Tensor-Network Methods: Structure, Applications and Holography: MERA exercises

Given the Pauli matrices,

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \; Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{1}$$

we define the (3-body) Hamiltonian H_1 as,

$$\begin{aligned} H_1 &= \sum_r h_1(r, r+1, r+2) \\ &= \sum_r (X(r)Z(r+1)X(r+2) - X(r)X(r+1)). \end{aligned} \tag{2}$$

Note that H is equivalent to the critical Ising model (transformed through a shallow unitary circuit). A scale invariant MERA that approximates the ground state of H_1 is defined from the following pair of unitary gates,

$$\begin{aligned} w &= \tfrac{\sqrt{3}+\sqrt{2}}{4} II + \tfrac{\sqrt{3}-\sqrt{2}}{4} ZZ + \tfrac{i(1+\sqrt{2})}{4} XY + \tfrac{i(1-\sqrt{2})}{4} YX \\ u &= \tfrac{\sqrt{3}+2}{4} II + \tfrac{\sqrt{3}-2}{4} ZZ + \tfrac{i}{4} XY + \tfrac{i}{4} YX, \end{aligned} \tag{3}$$

as depicted in Fig. 1. The convention in use here is such that the Hamiltonian transforms as $W^\dagger U^\dagger (H) UW$.

Exercise 1a: Coarse-grain the Hamiltonian through one layer of the MERA using the ascending superoperators (see Fig. 2) defined from the tensors in Eq. 3, as to obtain

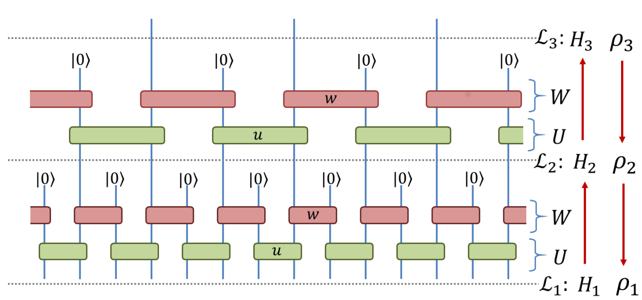

FIG. 1. Depiction of a binary scale-invariant MERA.

$h_2 = \mathcal{A}_L(h_1) + \mathcal{A}_R(h_1)$. Form a periodic Hamiltonian on three sites of $\mathcal{L}_2$ from the coarse-grained coupling h_2 and its two periodic translations. What is the ground energy per site $E6_{\text{grnd}}$ (counting relative to the original sites in $\mathcal{L}_0$) and energy gap $\Delta E6$?

[**Ans:** $E6_{\text{grnd}} = -1.26469122$, $\Delta E6 = 0.27007439$]

Exercise 1b: Repeat the calculations of ground energy and gap after a second and third coarse-graining of the Hamiltonian (i.e. forming periodic Hamiltonians on three sites of $\mathcal{L}_3$ and $\mathcal{L}_4$ using h_3 and h_4 respectively). You should see that the ground energy density tends to a constant (within few percent of the exact energy density $E_{\text{ex}} = -4/\pi \approx -1.2732$) and that the energy gap closes linearly with system size.

[**Ans:** $E12_{\text{grnd}} = -1.2480926901$, $\Delta E12 = 0.1297782430$, $E24_{\text{grnd}} = -1.243754362$, $\Delta E24 = 0.0629028140$]

Exercise 2: Form a periodic Hamiltonian on three sites of $\mathcal{L}_4$ using h_4 (as obtained in the previous exercise) and diagonlise for its ground state, which can be formed into a density matrix ρ_4. Use the descending superoperators (see Fig. 3) to compute the average (3-site) reduced density matrices $\rho_4 \to \rho_3 \to \rho_2 \to \rho_1$. Evaluate the energy density $E = \text{tr}\,(\rho_1 h_1)$ and check that it exactly matches the value $E24_{\text{grnd}}$ from the previous exercise. What is the average magnetization $\langle Z \rangle$ of the 24 site system?

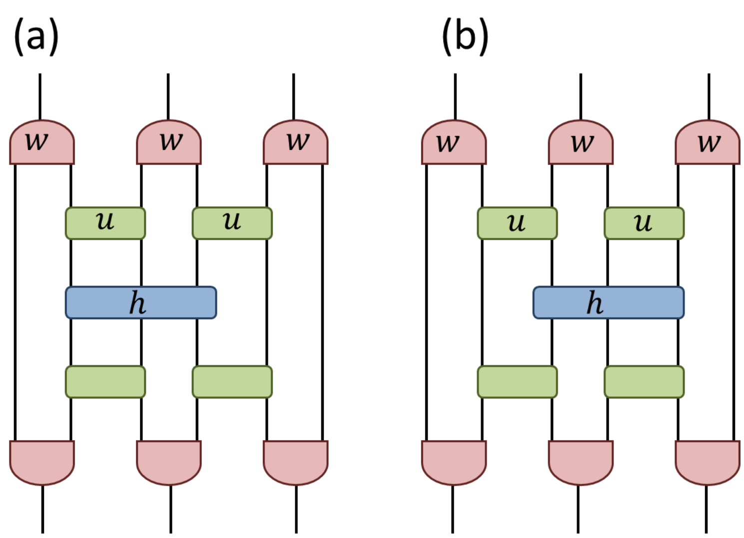

FIG. 2. The binary MERA ascending superoperators, with (a) depicting $\mathcal{A}_L$ and (b) depicting $\mathcal{A}_R$.

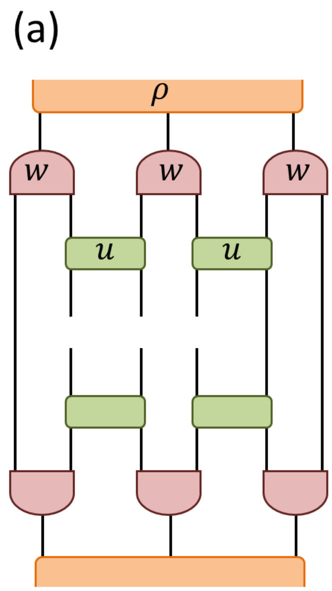

FIG. 3. MERA descending superoperators, with (a) depicting $\mathcal{D}_L$ and (b) depicting $\mathcal{D}_R$.

[**Ans:** $\langle Z \rangle = -0.18623583$]

Exercise 3: Using the Hamiltonian coupling h_1 and the density ρ_2 compute the environment Γ_u of a disentangler (see Fig. 4) from the lowest MERA layer, then update this u using the SVD approach. Repeat for 50 iterations. What is the new ground energy $\tilde{E}24_{\mathrm{grnd}}$ with this updated disentangler?

[**Ans:** $\tilde{E}24_{\mathrm{grnd}} = -1.256168190831$]

Exercise 4: Using the Hamiltonian coupling h_1 and the density ρ_2 compute the environment Γ_w of an isometry (see Fig. 5) from the lowest MERA layer, then update this w using the SVD approach. Repeat for 50 iterations. What is the new ground energy $\tilde{E}24_{\mathrm{grnd}}$ with this updated isometry (and disentangler from the previous exercise)?

[**Ans:** $\tilde{E}24_{\mathrm{grnd}} = -1.2596107857$]

Exercise 5: Combine everything from the previous exercises to build a full variational MERA algorithm. What is the optimised ground energy $\tilde{E}24_{\mathrm{grnd}}$ for a $\chi = 2$ MERA of three layers (i.e. equal to 24 spins) for the Hamiltonian H_1 of Eq. 2?

[**Ans:** $\tilde{E}24_{\mathrm{grnd}} = -1.2704707880$]

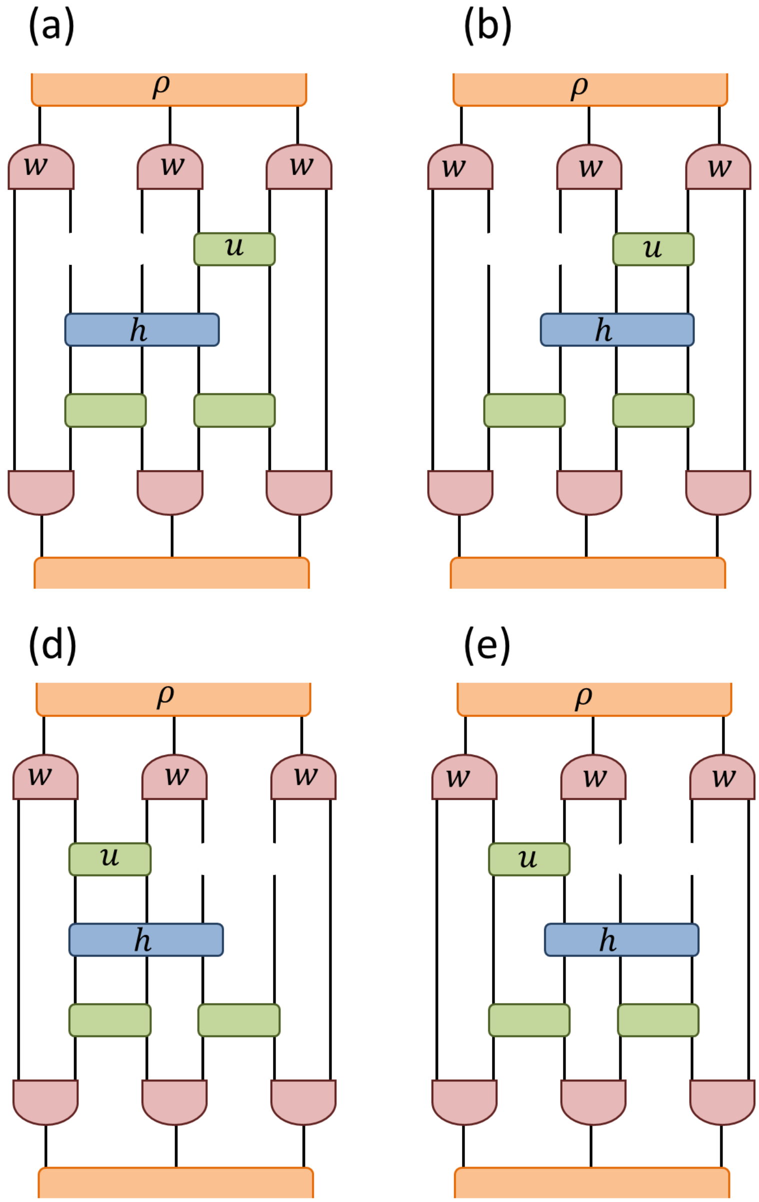

FIG. 4. Environments of a disentangler in a binary MERA.

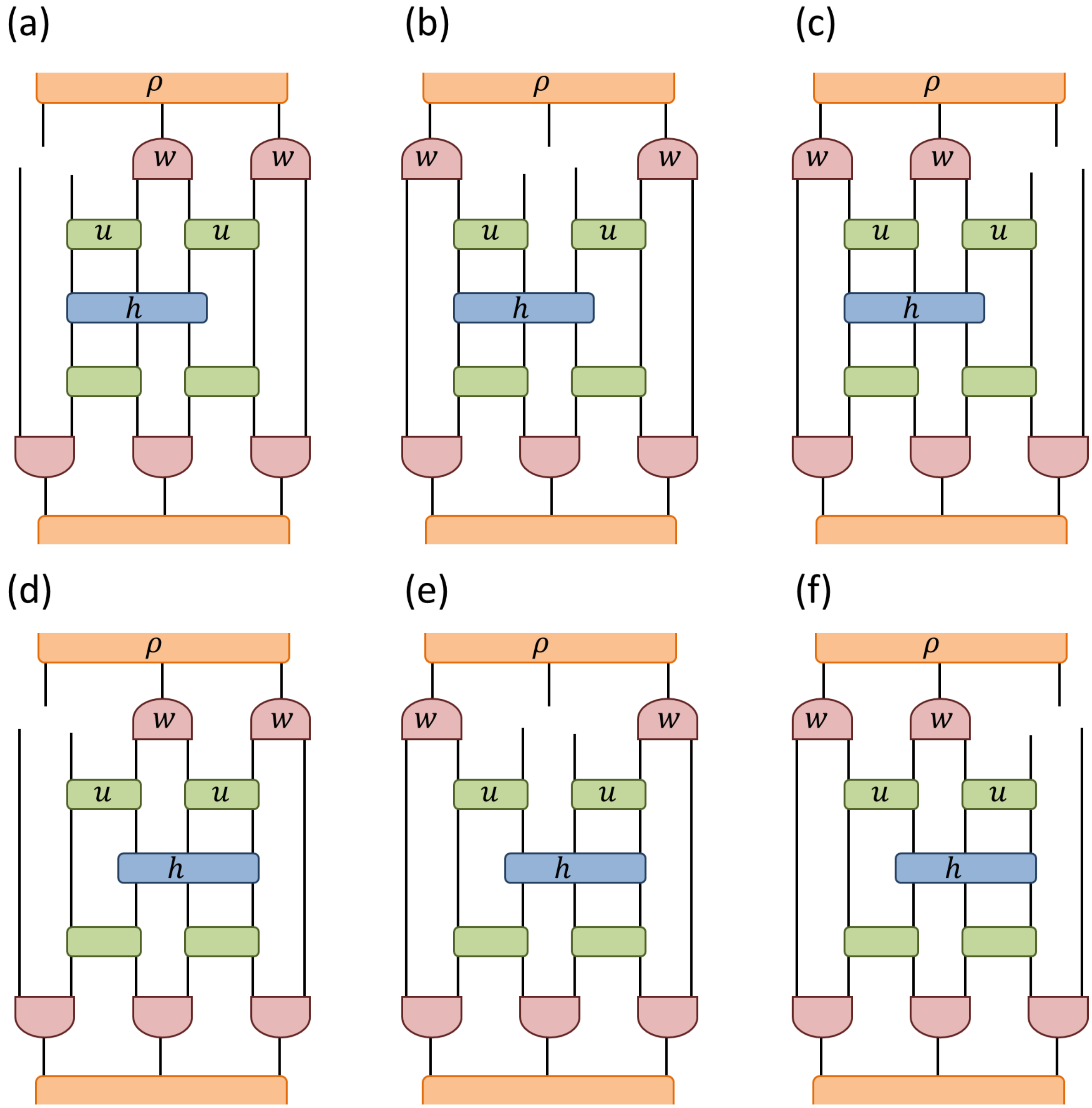

FIG. 5. Environments of an isometry in a binary MERA.