Equivariant Cohomology: A Primer

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1 The Borel Construction

Let G be a discrete group and let $G \times X \to X$ be an action. The *Borel construction* of the equivariant cohomology $H_G^*(X)$ goes as follows. Let EG be a CW complex on which G acts (from the left) freely and discretely. (This means that for every point $x \in EG$ there is a neighborhood $x \in U_x \subset EG$ such that if $g \in G$ has the property that $gU_x \cap U_x \neq \emptyset$ then g is the identity element of the group.) Then we form $EG \times X$ with the group action being the product of the given actions of G on EG and on X; i.e., $g(z, x) = (gz, gx)$. This is a free and discrete action. Let $X_G = G\backslash(EG \times X)$. By definition $H_G^*(X) = H^*(X_G)$. In order for this to be a reasonable definition two things have to be checked:

1. There is a contractible CW complex EG with a free and discrete action of G.
2. Given two such EG and EG' there is a G-equivariant map $EG \to EG'$.

For any such EG the quotient space $BG = G\backslash EG$ has fundamental group G and trivial higher homotopy groups (since $EG \to G\backslash EG$ is the universal covering). The equivariant map in Item 2 induces a map $G\backslash EG \to G\backslash EG'$ which induces an isomorphism on the homotopy groups and hence an isomorphism on the cohomology groups. If we have a G-action on X, then $G\backslash(EG \times X) \to G\backslash EG$ is a locally trivial fibration with fiber X. From the homotopy long exact sequence we see that if EG and EG' are given as in Item 2, then the map $G\backslash(EG \times X) \to G\backslash(EG' \times X)$ (induced by the map $EG \to EG'$ given in Item 2 and the identity on X) is an isomorphism on the homotopy groups and hence on the cohomology groups. This shows that these two properties determine the equivariant cohomology up to canonical isomorphism.

Proposition 1.1. *If G is a discrete group acting freely and properly discontinuously on X then $H^*_G(X) = H^*(G\backslash X)$.*

Proof. If the action of G on X is free and discrete then the projection onto the second factor $EG \times X \to X$ induces a locally trivial fibration $\pi\colon G\backslash(EG\times X) \to G\backslash X$. The fiber of this fibration is EG, which, recall, is contractible. Hence, π induces an isomorphism on the cohomology: $\pi^*\colon H^*(G\backslash X) \to H^*(X_G)$. □

In general the projection to the second factor does not have good properties when we divide out by G but the projection to the first factor always does. It induces a map $G\backslash(EG \times X) \to BG$ which is a locally trivial fibration with with fiber X. Hence the Serre spectral sequence has E_2-term $H^*(BG; H^*(X))$ and converges to $H^*_G(X)$, Of course, in the E_2-term the coefficients are the locally trivial system induced by the action of $\pi_1(BG) = G$ on $H^*(X)$.

1.1 Milnor's construction of EG

Recall that if X and Y are spaces, the *join* $X * Y$ is obtained from $X \times I \times Y$ by collapsing $X \times \{0\} \times Y$ to X and collapsing $X \times \{1\} \times Y$ to Y. Thus, X and Y are the subspaces over the endpoints and the join lines consists of intervals connecting a point of X to a point of Y with its zero-end identified with the point of X and its one-end identified with the point of Y and with the interiors of the intervals disjoint and the topology being the quotient topology from the product. Notice that fixing a point $y_0 \in Y$ the joint of X with y_0 is naturally a subspace of $X * Y$ isomorphic to the cone over X whose boundary is the natrual inclusionof $X \subset X * Y$. This shows that the inclusion $X \subset X * Y$ is homotopic to a point map of X to $X * Y$.

We have already seen one example of EG: for $G = \mathbb{Z}/2\mathbb{Z}$ we take $EG = S^\infty$ with the action being the antipodal action. Let me describe this in a way that generalizes to the Milnor construction. Start with G which we identify with $S^0 = \{\pm 1\} \subset \mathbb{R}^1$. There is a natural isomorphism of S^1 with the join of G with itself, denoted $G * G$. More generally, $\underbrace{G * G * \cdots * G}_{k-\text{times}}$ is naturally identified with S^{k-1}, so that S^∞ is the limit of the iterated join of G with itself k-times as $k \mapsto \infty$. The antipodal action is the action of G on the join is given by the iterated join of the natural action of G on itself by left multiplication.

The Milnor construction for any discrete group G is completely parallel to this. We form the infinite iterated joint $EG = G * G * \cdots$ (meaning the direct limit of finite iterated joins of G with itself with the weak topology). There is a natural action of G on this join which is easily seen to be free and discrete. The fact that we are taking infinite joins means that EG is contractible. As we indicated above, the inclusion

$$\underbrace{G * \cdots * G}_{k-\text{times}} \subset \underbrace{G * \cdots * G}_{(k+1)-\text{times}}$$

is homotopic to a point map and hence induces the trivial map on all homotopy groups. This implies that the limit has trivial homotopy groups. Also, there is a natural CW structure on EG whose k-skeleton is the $(k+1)$-fold iterated join. From now on, when we write EG we mean the Milnor construction, and when we write BG we mean the quotient $G\backslash EG$.

1.2 Uniqueness of EG

Now suppose that we have another space EG' on which G acts freely and discretely, then the quotient space $(BG)' = G\backslash EG'$ has fundamental group G and trivial higher homotopy groups. Since BG is a CW complex with the same homotopy groups, obstruction theory tells us there is a map $BG \to (BG)'$ inducing an isomorphism of homotopy groups and compatible with the identification of the fundamental group of each with G. Passing to the universal covers produces an equivariant map $EG \to EG'$ as required.

2 Group Cohomology

The k-cells of EG are labeled by the ordered sequences $(g_0, \ldots, g_k)$ of elements of G. The boundary map is given by

$$\partial(g_0, \ldots, g_k) = \sum_{i=0}^{k} (-1)^i (g_0, \ldots \hat{g}_i, \ldots, g_k).$$

When we pass to $BG = G\backslash EG$ it is natural to label the image of the cell $(g_0, \ldots g_k)$ by $(h_1, \ldots, h_k)$ where $h_i = g_{i-1}^{-1} g_i$. [Notice that if $(g'_0, \ldots, g'_k)$ and $(g_0, \ldots, g_k)$ have the same image in BG then there is $g \in G$ such that $g'_i = g g_i$ for all $0 \le i \le k$. Hence $h'_i = h_i$ for all $1 \le i \le k$.]

Then the formula for the boundary map in BG is

$$\partial(h_1, \ldots, h_k) =$$
$$= (h_2, \ldots, k_k) + \sum_{i=1}^{k-1} (-1)^i (h_1, \ldots, h_{i-1}, h_i h_{i+1}, h_{i+2}, \ldots, h_k) + (-1)^k (h_1, \ldots, h_k).$$

There is a proviso that the first term involves a shift around the loop h_1. This is the chain complex, denoted $C_*(BG)$, associated with the quotient CW structure on BG. The cohomology of G with coefficients in an abelian group A with trivial G-action, which is denoted $H^*(BG;A)$ is by definition the cohomology of the cochain complex $\mathrm{Hom}(C_*(BG),A)$ where the differential is the algebraic dual of the boundary map. If A has a G action then the cohomology $H^*BG;A)$ is the cohomology of $\mathrm{Hom}(BG;A)$ where now the differential is given by

$$\delta\varphi(h_1,\ldots,h_k) =$$
$$h_1\varphi(h_2,\ldots,h_k)+\sum_{i=1}^{k-1}(-1)^i\varphi(h_1,\ldots,h_{i-1},h_ih_{i+1},h_{i+2},h_k)+(-1)^k(h_1,\ldots,h_{k-1}),$$

reflecting the fact that the first boundary map involves transport around the loop h_1. In both cases these algebraically defined cohomology groups agree with the usual topological cohomology groups of BG with either constant coefficients (action of G on A trivial) or a system of local coefficients (action of G on A non-trivial).

Exercises: 1. Let A be an abelian group with an action of G. Show that the 1-cocycles with values in a G module A are the *crossed homomorphism* $G \to A$, that is to say functions $\varphi\colon G \to A$ satisfying $\varphi(h_1h_2) = \varphi(h_1) + h_1 * \varphi(h_2)$. These form an abelian group under addition of values of functions. Since there are no non-zero coboundaries, this is the first cohomology of G with coefficients in the G-module A.

2. Let A be an abelian group with a G action. Show that $H^2(G;A)$ classifies extensions, up to equivalence,

$$\{1\} \to A \to E \to G \to \{1\}$$

where A is a normal subgroup on which the conjugation action of action of E factors through the given action of G on A.