The d-orbifold programme. Lecture 4 of 5: D-manifolds and d-orbifolds with corners. Stratified manifolds and 'bubbling'

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For d-manifolds and d-orbifolds *with corners*: you can find a preliminary version in the book at http://people.maths.ox.ac.uk/∼joyce/dmanifolds.html, but I am rewriting this, for reasons I will explain.
'Stratified manifolds' are work in progress, no papers yet.

Plan of talk:

1. Introduction

In symplectic geometry, one considers moduli spaces $\mathcal{M}$ of J-holomorphic curves $u : \Sigma \to S$, for (S, ω) a symplectic manifold with almost complex structure J. For counting problems (Gromov–Witten, etc.) it is essential that $\mathcal{M}$ be compact. But if we consider only nonsingular Riemann surfaces Σ, we generally get noncompact $\mathcal{M}$. To get compactified moduli spaces $\bar{\mathcal{M}}$, we must include Riemann surfaces Σ with singularities (nodes).

Two closely related problems are moduli spaces of J-holomorphic curves with ends in Symplectic Field Theory including curves which 'stretch' along an infinite cylinder (we call this 'neck stretching'), and (simpler) moduli spaces of gradient flow lines in Morse homology, including 'broken flow lines'.

These singularities occur either in *real codimension one* (boundary nodes of Riemann surfaces with boundary; 'neck stretching' in SFT; 'broken flow lines' in Morse homology), or *complex codimension one* (interior nodes in Riemann surfaces).

For simplicity, this lecture will discuss only the real codimension one case. (One way to handle the complex codimension one case is to reduce it to the real codimension one case by doing a real blow up, replacing each singular node by an $\mathcal{S}^1$.)

This kind of real codimension one singularity is closely related to *boundaries* and *corners* of moduli spaces. For example, one would like the moduli space $\bar{\boldsymbol{\mathcal{M}}}$ of stable J-holomorphic curves in M with boundary in a Lagrangian L to be a d-orbifold with corners, where the boundary $\partial\bar{\boldsymbol{\mathcal{M}}}$ is the stratum with one node, and more generally the k-corners $C_k(\bar{\boldsymbol{\mathcal{M}}})$ is the stratum with k boundary nodes for $k = 0, 1, 2, \ldots$.

Modelling such moduli spaces $\bar{\mathcal{M}}$ analytically near $\Sigma \in \bar{\mathcal{M}}$ with nodes is rather messy (Hofer: the 'analytical chamber of horrors'), and apparently *not smooth*: the obvious constructions yield Kuranishi neighbourhoods (V, E, s) on $\bar{\mathcal{M}}$ in which the section $s : V \to E$ is not smooth normal to the nodal stratum $V_{\rm node} \subset V$, but only continuous. (Curiously, the algebraic geometry version does not suffer from this problem.) Non-smooth sections $s : V \to E$ would be *bad* in our C^∞-geometry approach.

In the polyfold picture, one deals with this using a *gluing profile* φ: basically, one changes the smooth structure on V along $V_{\rm node}$ in the normal directions, to get a new manifold $\tilde{V}$ with the same topological space as V, such that s is smooth w.r.t. the smooth structure on $\tilde{V}$. Roughly, $\tilde{V} = V \times_{r^2,[0,\infty),\varphi} [0, \infty)$, where $r : V \to [0, \infty)$ is the distance from $V_{\rm node}$, and $\varphi : [0, \infty) \to [0, \infty)$ is $\varphi(0) = 0$, $\varphi(x) = e^{-1/x}$, $x > 0$.

I will try to make the case that the reason (or one reason) why such 'bubbling' problems yield apparently non-smooth moduli spaces, is that *the conventional smooth structure on manifolds with boundary and corners is the wrong one for this problem*.
Instead, I want to introduce a new category of 'stratified manifolds', which are basically manifolds with corners, but with an exotic smooth structure at their boundary.
I hope there will be some kind of 'stratified analysis' explaining why stratified manifolds occur in such moduli problems. There appear to be close connections here with the work of Richard Melrose on analysis on manifolds with boundary and corners, but I don't understand them yet.
I will integrate stratified manifolds into my theory of d-orbifolds. But in fact I expect they are 'platform independent': one ought to be able to use the same ideas in Kuranishi spaces or polyfolds to better understand smooth structures around singular curves.

2. Manifolds with (generalized) corners

2.1. Two kinds of tangent bundle $TX, {}^bTX$

To explain stratified manifolds, I begin by discussing ordinary manifolds with corners. Manifolds with corners X are locally modelled on $[0,\infty)^k \times \mathbb{R}^{n-k}$, so we can choose local coordinates $(x_1,\ldots,x_n)$ with $x_1,\ldots,x_k \in [0,\infty)$ and $x_{k+1},\ldots,x_n \in \mathbb{R}$.
There are two notions of tangent bundle: the (*ordinary*) *tangent bundle* TX, which has local basis of sections $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}$, and the *b-tangent bundle* ${}^bTX$, with local basis of sections $x_1\frac{\partial}{\partial x_1},\ldots,x_k\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_{k+1}},\ldots,\frac{\partial}{\partial x_n}$. There is a natural projection $\pi : {}^bTX \to TX$ which is an isomorphism on the interior $X^\circ = X\setminus\partial X$, but not an isomorphism over ∂X.
The importance of b-tangent bundles is stressed by Richard Melrose (see e.g. arXiv:1107.3320).

Similarly, we have the (*ordinary*) *cotangent bundle* T^*X with local basis of sections $\mathrm{d}x_1,\ldots,\mathrm{d}x_n$, and the *b-cotangent bundle* ${}^bT^*X$ with local basis of sections $x_1^{-1}\mathrm{d}x_1,\ldots,x_k^{-1}\mathrm{d}x_k,\mathrm{d}x_{k+1},\ldots\mathrm{d}x_n$, and a natural projection $\pi : T^*X \to {}^bT^*X$, an isomorphism over X°.

Principle

When working with manifolds with corners, it is better to use the b-versions ${}^bTX, {}^bT^*X$ *than* TX, T^*X.

For example: on a manifold without boundary $C^\infty(TX)$ is the Lie algebra of the diffeomorphism group $\mathrm{Diff}(X)$.
On a manifold with corners, $C^\infty({}^bTX)$ is the vector fields on X tangent to each boundary and corner stratum, so it is the Lie algebra of the group $\mathrm{Diff}^c(X)$ of diffeomorphisms preserving the boundary and corner strata; but $C^\infty(TX)$ is not the Lie algebra of a sensible infinite-dimensional Lie group.

2.2. Smooth maps of manifolds with corners

What should we mean by a smooth map $f: X \to Y$ of manifolds with corners X, Y? Obviously (???), writing $f = (f_1, \ldots, f_n)$ in local coordinates $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_n)$ on X, Y, we want all derivatives of $f_i(x_1, \ldots, x_m)$ to exist for $i = 1, \ldots, n$. But should we also impose extra conditions on f over the boundary and corners of X, Y, e.g. do we want f to map $\partial^k X \to \partial^k Y$?

There are several different definitions of smooth maps of manifolds with corners, one of which is due to me (arXiv:0910.3518), and used in the current version of the d-manifolds book.

Unfortunately, for reasons I'll explain later, my definition is the wrong one for the 'representable 2-functor' approach. Instead we need a more general notion, which Melrose calls 'b-maps'.

For example, $f: [0,\infty)^2 \to [0,\infty)$, $f(x,y) = xy$ is a b-map, but not smooth in my sense. From now on, 'smooth' means 'b-map'.

2.3. Manifolds with generalized corners (g-corners)

In fact, to get categories in which suitable fibre products exist, we need to generalize our notion of manifold with corners.

A *monoid* is essentially an abelian group without inverses. For example, $\mathbb{N}, \mathbb{Z}$ under addition, and $[0,\infty)$ under multiplication, are monoids. Manifolds with corners are locally modelled on $[0,\infty)^k \times \mathbb{R}^{n-k}$, and we can identify $[0,\infty)^k \times \mathbb{R}^{n-k}$ with $\mathrm{Hom}_{\mathbf{Monoids}}(\mathbb{N}^k \times \mathbb{Z}^{n-k}, [0,\infty))$. Here $\mathbb{N}^k \times \mathbb{Z}^{n-k}$ is a 'toric monoid' (a class of monoids classifying affine toric varieties).

One can define n-dimensional *manifolds with generalized corners* (*g-corners*) to be locally modelled on $\mathrm{Hom}_{\mathbf{Monoids}}(P, [0,\infty))$ for P a rank n toric monoid. (Not yet published, but inspired by Kottke–Melrose arXiv:1107.3320, 'interior binomial varieties'.)

- Manifolds with corners, and with g-corners, form categories $\mathbf{Man^c} \subset \mathbf{Man^{gc}}$ with morphisms smooth maps (b-maps).
- B-tangent bundles ${}^bTX$ work nicely for manifolds with g-corners, but tangent bundles TX do not. (The rank of TX jumps at generalized corners, so TX is not a vector bundle.) This is an example of the Principle in §2.1.
- Mau–Wehrheim–Woodward study moduli spaces of *quilted holomorphic discs* which have non-simplicial corner structures. Their moduli spaces are in fact *manifolds with g-corners*, not manifolds with corners.

2.4. Interior maps, b-transverse fibre products

Let $f : X \to Y$ be a smooth map of manifolds with (g-)corners. Call f *interior* if $f(X^\circ) \subset Y^\circ$. That is, f does not map X into a boundary stratum of Y. Write $\mathbf{Man^c_i}, \mathbf{Man^{gc}_i}$ for the categories of manifolds with (g-)corners and interior maps.
If f is interior, there is a natural map ${}^b\mathrm{d}f : {}^bTX \to f^*({}^bTY)$.
We call interior $g : X \to Z$, $h : Y \to Z$ *b-transverse* if ${}^b\mathrm{d}g|_x \oplus {}^b\mathrm{d}h|_y : {}^bT_xX \oplus {}^bT_yY \to {}^bT_zZ$ is surjective for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z \in Z$.

Theorem (unpublished; compare Kottke–Melrose arXiv:1107.3320)

If $g : X \to Z$, $h : Y \to Z$ are b-transverse then a fibre product $X \times_{g,Z,h} Y$ exists in $\mathbf{Man^{gc}_i}$.

Note that fibre products exist in $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^c_i}$ only under complicated extra combinatorial conditions on g, h; it is only in $\mathbf{Man^{gc}_i}$ that fibre products exist under weak conditions.
I will also introduce d-manifolds and d-orbifolds with (g-)corners $\mathbf{dMan^c} \subset \mathbf{dMan^{gc}}$, $\mathbf{dOrb^c} \subset \mathbf{dOrb^{gc}}$. Fibre products by interior 1-morphisms exist under weak conditions ('bd-transversality') in the 2-categories $\mathbf{dMan^{gc}_i}, \mathbf{dOrb^{gc}_i}$ with interior 1-morphisms.

This is important for the 'representable 2-functor' approach. Recall from Lecture 3 that to define the moduli 2-functor $F : (\mathbf{dMan^{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$, we needed the fibre product $\mathbf{X} \times_{\pi,\mathbf{S},\mathbf{f}} \mathbf{T}$ to exist in $\mathbf{dMan}$ for $\pi : \mathbf{X} \to \mathbf{S}$ a submersion and $\mathbf{f} : \mathbf{T} \to \mathbf{S}$ arbitrary. To include corners, we need $\mathbf{X} \times_{\pi,\mathbf{S},\mathbf{f}} \mathbf{T}$ to exist in $\mathbf{dMan^{gc}_i}$ for all $\pi : \mathbf{X} \to \mathbf{S}$ a submersion and $\mathbf{f} : \mathbf{T} \to \mathbf{S}$ interior. This works in $\mathbf{dMan^{gc}_i}$, but not in $\mathbf{dMan^c}, \mathbf{dMan^{gc}}, \mathbf{dMan^c_i}$.

3. C^∞-rings and C^∞-schemes with corners

To define d-manifolds and d-orbifolds, we began with C^∞-rings and C^∞-schemes. To define d-manifolds and d-orbifolds with (generalized) corners, where should we start?
We defined C^∞-ring in Lectures 1-2 as a set $\mathfrak{C}$ with operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$.
Here is an alternative, equivalent definition: write **Euc** for the full subcategory of manifolds **Man** with objects the Euclidean spaces $\mathbb{R}^n$, $n = 0, 1, 2, \ldots$. Note that **Euc** is closed under products, as $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Then a *$C^\infty$-ring* may be defined to be a product-preserving functor $F : \mathbf{Euc} \to \mathbf{Sets}$. This relates to the previous definition by putting $\mathfrak{C} = F(\mathbb{R})$. As F is product-preserving, $F(\mathbb{R}^n) = \mathfrak{C}^n$. So if $f : \mathbb{R}^n \to \mathbb{R}$ is a morphism in **Euc**, then $\Phi_f := F(f) : \mathfrak{C}^n \to \mathfrak{C}$ is a morphism in **Sets**, that is, a map $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$. The identities on the Φ_f follow as F is a functor.

The definition of C^∞-ring with corners

We have categories $\mathbf{Man^c}, \mathbf{Man^c_i}$ of manifolds with corners, with morphisms smooth maps (b-maps). Define $\mathbf{Euc^c}, \mathbf{Euc^c_i}$ to be the full subcategories of $\mathbf{Man^c}, \mathbf{Man^c_i}$ with objects $[0,\infty)^k \times \mathbb{R}^{n-k}$, for $0 \leqslant k \leqslant n$. They are closed under products.

Define a *C^∞-ring with corners* to be a product-preserving functor $F : \mathbf{Euc^c} \to \mathbf{Sets}$. Since $\mathbf{Euc^c}$ is generated under products by $\mathbb{R}$ and $[0,\infty)$, we can give an equivalent definition of a C^∞-ring with corners in terms of sets $\mathfrak{C} = F(\mathbb{R})$ and $\mathfrak{C}_\partial = F([0,\infty))$, with operations $\Phi_f : \mathfrak{C}_\partial^k \times \mathfrak{C}^{n-k} \to \mathfrak{C}$ for all smooth $f : [0,\infty)^k \times \mathbb{R}^{n-k} \to \mathbb{R}$ and $\Psi_f : \mathfrak{C}_\partial^k \times \mathfrak{C}^{n-k} \to \mathfrak{C}_\partial$ for all smooth $f : [0,\infty)^k \times \mathbb{R}^{n-k} \to [0,\infty)$, satisfying many identities.

Similarly, define an *interior C^∞-ring with corners* to be a product-preserving functor $F : \mathbf{Euc^c_i} \to \mathbf{Sets}$.

Introduction
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C^∞-rings and C^∞-schemes with corners
Stratified manifolds
Using stratified (d-)manifolds to model bubbling

I propose to define 'things with corners' by replacing C^∞-rings by C^∞-rings with corners throughout our theory. So we define *C^∞-schemes with corners* $\mathbf{C^\infty Sch^c}$, into which manifolds with (g-)corners embed as full subcategories $\mathbf{Man^c}, \mathbf{Man^{gc}} \subset \mathbf{C^\infty Sch^c}$. Then we define 2-categories of *d-spaces with corners* $\mathbf{dSpa^c}$ and *d-stacks with corners* $\mathbf{dSta^c}$, special kinds of derived C^∞-schemes and C^∞-stacks with corners, and full 2-subcategories $\mathbf{dMan^c}, \mathbf{dMan^{gc}} \subset \mathbf{dSpa^c}$ of *d-manifolds with (g-)corners*, and $\mathbf{dOrb^c}, \mathbf{dOrb^{gc}} \subset \mathbf{dSta^c}$ of *d-orbifolds with (g-)corners.*

This new 'C^∞-geometry with corners' is similar to log geometry in algebraic geometry. I learnt a lot from Gillam and Molcho, 'Log differentiable spaces and manifolds with corners' (preprint, 2013).

Note: the current version of my book instead defines d-manifolds with corners using 1-morphisms $\mathbf{i_X} : \partial \mathbf{X} \to \mathbf{X}$ in $\mathbf{dSpa}$. This works, but the new approach is prettier and better for 'g-corners'.

4. Stratified manifolds
4.1. Motivation

Let us return to the definition of smooth map $f : X \to Y$ of manifolds with corners. Choosing connections ∇ on TX, TY, we can write it like this: a continuous map $f : X \to Y$ is smooth if $\nabla^k f$ exists as a continuous section of $\bigotimes^k T^*X \otimes f^*(TY)$ for all $k \geqslant 1$, and f satisfies additional conditions over $\partial^k X, \partial^l Y$ (e.g. f preserves stratifications into boundary strata).

However, *this violates our Principle in* §2.1: if possible we should work with ${}^bT^*X, {}^bTY$, not T^*X, TY.

So: let us make a provisional definition that a continuous map $f : X \to Y$ is *b-smooth* if ${}^b\nabla^k f$ exists as a continuous section of $\bigotimes^k {}^bT^*X \otimes f^*({}^bTY)$ for all $k \geqslant 1$.

How do b-smooth maps differ from smooth maps?

Example: b-smooth maps $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$

For example, let $[0,\infty) \times \mathbb{R}^n$ have coordinates $(r, x_1, \ldots, x_n)$ for $r \in [0,\infty)$ and $x_1, \ldots, x_n \in \mathbb{R}$. Then a function $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is b-smooth if:

- f is continuous;
- f is smooth on $(0,\infty) \times \mathbb{R}^n$ in the usual sense; and
- $r^a \frac{\partial}{\partial r^a} \frac{\partial^{b_1}}{\partial x_1^{b_1}} \cdots \frac{\partial^{b_n}}{\partial x_n^{b_n}} f(r, x_1, \ldots, x_n)$, considered as a smooth function on $(0,\infty) \times \mathbb{R}^n$, extends to a continuous function on $[0,\infty) \times \mathbb{R}^n$, for all $a, b_1, \ldots, b_n \geqslant 0$.

So, for example, $f(r, x_1, \ldots, x_n) = r^\alpha$ is a b-smooth function for all $\alpha > 0$. This is rather different to ordinary smooth functions on $[0,\infty) \times \mathbb{R}^n$ — no Taylor series at $r = 0$.

When doing analysis in 'bubbling' problems, it is typical to see estimates like

$$r^a \frac{\partial}{\partial r^a} \frac{\partial^{b_1}}{\partial x_1^{b_1}} \cdots \frac{\partial^{b_n}}{\partial x_n^{b_n}} f(r, x_1, \ldots, x_n) = O(r^\alpha) \qquad (1)$$

for some $\alpha > 0$ and all $a, b_1, \ldots, b_n \geqslant 0$, where $f(r, x_1, \ldots, x_n)$ is some function in the problem (e.g. f could be the Kuranishi section s in a Kuranishi neighbourhood (V, E, s)) and r is the 'neck length', or distance to the singular stratum, so that $r = 0$ is the bubbling locus. The 'non-smoothness' of moduli spaces $\bar{\mathcal{M}}$ at curves with nodes referred to in §1 happens because f satisfying (1) need not be smooth (in the conventional sense) at $r = 0$.

However, $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ satisfying (1) *is b-smooth*. This suggests that some form of b-smoothness for manifolds with corners may be better than smoothness for 'bubbling' problems.

4.2. Stratified manifolds — the rough idea

I intend to define a new category $\mathbf{Man^{st}}$ of *stratified manifolds*. The definition is close to that of manifolds with corners. First, one defines *b-smooth maps* $f : U \to V$ between open subsets $U \subseteq [0, \infty)^k \times \mathbb{R}^{m-k}$ and $V \subseteq [0, \infty)^l \times \mathbb{R}^{n-l}$. The definition is like that of b-map, except (roughly) that one uses $x^a \frac{\partial^a}{\partial x^a}$ rather than $\frac{\partial^a}{\partial x^a}$ for variables $x \in [0, \infty)$. I will also include $O(r^\alpha)$ decay conditions like in (1), rather than continuity of b-derivatives.

Then one defines an m-dimensional *stratified manifold* to be a Hausdorff, second countable topological space X with a maximal atlas of charts by open sets of $[0, \infty)^k \times \mathbb{R}^{m-k}$ whose transition functions are b-smooth with b-smooth inverses. A map $f : X \to Y$ of stratified manifolds is *b-smooth* if on charts it is modelled on b-smooth maps between $U \subseteq [0, \infty)^k \times \mathbb{R}^{m-k}$, $V \subseteq [0, \infty)^l \times \mathbb{R}^{n-l}$.

Relation with ordinary manifolds with corners

Since ordinary smooth maps $f : U \to V$ between open $U \subseteq [0,\infty)^k \times \mathbb{R}^{m-k}$ and $V \subseteq [0,\infty)^l \times \mathbb{R}^{n-l}$ are b-smooth (but not vice versa), there is a natural functor $F_{\mathbf{Man^c}}^{\mathbf{Man^{st}}} : \mathbf{Man^c} \to \mathbf{Man^{st}}$.
One can also go the other way. To a first approximation, if you choose a *gluing profile function* φ as in polyfold theory, you can define a smoothing functor $F_\varphi : \mathbf{Man^{st}} \to \mathbf{Man^c}$.
This only works provided we include suitable decay estimates in the definition of b-smooth function, like the $O(r^\alpha)$ term in (1).
Also, the smoothing functor F_φ does not smooth *all* b-smooth maps $f : X \to Y$, but only those satisfying an extra condition over $\partial^k X, \partial^l Y$ – call these *simple* maps. So F_φ is really a functor $F_\varphi : \mathbf{Man^{st}_{si}} \to \mathbf{Man^c}$, for $\mathbf{Man^{st}_{si}} \subset \mathbf{Man^{st}}$ the subcategory of stratified manifolds and simple b-smooth maps.

On smoothing functors and gluing profiles

For example, $f : [0,\infty)^2 \to [0,\infty)$, $f : (x,y) \mapsto xy$, is not simple, and is not smoothed by F_φ. For one choice of φ, we have $F_\varphi(f) : (x,y) \mapsto xy/(x+y)$, which is not smooth.
This is important, as it shows a limitation of the gluing profile approach, and that stratified manifolds may be useful; if F_φ was defined on all of $\mathbf{Man^{st}}$, we could apply F_φ implicitly from the beginning, work in $\mathbf{Man^c}$, and not bother with stratified manifolds.

Maps like $f : [0,\infty)^2 \to [0,\infty)$, $f : (x,y) \mapsto xy$ occur in *forgetful morphisms* $F_i : \overline{\mathcal{M}}_{k+1} \to \overline{\mathcal{M}}_k$ forgetting boundary marked points in moduli spaces of J-holomorphic curves with boundary in a Lagrangian. Thus I claim: if you use the polyfold gluing profile approach, *such forgetful morphisms will not be smooth*. But if you use a stratified manifolds approach, they should be b-smooth.

4.3. Stratified d-manifolds and d-orbifolds

I propose to include stratified manifolds in the d-manifolds and d-orbifolds picture from the beginning. We follow the same method as for d-manifolds and d-orbifolds with corners: define $\mathbf{Euc^{st}}$ to be the full subcategory of $\mathbf{Man^{st}}$ with objects $[0,\infty)^k \times \mathbb{R}^{n-k}$ for $0 \leqslant k \leqslant n$, define a *stratified C^∞-ring* to be a product-preserving functor $F : \mathbf{Euc^{st}} \to \mathbf{Sets}$, and then do everything over stratified C^∞-rings.

Is all this really worth the effort? One pay-off should be that forgetful morphisms $F_i : \bar{\mathcal{M}}_{k+1} \to \bar{\mathcal{M}}_k$ forgetting boundary marked points will be 1-morphisms of stratified d-orbifolds, whereas we could not define F_i as a morphism using ordinary d-orbifolds with corners and the gluing profile method.

However, *I am hoping for more*: I believe stratified manifolds may somehow simplify and improve the analysis. I don't yet understand how, but the work of Richard Melrose looks like the place to start.

5. Using stratified (d-)manifolds to model bubbling
5.1. Bubbling and representable 2-functors

Start with an elementary observation: once we have good theories of d-manifolds and d-orbifolds with corners, or stratified d-manifolds and d-orbifolds, it is actually *easy* to include bubbling, J-holomorphic curves with nodes, and so on, in the 'representable 2-functor' approach, at least up to a point. That is, we can define moduli 2-functors for moduli spaces of J-holomorphic curves including singular curves, in a natural and straightforward way.

Proving that these 2-functors are representable (if indeed they are) is likely to be more difficult. One reason for introducing stratified d-manifolds is that the naïve definition of the moduli 2-functor using d-manifolds with corners is probably not representable.

To define a moduli 2-functor $F : \mathbf{dMan^{c,aff}} \to \mathbf{Groupoids}$ or $F : \mathbf{dMan^{st,aff}} \to \mathbf{Groupoids}$ representing a moduli space of J-holomorphic curves, including curves with boundary nodes, we have to define the groupoids $F(\mathbf{S})$ of families $(\mathbf{X}, \boldsymbol{\pi}, \mathbf{u}, j)$ of J-holomorphic curves with nodes over a base $\mathbf{S}$ which is a d-manifold with corners, or stratified d-manifold.

The key point is this: the curves $\boldsymbol{\pi}^{-1}(s)$ in the family for $s \in \mathbf{S}$ should have k boundary nodes over points s in the codimension k boundary stratum $C_k(\mathbf{S})$ in $\mathbf{S}$.

The b-submersions $\boldsymbol{\pi} : \mathbf{X} \to \mathbf{S}$ include factors like $f : [0,\infty)^2 \to [0,\infty)$, $f(x,y) = xy$. (Note that this was the function we could not smooth using gluing profiles. Also, it is a Melrose 'b-map', but not smooth in my sense in arXiv:0910.3518. This is why I'm rewriting the corners material in my book.)

5.2. Example: 'broken flow lines' in Morse homology

Suppose $f : M \to \mathbb{R}$ is a Morse function on a manifold. Morse homology involves studying moduli spaces of gradient flow lines γ of f going between $p, q \in \mathrm{Crit}(f)$. A family of such flow lines γ_t, $t \in (0,\infty)$ can limit as $t \to 0$ to a union of two flow lines $p \to r$ and $r \to q$ for $r \in \mathrm{Crit}(f)$, a 'broken flow line'. How should we define a moduli functor $F : \mathbf{dMan^{c,aff}} \to \mathbf{Groupoids}$ for the moduli space of flow lines, including broken flow lines?

We need a notion of *family of flow lines over* $\mathbf{S}$ for $\mathbf{S} \in \mathbf{dMan^{c,aff}}$. For example, define a 2-manifold with corners X by

$$X = \{(x,y) \in [0,\infty)^2 : -1 \leqslant x - y \leqslant 1\},$$

and define $\pi : X \to [0,\infty)$ by $\pi(x,y) = xy$.

Think of $\pi : X \to [0,\infty)$ as a family of flow lines in M over the base $[0,\infty)$, where $\pi^{-1}(t)$ for $t > 0$ is a single flow-line isomorphic to $[0,1]$, and $\pi^{-1}(0)$ is a 'broken flow line'. Near the singularity, π is locally modelled on $xy : [0,\infty)^2 \to [0,\infty)$. Higher-dimensional situations such as neck-stretching and bubbling of boundary nodes can be modelled in a similar way.