TWO AND TWO MAKE FOUR

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If we have four objects—say A, B, C, D—there are just three ways of dividing them into pairs

$$(AB)(CD) \quad (AC)(BD) \quad (AD)(BC).$$

The salient property is that 3 is less than 4. This simple fact expresses something special about the number 4. For example if we take 6 objects there are 10 ways to divide them into triples; there are 35 ways to divide 8 objects into quadruples, 126 ways to divide 10 objects into quintuples and so on. We will discuss two famous applications of this special property of 4: one going back five centuries and one underlying important concepts in contemporary differential geometry and physics.

THE QUARTIC EQUATION

We are all familiar with the quadratic formula. The solutions of an equation $at^2 + bt = c = 0$ are

$$\frac{-b/2 \pm \sqrt{b^2 - 4ac}}{2a}.$$

The solution of quadratic equations (but not in such algebraic notation) goes back many millenia. The ancient Greeks studied conic curves, the curves obtained by intersecting a plane with a cone or cylinder. These are ellipses, hyperbolae and parabolas, with some exceptional, "singular" cases when we get one or two straight lines. Algebraically, in terms of co-ordinates (x, y) on the plane, a conic is given by an equation $Q(x, y) = 0$ where

$$Q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F,$$

for constants $A, \ldots, F$. The problem of finding the intersection of a conic with a given line becomes a quadratic equation in a single variable t, which is solved by the quadratic formula.

What about equations of higher degree, involving higher powers of t: can we find formulae for the solutions? We move on to the beginning of the modern era in mathematics and the successful solution of cubic equations in 16th century Bologna. For our purposes here, let us just say that there is a formula to solve any cubic equation (that is, involving at most the power t^3). Adjusting t by the addition of a constant, one can reduce to the case of an equation $t^3 - pt - q = 0$ and the formula is

$$\sqrt[3]{q/2 + \sqrt{q^2/4 - p^3/27}} + \sqrt[3]{q/2 - \sqrt{q^2/4 - p^3/27}}.$$

We want to focus on the next case: quartic equations, involving t^4. Remarkably, the solution of quartic equations came hard on the heels of the cubics. (The solutions were both published in a book Ars Magna by Cardano, in 1545, the quartic case being due to Ludovico Ferrari.) This came about because the quartic equation can be reduced to the cubic case. In general, an equation of degree n has n solutions or "roots" (some of which may coincide). To understand this properly one needs to use complex numbers but we do not need to go into that, so we just assume that the roots exist as some kind of numbers and write the four roots of a quartic as t_1, t_2, t_3, t_4. Then we can form three expressions by dividing these four roots into pairs

$$\lambda_1 = (t_1 + t_2)(t_3 + t_4)$$
$$\lambda_2 = (t_1 + t_3)(t_2 + t_4)$$
$$\lambda_3 = (t_1 + t_4)(t_2 + t_3).$$

The key point now is that $\lambda_1, \lambda_2, \lambda_3$ are the roots of a cubic equation, with co-efficients that can be written down in terms of those of the quartic. We first solve this cubic equation to find the λ_i and then we can recover the roots $t_1, \ldots, t_4$ from those.

This can all be expressed more geometrically by setting up our quartic equation as the problem of finding the intersection of two conics. Suppose that our quartic equation is

$$at^4 + bt^3 + ct^2 + dt + e = 0$$

Divide by t^2, so the equation is

$$at^2 + bt + c + dt^{-1} + et^{-2} = 0.$$

Now let $Q_0(x, y) = xy - 1$ so the conic $Q_0(x, y) = 0$ is a hyperbola, which we can parametrise by a variable t as $x = t, y = t^{-1}$. Take another conic $Q_1(x, y) = 0$ where

$$Q_1(x, y) = ax^2 + bx + c + dy + ey^2.$$

Then the roots of our quartic equation yield the intersection points of these two conics. Now for any fixed number λ the conic with equation $Q_0 + \lambda Q_1$ also passes through these same four intersection points: we have what is called in algebraic geometry a "pencil of conics" through the four points. In this pencil there are three singular conics: if the four intersection points are divided into pairs then we get two lines, one for each pair and these two lines form a singular conic. Algebraically, the condition for the conic defined by the equation $(Q_0 + \lambda Q_1) = 0$ to be singular is the vanishing of the determinant of a 3×3 matrix and this yields a cubic equation in λ with the three roots $\lambda_1, \lambda_2, \lambda_3$. So the procedure is to first find a root of this cubic; identify the lines making up the resulting singular conic, and finally get the solutions of the original problem as the intersections of the lines with the hyperbola.

One might have hoped to go on to find more and more complicated formulae solving equations of arbitrarily high degree, but that turns out to be impossible. (By this we mean formulae involving only the ordinary arithmetic operations and taking roots $\sqrt[r]{\ }$.) This is nowadays understood as a part of Galois Theory, initiated by Évariste Galois before his death in a duel in 1832. There is a profound connection between the solubility of equations and the symmetries between the different roots. For each n the permutation group Σ_n is the group of one-to-one mappings from a set of n elements to itself. Galois tells us that the solubility of a general equation of degree n is expressed in terms of the permutation groups Σ_n The solubility of an equation of degree n is equivalent to a special property of Σ_n, and this fails for $n \geq 5$, so there is no formula. From this point of view the solution of quartic equations $(n = 4)$ hinges on the existence of a group homomorphism from Σ_4 onto Σ_3. This just expresses the fact that if we permute the four objects A, B, C, D we permute the three decompositions into pairs. For $n > 4$ there is no similar homomorphism (all one can do is to map the group Σ_n to the group of order two by the "sign" of a permutation).

4-DIMENSIONAL SPACE

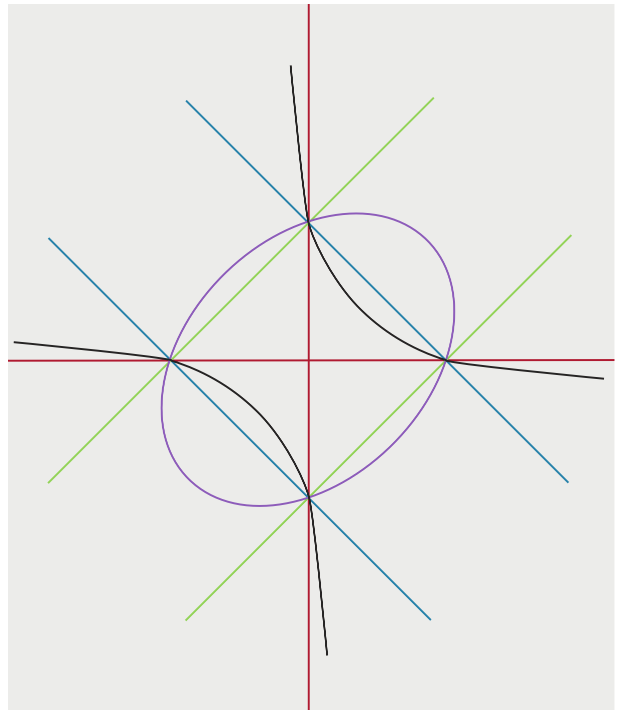
We change direction and consider spaces of dimension n rather than finite sets. Take $n = 2$ so we have the plane $\mathbf{R}^2$. The rotations of the plane (fixing the origin) form a group $SO(2)$ which can be identified with a circle (the angle of rotation). Similarly, the rotations of 3-dimensional space $\mathbf{R}^3$ are a group $SO(3)$ which is much more complicated. The rotations themselves form a 3-dimensional object, since we have to specify an axis of rotation (2 parameters) and an angle (1 parameter). In general for each dimension n we have a group $SO(n)$ of rotations of $\mathbf{R}^n$. The special fact about dimension 4 is that there is a homomorphism from $SO(4)$ to $SO(3)$–roughly analogous to the homomorphism $\Sigma_4 \to \Sigma_3$ of finite permutation groups that we discussed above. In fact there are two such homomorphisms (interchanged by changing the orientation of $\mathbf{R}^4$) which we can put together to make a map

$$\rho : SO(4) \to SO(3) \times SO(3),$$

and this is very close to being a 1-1 correspondence: it only fails to be so because for any $R \in SO(4)$ the negative $-R$ is also a rotation, and these map to the same pair in $SO(3) \times SO(3)$. So up to this small ambiguity of sign we could say that rotations of 4-space decompose into "right handed" and "left-handed" parts, each of which is a rotation of a 3-dimensional space.

This is a very special property of 4-dimensions: nothing like it happens in other dimensions. If we were 4-dimensional beings it would be wired into our brains–perhaps pure right (or left) handed rotation would be a prized achievement in 4-dimensional gymnastic competitions. (Of course, you might say that we are 4-dimensional beings because we live in 4-dimensional space-time. That involves Lorentzian, rather than Euclidean, geometry and works out differently—the special feature is that the Lorentz group in $3+1$ dimensions can be described by 2×2 matrices with complex numbers as entries. But we will not go into that here.)

One way to see the map ρ is by considering the space of 2-dimensional planes through the origin in $\mathbf{R}^4$, clearly analogous to the pairs in a



Intersection of conics

set of four objects. More precisely we want to take 2-dimensional planes with a given orientation. The set of these planes forms what is called the Grassmann manifold $\mathrm{Gr}_2(\mathbf{R}^4)$ and it is not hard to see that it is a 4-dimensional object. There is nothing special about 2 and 4 here–we can just as well consider k-dimensional subspaces of $\mathbf{R}^n$ for any k, n. What is special is that $\mathrm{Gr}_2(\mathbf{R}^4)$ is a product of two-dimensional spheres $S^2 \times S^2$. Now $SO(4)$ acts on $\mathrm{Gr}_2(\mathbf{R}^4)$—a rotation of the 4-dimensional space moves one plane to another one—and this action is just given by ordinary rotations of the two copies of S^2. So, if one accepts these assertions, one gets the map ρ. Pure left-handed rotations of $\mathbf{R}^4$ fix one copy of S^2 and pure right-handed rotations the other.

To get to the bottom of things we need some algebraic machinery. We will use "exterior" or "Grassmann" algebras introduced by the mathematician Hermann Grassmann in 1844. (Another route to the same end would be to discuss a different kind of algebra, the "quaternions" discovered by Hamilton about the same time.) Work in $\mathbf{R}^n$ with co-ordinates $x_1, \ldots x_n$. Then we have the algebra of polynomials (just as we considered in Section 1 for the case $n = 2$) which can be thought of as formal expressions generated by the x_i subject to the commutative condition $x_i x_j = x_j x_i$. For the exterior algebra we do the same but impose anti-commutativity. The product is denoted by the symbol $\wedge$ and we impose $x_i \wedge x_j = -x_j \wedge x_i$. Just as a polynomial can be written as a sum of terms of different degrees, so also for the elements of the exterior algebra: for each k we have a space $\Lambda^k \mathbf{R}^n$ of "exterior forms of degree k". The elements of $\Lambda^k(\mathbf{R}^n)$ can be expressed as linear combinations of a standard basis

$$x_{i_1} \wedge x_{i_2} \wedge \ldots x_{i_k},$$

corresponding to subsets $\{i_1, \ldots i_k\}$ of size k of $\{1, \ldots, n\}$. So the dimension of $\Lambda^k(\mathbf{R}^n)$ is the binomial co-efficient

$$\frac{n \times (n-1) \ldots \times (n-k+1)}{k \times (k-1) \ldots . \times 1},$$

the number of ways of choosing k objects out of n.

This exterior algebra is tailored to study the geometry of subspaces. Let P be a k-dimensional subspace of $\mathbf{R}^n$ and choose an orthonormal basis of vectors $v_1, \ldots v_k$ for P. Then the wedge product $v_1 \wedge \ldots v_k$ lies in $\Lambda^k \mathbf{R}^n$ and the algebra is designed so that changing the choice of basis changes this at most by a sign. The choice of an orientation of P fixes this sign ambiguity. The upshot is that we map the Grassmann manifold $\mathrm{Gr}_k(\mathbf{R}^n)$ into $\Lambda^k \mathbf{R}^n$.

If we have a k-dimensional subspace $P \subset \mathbf{R}^n$ the orthogonal complement of P is an $(n-k)$-dimensional subspace. The corresponding construction in exterior algebra is a map

$$* : \Lambda^k \mathbf{R}^n \to \Lambda^{n-k} \mathbf{R}^n.$$

Up to a sign this takes a basis element $x_{i_1} \wedge \ldots x_{i_k}$ to $x_{j_1} \wedge \ldots \wedge x_{j_{n-k}}$ where $\{j_1, \ldots j_{n-k}\}$ is the complement of $\{i_i, \ldots, i_k\}$ in $\{1, \ldots, n\}$. Now we are almost done. If we take $n = 4$ and $k = 2$ the map $*$ takes $\Lambda^2 \mathbf{R}^4$ to itself and the reader can check that $**$ is the identity. This means that $\Lambda^2 \mathbf{R}^4$ is the sum of two pieces, called the self-dual and anti self-dual pieces, on which $*$ is ± 1 respectively

$$\Lambda^2 \mathbf{R}^4 = \Lambda^2_+ \oplus \Lambda^2_-.$$

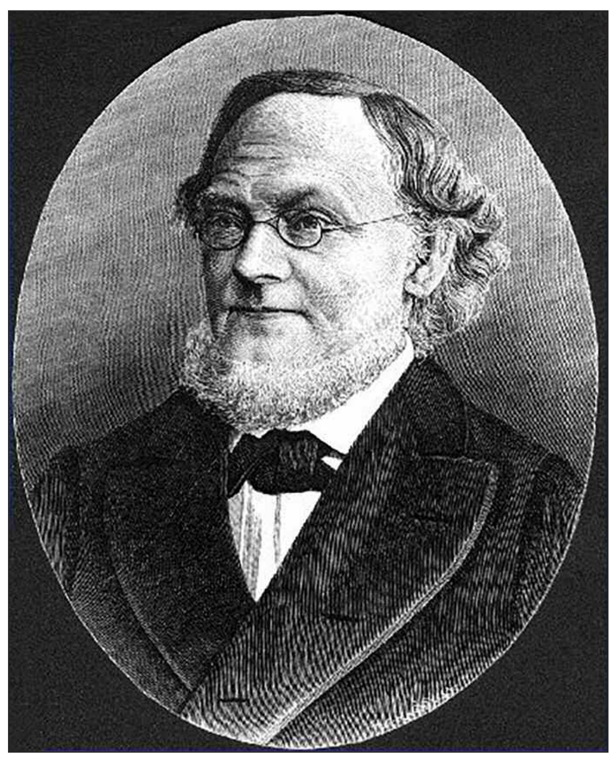
These two pieces are each 3-dimensional. Explicitly, a basis of λ^2_+ is given by

$$x_1 \wedge x_2 + x_3 \wedge x_4 \ , \ x_1 \wedge x_3 + x_4 \wedge x_2 \ , \ x_1 \wedge x_4 + x_2 \wedge x_3,$$

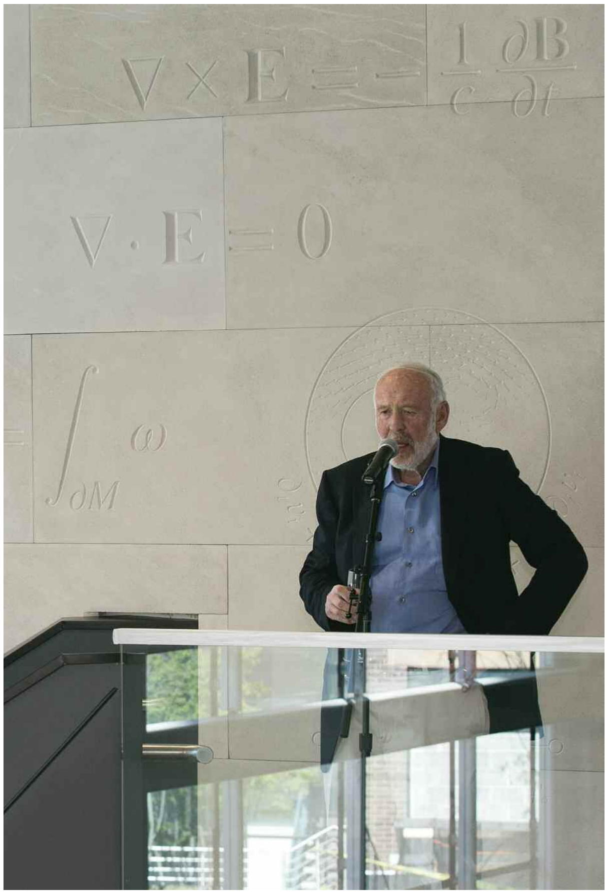
and of Λ^2_- by

$$x_1 \wedge x_2 - x_3 \wedge x_4 \ , \ x_1 \wedge x_3 - x_4 \wedge x_2 \ , \ x_1 \wedge x_4 - x_2 \wedge x_3;$$

these corresponding to the three ways of dividing $\{1, 2, 3, 4\}$ into pairs. Now we see the map ρ: it is given by the action of a rotation of $\mathbf{R}^4$ on these two 3-dimensional spaces $\Lambda^2_\pm$.



Hermann Grassmann



Jim Simons (left) speaks at the Dedication of the Iconic Wall, standing before Maxwell's Equations. May 8, 2015.
Photo Stony Brook University

If we write an element $\omega \in \Lambda^2\mathbf{R}^4$ as a sum of self-dual and anti-self-dual pieces $\omega = \omega_+ + \omega_-$ then one finds that the image of the Grassmann manifold is given by the condition that these two pieces have length 1

$$|\omega_+| = |\omega_-| = 1,$$

which exhibits $\mathrm{Gr}_2(\mathbf{R}^4)$ as a product of a pair of 2-spheres.

This decomposition of the exterior 2-forms is important in differential geometry and mathematical physics. If we single out one "time direction" $x_4 = t$ in $\mathbf{R}^4$ then an exterior 2-form can be written in terms of a pair of vectors $\mathbf{E}, \mathbf{B}$ in 3-space as

$$B_1(x_2 \wedge x_3) + B_2(x_3 \wedge x_1) + B_3(x_1 \wedge x_2) + \\ + E_1(t \wedge x_1) + E_2(t \wedge x_2) + E_3(t \wedge x_3).$$

In a relativistic treatment of electromagnetism, the electromagnetic field at each point is an exterior 2-form on space-time. Then when a time direction is singled out the vectors $\mathbf{E}, \mathbf{B}$ are the usual electric and magnetic fields. The $*$-operation interchanges $\mathbf{E}, \mathbf{B}$ and can be seen as a symmetry between electricity and magnetism, particular to 4 dimensions. This is not precisely a symmetry on Lorentzian space-time; it does not exactly preserve Maxwell's equations because some signs change. But in the Euclidean version of Maxwell's theory the symmetry becomes exact. The same holds for the generalisations of Maxwell's equations to the Yang-Mills equations of modern particle physics. One can define special "instanton" solutions of the Yang-Mills equations which exploit this particular feature of dimension 4, and various relatives such as the Seiberg-Witten equations.

Another mathematical area where dimension 4 stands out is topology, more precisely the differential topology of 4-dimensional manifolds. There are many fundamental questions one can pose about n-dimensional manifolds which are broadly understood for n not equal to 4 but are completely out of reach in this special dimension. But many surprising phenomena have been detected–for example infinite families of inequivalent manifolds which look identical from the point of view of classical topological tools. These manifolds are distinguished by certain topologically-invariant properties of the instanton solutions to the Yang-Mills or Seiberg-Witten equations defined on them. In some mysterious way, which we only have glimpses of, the special geometry of the rotations in four dimensions, of the 2-planes and exterior forms is bound up with these topological phenomena. **SC**