_{\mathcal{K}} \arrow[r, "\pi_{\mathcal{K}}"] & |\mathcal{K}| \arrow[r, "\pi"] & X, \end{tikzcd} \tag{6.2.3}$$

where $\tau : |\mathbf{E}_{\mathcal{K}}| \to E$ is induced by $\tau\big((e_i)_{i\in I}\big) = \sum_{i\in I} \tau_i(e_i)$ and $\pi : |\mathcal{K}| \to X$ by the projection $(\vec{e}, x) \mapsto x$.

We now show that $\mathcal{K}$ has a reduction $\mathcal{V}$ such that each section $\nu : X \to E$ that is transverse to the zero section lifts to to an admissible section of $\mathrm{pr} : \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}$.

Proposition 6.2.3. *Let $\pi_X : E \to X$ be a nontrivial bundle over a manifold X with atlas $\mathcal{K}$ as above. Let $(Z_I \sqsubset F_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ be any reduction of the footprint cover and $V_I := \big\{(\vec{e}, x) \in U_I \ |x \in Z_I\big\}$ the associated reduction of $\mathcal{K}$. Then any section $\nu^X : X \to E$ of π_X lifts to a functor $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ whose zero set can be identified with $(\nu^X)^{-1}(0)$. Moreover, if ν^X is transverse to 0, so is its lift. Therefore $[X]^{vir}_{\mathcal{K}} = \chi(E)$.*

Proof. The first step is to choose a smooth partition of unity $(\beta_i)_{i=1,\dots,N}$ on $\bigcup_I Z_I$ such that

$$x \in Z_J \Longrightarrow \sum_{i\in J} \beta_i(x) = 1. \tag{6.2.4}$$

For this, fix a metric on X, and define $\rho_i(x) := d(x, \bigcup_{i\notin J} \overline{Z}_J)$. For each x the set of I such that $x \in Z_I$ is totally ordered and so can be written as a chain $I_1^x \subsetneq I_2^x \subsetneq \cdots \subsetneq I_q^x$ for some $q \geq 1$. Therefore $\rho_j(x) > 0$ for $j \in I_1^x$. Hence the function $\beta_i(x) := \frac{1}{\sum_j \rho_j(x)}\rho_i(x)$ is well defined. Moreover (6.2.4) holds because

$$x \in Z_J \Rightarrow \beta_i(x) = 0, \forall i \notin J.$$

Next define $V_I := \big\{(\vec{e}, x) \in U_I \mid x \in Z_I\big\}$. These sets $(V_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ form a reduction of $\mathcal{K}$. Further, given a section $\nu^X : X \to E$ there is an associated functor $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ defined by

$$\nu_I(x) := \big(\nu_I^i(x)\big)_{i\in I} \ = \ \Big(\beta_i(x)\tau_i^{-1}\big(\nu^X(x)\big)\Big)_{i\in I} \in E_I.$$

These sections are compatible with the coordinate transformations and have the property that for each $x \in Z_I$ we have $\sum_i \tau_i(\nu_I^i(x)) = \nu^X(x)$. On the other hand, by definition of U_I and s_I the elements $\vec{e} = (e_i)_{i\in I}$ in $\operatorname{im} s_I(x) \subset E_I$ have the property that $\sum_i \tau_i(e_i,x) = 0$. Therefore $s_I|_{V_I} + \nu_I(x) = 0$ precisely if $\nu^X(x) = 0$. Further, the fact that ν^X is transverse to the zero section easily implies that $s_I|_{V_I} + \nu_I$ is also transverse to zero. Similarly, one can check that the orientation of $E \to X$ induces an orientation of $\mathcal{K}$ and that the induced orientations on the zero sets agree. Since the zero set is compact, this completes the proof. □

Remark 6.2.4. If X is a GW moduli space and E is isomorphic to the bundle of cokernels, then it is not hard to build a GW atlas with basic charts isomorphic to $\mathbf{K}_i$ as above. Note that the section s_i as geometrically defined in (5.1.21) is zero since there are no solutions of the equation $\overline{\partial}_J f = \lambda(\vec{e})|_{\mathrm{graph\,f}}$ with $\vec{e} \neq 0$. One can check further that the sum charts have the form described above. Hence the atlas constructed in Proposition 6.2.3 is isomorphic to a GW atlas in the sense of Definition 5.2.5.

The case with isotropy. Now suppose given an orbibundle $E \to X$ with fiber E_0. By Proposition 6.1.3 we may suppose that X has a Kuranishi atlas $\mathcal{K}_X$ with charts $(W_I, \Gamma_I, \psi_I^X)$ and footprint cover $(F_i)_{1\le i\le N}$. After refinement and possible adjustment of the groups Γ_i, we may assume that for each i the orbibundle $E|_{F_i}$ pulls back to a trivial bundle $(\psi_i^X)^*(E|_{F_i})$ on which Γ_i acts by a product action, and then choose Γ_i-equivariant trivializations

$$\tau_i : E_i \times W_i \stackrel{\cong}{\to} (\psi_i^X)^*(E|_{F_i}),$$

where we denote the fiber (which is isomorphic to E_0) by E_i to emphasize that it supports an action of Γ_i. We then define an atlas essentially as before, incorporating the groups in the natural way. Thus we define

$$\mathbf{K}_I := \bigl(U_I, \Gamma_I := \prod_{i\in I}\Gamma_i, E_I := \prod_{i\in I} E_i, s_I, \psi_I\bigr) \tag{6.2.5}$$

where

$$U_I = \Bigl\{(\vec{e},x) \in E_I \times W_I \;|\; \sum_{i\in I}\tau_i(e_i,x) = 0\Bigr\}, \quad s_I(\vec{e},x) = \vec{e}, \quad \psi_I(\vec{0},x) = \psi_I^X(x) \in F_I.$$

The coordinate changes have domains $\widetilde{U}_{IJ} := \widehat{\phi}_{IJ}(E_I) \times W_J \subset U_J$ and are induced by the obvious projections

$$\rho_{IJ} : \bigl((e_i)_{i\in I}, (0_j)_{j\in J\smallsetminus I}, x\bigr) \mapsto \bigl((e_i)_{i\in I}, \rho_{IJ}^X(x)\bigr) \in U_I.$$

As before, $\mathcal{K}$ is tame.

In general, the Euler class of an oriented orbibundle may be represented by the zero set of a multisection, which is a weighted branched manifold. As explained in §3.2, when dealing with atlases one can always use multisections with controlled branching, that are constructed as follows. Choose a reduction $\underline{Z_I} \sqsubset \underline{W_I}$ of the footprint cover (cf. Lemma 4.1.12), and define $Z_I := (\psi_I^X)^{-1}(\underline{Z_I})$. Consider a family of maps $\nu_I^X : Z_I \to E_0$ that are *not* Γ_I-equivariant, but that satisfy the compatibility condition: $\nu_J^X = \nu_I^X \circ \rho_{IJ}^X : Z_J \to E_0$, and define the multisection with branches $(\gamma\nu_J^X)_{\gamma\in\Gamma_J}$, each weighted by $\frac{1}{|\Gamma_J|}$. By the results in [M07], one can represent $\chi(E) \in H_d(X;\mathbb{Q})$ by the zero of such a multisection, provided that all branches are transverse to 0. The proof of

Proposition 6.2.3 now carries through to show that every multisection $\left(\nu_I^X : Z_I \to E_0\right)_I$ of this kind may be lifted via a partition of unity to a section $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ in the sense of Definition 3.2.9. Hence, as before, the VFC defined by the Kuranishi atlas $\mathcal{K}$ is the Euler class of $E \to X$. Further details are left to the interested reader.

Remark 6.2.5. If X is more complicated, for example a union of strata each of which has fixed dimension and cokernel bundle of constant rank, then one should first build local atlases that model each stratum separately, and then put them together via the gluing parameters. Suppose, for example, that X is a compact $2k$-dimensional manifold that contains a codimension 2 submanifold Y consisting of curves with one node, and that the cokernels have constant rank $2r$ so that they form bundles $E^Y \to Y$ and $E^X \to X \setminus Y$. The complex line bundle L_Y over Y formed by the gluing parameter at the node is the normal bundle to Y in X. Let $\mathcal{N}(Y) \subset X$ be a neighbourhood of Y that forms a disc bundle $\pi_Y : \mathcal{N}(Y) \to Y$. Notice that the restriction of E^X to $\mathcal{N}(Y)$ may not simply be the pullback $\pi_Y^*(E^Y)$ because when one glues with the family of parameters $a = \varepsilon e^{2\pi i \theta}, \theta \in S^1$, one of the components twists by $2\pi\theta$ relative to the other. In many situations the bundles E^X, E^Y have a natural complex structure that is preserved by the S^1 action in the fibers of $\partial\mathcal{N}(Y) \to Y$, and E^Y decomposes into a finite sum $\oplus_k E_k^Y$ so that $E^X|_{\mathcal{N}(Y)} = \oplus_k \pi_Y^*\left(E_k^Y \otimes_{\mathbb{C}} (L_Y)^{\otimes k}\right)$. One should then build the atlas over X in stages, with one atlas over a neighbourhood $\mathcal{N}_1(Y)$ of Y with obstruction bundle E^Y and gluing parameters in L_Y, another over $X \setminus \mathcal{N}_2(Y)$ with obstruction bundle E^X, and appropriate sum charts over the deleted neighbourhood $\mathcal{N}_1(Y) \setminus \mathcal{N}_2(Y)$. The interaction of the gluing parameters and the obstruction bundles will be seen in the structure of these sum charts.

6.3. S^1 **actions.** Finally we reprove a result from [M00].

Proposition 6.3.1. *Let $M = (S^2 \times M_1, \omega_0 \times \omega_1)$, and let $A = [S^2 \times pt] + B$, where $B \in H_2(M_1)$. Then*

$$\langle pt, c\rangle_{0,2,A} = 0, \quad \forall B \neq 0, c \in H_*(M_1). \tag{6.3.1}$$

This statement about 2-point Gromov–Witten invariants immediately implies that the Seidel element corresponding to the trivial loop in $\mathrm{Ham}(M_1, \omega_1)$ is the identity. (Cf. [M00] or [MS, Chapter 12.5] for information on the Seidel representation.) The key idea of the proof is that the manifold M supports an S^1 action that rotates the S^2 factor with fixed points $0, \infty$. If $B \neq 0$ and we choose $J = j \times J_1$ to be a product, then the elements in the top stratum of the moduli space $\overline{\mathcal{M}}_{0,2}(M, A, J)$ are simply graphs of non constant J_1-holomorphic maps to M_1. Therefore the action of S^1 on this stratum is nontrivial. Since we can place the constraints in the fixed fibers over 0 and ∞, it should be impossible to find isolated regular solutions of the equation. The difficulty with this argument is that the S^1 action does have fixed points on the compactified moduli space $\overline{\mathcal{M}}_{0,2}(M, A, J)$, and it is not clear what effect these might have on the invariant.

We begin by discussing the abstract situation.

Definition 6.3.2. *Suppose that S^1 acts on X. Then we say that a Kuranishi atlas on X supports an S^1 action if the following conditions hold:*

- *The action of S^1 on each domain U_I is smooth and commutes with the action of Γ_I;*
- *S^1 acts trivially on E_I;*
- *the maps s_I and ψ_I are S^1 equivariant.*
- *the subsets $U_{IJ} \subset U_I, \widetilde{U}_{IJ} \subset U_J$ are S^1-invariant and the covering map ρ_{IJ} : $\widetilde{U}_{IJ} \to U_{IJ}$ commutes with S^1-action.*

Further we say that the action has fixed points of codimension at least 2, if both the domains U_I and X have a codimension 2 stratum that is respected by the footprint maps and contains all fixed points of the action.

Lemma 6.3.3. *Let $\mathcal{K}$ be a a 0-dimensional Kuranishi atlas that supports an S^1 action that is compatible with the footprint maps to the S^1-space X and is trivial on the obstruction spaces as above. Suppose further that this action has fixed points of codimension at least 2. Then $[X]^{vir}_{\mathcal{K}} = 0$.*

Proof. First note that we may construct the taming to consist of S^1 invariant sets, because the main step in the construction is Lemma 2.3.5 which applies to any complete metric space and hence in particular to quotients such as U_I/S^1. For a similar reason, we may suppose that the reduction consists of S^1 invariant sets; cf. Lemma 4.1.12. It remains to construct ν_I inductively over I (by the method explained in Proposition 2.4.10) so that $s_I + \nu_I$ has no zeros. Because S^1 acts trivially on the obstruction spaces, we may assume that $\nu_I : V_I \to E_I$ factors through V_I/S^1, which is the quotient of a k-dimensional manifold by a smooth action of S^1, and hence a CW complex of dimension $k-1$. Since E_I has dimension k, we can extend any nonzero section that is defined on a closed subset of V_I/S^1 to a section that is nonzero everywhere. This completes the proof. □

Proof of Proposition 6.3.1. It remains to construct an appropriate Kuranishi atlas. This requires some care. To reduce the dimension to 0 we consider the cut down moduli space

$$X_c = \big\{[\Sigma, \mathbf{z}, f] \in \overline{\mathcal{M}}_{0,2}(A, S^2 \times M_1, j \times J_1) : f(z_0) \in \{0\} \times M, f(z_\infty) \in \{\infty\} \times Z_c\big\}$$

as in §5.3 [b], where Z_c is a manifold with $\dim Z_c + 2c_1(B) = \dim M_1$. We consider X_c to be a stratified space as in Remark 5.1.4. It supports an S^1 action that is free on the top stratum. Indeed, each stratum of X_c has exactly one component that is a graph over S^2, and the stratum contains a fixed point only if this component is the constant map. In order that S^1 act on each basic chart with free action on the top stratum, we must choose both the obstruction spaces E_i and the slicing conditions to be S^1 invariant. As far as the obstruction spaces go, this is easy since we can choose the linear map $\lambda_i : E_i \to \mathcal{C}^\infty\big(\mathrm{Hom}^{0,1}_J(\mathcal{C}|_\Delta \times S^2 \times M_1)$ of (5.1.12) to take values in $\mathcal{C}^\infty\big(\mathrm{Hom}^{0,1}_J(\mathcal{C}|_\Delta \times M_1)$. Note that these do suffice for regularity because all components in the fiber are spheres so that the trivial horizontal bundle does not contribute to the cokernel; cf. [MS, Proposition 6.7.9]. Further, we can use slicing manifolds Q_i of the form $\mathcal{U} \times Q_i'$, where $Q_i' \subset M_1$ has codimension 2 and $\mathcal{U} \subset S^2$ is open, to stabilize all fiberwise components of the domain Σ of $[\Sigma, f]$, and also the section component provided that this is not constant. Notice that if the section component is constant and if there is a bubble component in some fiber other than $0, \infty$, then the section component is stable, since we already have marked points at $0, \infty$, and it has at least one other nodal point. Hence the only case when we need to use a non S^1-invariant slicing manifold is when $[\Sigma, f]$ is a fixed point of the action, consisting of the graph of a constant function together with some bubbles in the fibers over $0, \infty$. The domains of these graph components can be stabilized by slicing with the fiber $Q_F := \{1\} \times M_1$. Even though Q_F is not itself S^1-invariant, we can build an S^1 invariant chart with center $[\Sigma, \mathbf{z}, f]$ using this slicing manifold as well as the invariant manifolds $\mathcal{U} \times Q_i'$, because, after renormalizing, the induced action of S^1 on the stabilized map $(\Sigma_{0,\mathbf{P}}, \mathbf{w}, \mathbf{z}, f)$ is trivial. Here the normalization $\mathbf{P}$ contains the three points $0, 1, \infty$ on the constant graph, where at least one of $0, \infty$ is a node (the other might be a marked point), while the point w_1 at 1 maps to Q_F. Since the center point of this chart is fixed by the S^1 action, it is possible to build the chart to be S^1 invariant. Thus all the basic charts can be constructed to support an S^1 action that is free on the top stratum. It follows that one can choose the domains of the sum charts to be S^1-invariant. As before there are no fixed points in the top stratum. Hence the result follows from Lemma 6.3.3. □

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References

[Bbk] N. Bourbaki, General Topology Ch 1–4, Elements of Mathematics, Springer, Berlin 1998.

[C] R. Castellano, in preparation.

[FO] K. Fukaya and K. Ono, Arnold conjecture and Gromov–Witten invariants, *Topology* **38** (1999), 933–1048.

[FOOO] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Intersection Theory, Anomaly and Obstruction, Parts I and II*, AMS/IP Studies in Advanced Mathematics, Amer. Math. Soc. and Internat. Press.

[FOOO12] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Technical detail on Kuranishi structure and Virtual Fundamental Chain*, arXiv:1209.4410.

[J12] D. Joyce, D-manifolds. d-orbifolds and derived differential geometry: a detailed summary, arxiv:1208.4948

[M00] D. McDuff, Quantum homology of Fibrations over S^2, *International Journal of Mathematics*, **11**, (2000), 665–721.

[M07] D. McDuff, Branched manifolds, groupoids, and multisections, *Journal of Symplectic Geometry* **4** (2007) 259–315.

[M14] D. McDuff, Lectures on Kuranishi atlases, videotaped at SCGP Workshop, March 2014.

Lecture 1: http://media.scgp.stonybrook.edu/presentations/20140317_McDuff.pdf
Lecture 2: http://media.scgp.stonybrook.edu/presentations/20140318_McDuff.pdf

[MS] D. McDuff and D.A. Salamon, *J-holomorphic curves and symplectic topology*, Colloquium Publications **52**, American Mathematical Society, Providence, RI, (2004), second edition (2012).

[MW12] D. McDuff and K. Wehrheim, Smooth Kuranishi atlases with trivial isotropy, arXiv:1208.1340

[MW14] D. McDuff and K. Wehrheim, Smooth Kuranishi atlases with isotropy, work in progress.

[MWss] D. McDuff and K. Wehrheim, Stratified smooth Kuranishi atlases, work in progress.

[Mo02] I. Moerdijk, Orbifolds as groupoids, an introduction. DG/0203100, in *Orbifolds in Mathematics and Physics*, ed Adem, *Contemp Math* **310**, AMS (2002), 205–222.

[P13] John Pardon, An algebraic approach to virtual fundamental cycles on moduli spaces of J-holomorphic curves, arxiv:1309.2370

[Y14] Dingyu Yang, The polyfold–Kuranishi correspondence I: a choice-independent theory of Kuranishi structures, arXiv:1402.7008