# Lecture 4: Diagonal approximations and Steenrod operations

November 1, 2018

## 1 An explicit formula for a chain homotopy

In this lecture we expand upon the acyclic carrier arguments from the last lecture and produce explicit formula for a chain homotopy between the Alexander-Whitney chain approximation to the diagonal and its image under the (signed) flip of factors map, denoted $T$ in the last lecture. Indeed, we shall produce all higher homotopies. The formulae we derive can be found in Steenrod's original paper but our approach uses some of the modern ways of expressing the information using step functions.

The diagrams for the Alexander-Whitney product formula are given in Figure 1. The interpretation of this diagram is that $Wh(|\Delta^n|))$ is an element $P(k,n) \in Sing_k(|\Delta^n|) \otimes Sing_{n-k}(|\Delta^n|)$ which is the tensor product of the singular simplex that is the inclusion of the face spanned by the vertices in the upper row with singular simplex that is the inclusion of the face spanned by the vertices in the second row. Then $Wh(|\Delta^n|) \in Sing_*(|\Delta^n|) \otimes Sing_*(|\Delta^n|)$ is $\sum_{k=0}^n P(k,n)$. Then we define

$$Wh\colon Sing_*(X) \to Sing_*(X) \otimes Sing_*(X)$$

to be the map given by

$$Wh(\sigma^n) = (\sigma^n)_* \otimes (\sigma^n)_*(Wh(|\Delta^n|).$$

Last time we showed that $Wh$ is a chain map and we defined the cup product by

$$\langle \alpha \cup \beta, \sigma \rangle = \langle \alpha \otimes \beta, Wh(\sigma) \rangle.$$

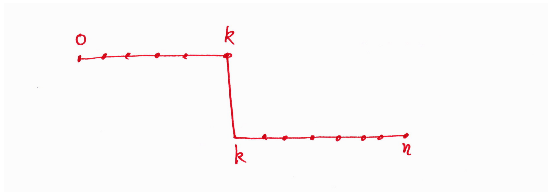


Figure 1: $P(k,n)$

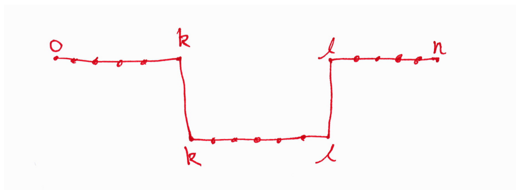


Figure 2: $P(k,l,n)$

### 1.1 An explicit chain homotopy

To define a chain homotopy between this product and its signed flip we consider all diagrams in Figure 2.

Notice that in each of the diagrams $P(k,\ell,n)$ the sum of the dimension of the face given by the vertices in the first row and the dimension of the face given by the vertices in the second row is $(n+1)$. As before, we associate to the diagram $P(k,\ell,n)$ an element, also denoted $P(k,\ell,n)$, of degree $(n+1)$ in $Sing_*(|\Delta^n|)\otimes Sing_*(|\Delta^n|)$ which is a tensor product of the inclusion of the face spanned by the vertices in the first row with the inclusion of the face spanned by the vertices in the second row.

We define

$$H_1(|\Delta^n|)=\sum_{0\leq k<\ell\leq n}\epsilon(k,\ell,n)P(k,\ell,n)\in Sing_*(|\Delta^n|)\otimes Sing_*(|\Delta^n|),$$

where $\epsilon(k,l,n)$ is $(-1)$ raised to the power $nk+n\ell+k\ell+k+\ell+1$.

Then, we define a linear map of degree $+1$

$$H_1\colon Sing_*(X)\to Sing_*(X)\otimes Sing_*(X)$$

by

$$H_1(\sigma^n)=(\sigma^n)_*\otimes(\sigma_n)_*H_1(|\Delta^n|).$$

**Lemma 1.1.**

$$\partial H_1+H_1\partial=T\circ Wh-Wh,$$

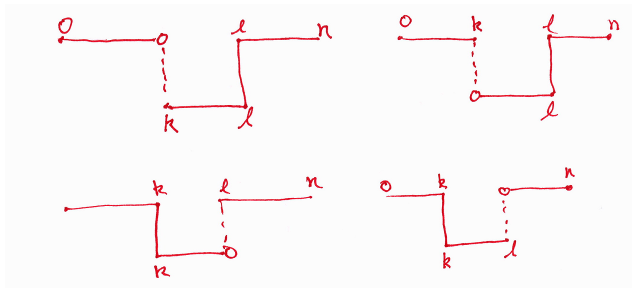


Figure 3: Type 1 Boundary Diagrams

*Proof.* I will work modulo two so that we can ignore the signs. (Interested readers can show that with the sign given above the arguments are valid in $\mathbb{Z}$ not just $\mathbb{Z}/2\mathbb{Z}$.) The term $\partial H_1(|\Delta^n|)$ is given by summing over all diagrams $P(k, \ell, n)$ of the diagrams derived by deleting a single vertex. The resulting diagrams are of two basic types as shown in Figures 3 and 4. The diagrams of Type 2 are exactly the same as the diagrams that appear when computing $H_1(\partial|\Delta^n|)$. The diagrams of Type 1 cancel out in pairs as shown in Figure 5 with the exception of the extremal diagrams shown Figure 6 and the degenerate diagrams shown in Figure 7. Each degenerate diagram except for the extremal degenerate diagrams (when the repeated vertex is either 0 or $n$) occurs twice (with opposite sign if we are working over $\mathbb{Z}$). So $\partial H_1(|\Delta^n|) + H_1(\partial|\Delta^n|)$ is the sum over the extremal diagrams and the extremal degenerate diagrams. The sum over the extremal diagrams produces all the terms of $T \circ Wh - Wh$ except those for which the repeated vertex is either 0 or $n$ (remember I am only computing mod 2). The last two terms are exactly the ones given by the two extremal degenerate diagrams. This proves

$$\partial H_1(|\Delta^n|) + H_1(\partial|\Delta^n|) = T \circ Wh(|\Delta^n|) - Wh(|\Delta^n|).$$

The lemma follows by naturality. $\square$

Last, week I showed that there was a chain homotpy between $T \circ Wh$ and $Wh$ without making one explicit. The formula for $H_1$ given here determines an explicit chain homotopy, but of course it is not the only one.

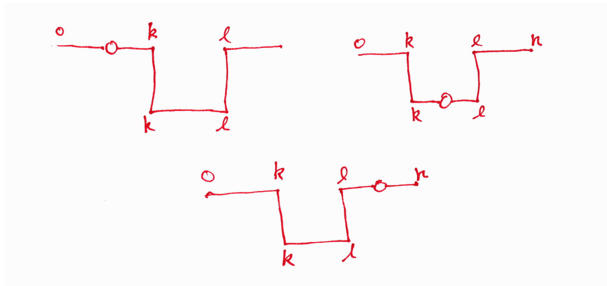


Figure 4: Type 2 Boundary Diagrams

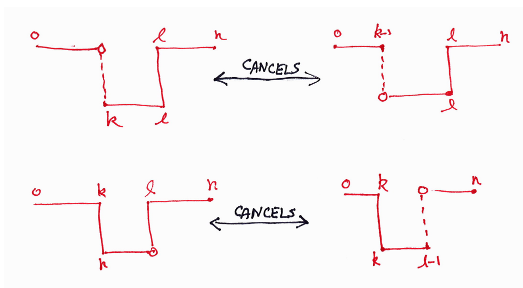


Figure 5: Cancelling of Type 1 Boundary Diagrams

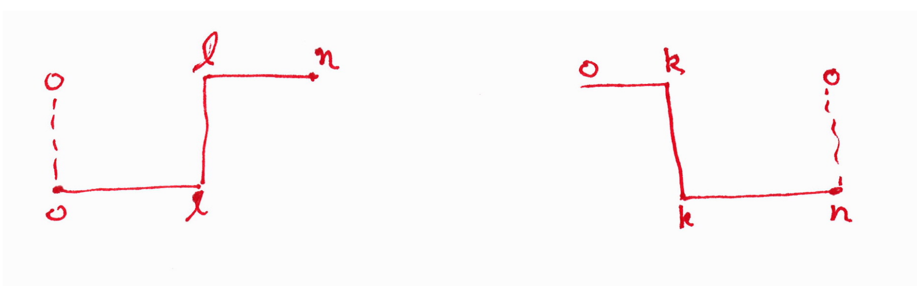


Figure 6: Extremal Type 1 Boundary Diagrams

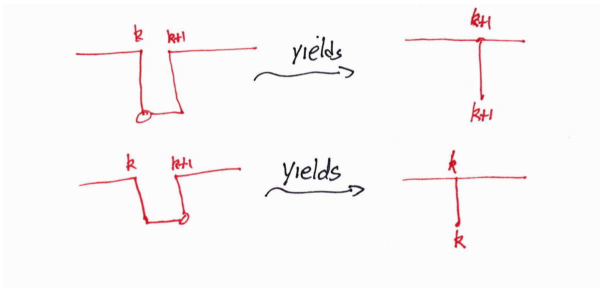


Figure 7: Degenerate Diagrams

### 1.2 Higher chain homotopies

We have defined $H_0 = Wh$ of degree 0 and $H_1$ a map of degree 1 both maps from $Sing_*(X)$ to $Sing_*(X) \otimes Sing_*(X)$ such that

$$\partial H_0 - H_0 \partial = 0$$

$$\partial H_1 + H_1 \partial = T \circ H_0 - H_0.$$

We shall define elements $H_k(|\Delta^n|) \in Sing_*(|\Delta^n|) \otimes Sing_*(|\Delta^n|)$ of degree $n+k$, and then we define

$$H_k \colon Sing_*(X) \to Sing_*(X) \otimes Sing_*(X)$$

of degree $+k$ by setting $H_k(\sigma) = (\sigma_* \otimes \sigma_*) H_k(|\Delta^n|)$. These are defined so that they satisfy

$$\partial H_k - (-1)^k H_k \partial = T \circ H_{k-1} - (-1)^{k-1} H_{k-1}.$$

For $0 \leq a_0 < a_1 < \cdots < a_k \leq n$ we define the diagram denoted in Figure 8.

This diagram corresponds to an element of degree $n+k$ in $Sing_*(|\Delta^n|) \otimes Sing_*(|\Delta^n|)$ given by the tensor product of the inclusion of the face spanned by the vertices in the first row with the inclusion of the face spanned by the vertices in the second row,

We also define signs $\epsilon(a_0, \ldots, a_k; n)$ by recursion. The recursive formulae are

$$\epsilon(a_0, \ldots, a_k; n) = (-1)^k \epsilon(a_0 + 1, \ldots, a_k + 1; n+1)$$
$$\epsilon(0, a_1 \ldots, a_k; n) = (-1)^{pq+k} \epsilon(a_1, \ldots, a_k : n-1)$$

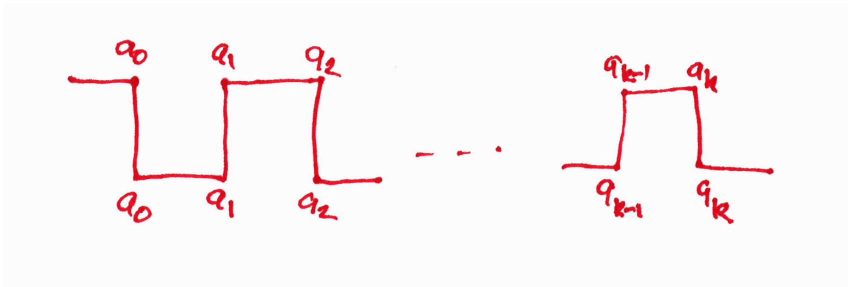


Figure 8: Diagram $P(a_1, \ldots, a_k; n)$

where in the second formula the diagram $P(0, a_1, \ldots, a_k)$ gives a term of bi-degree $(p+1, q)$ in $Sing_*(X) \otimes Sing_*(X)$. One can check that starting with $\epsilon(a_0 : n) = 1$ and applying these recursion relations one gets the signs $\epsilon(k, \ell; n)$ for the sum $H_1$ as stated above.

We define

$$H_k(|\Delta^n|) = \sum_{0 \le a_0 < \cdots < a_k \le n} \epsilon(a_0, \ldots, a_k; n) P(a_0, \ldots, a_k; n)$$

leading, as described above, to the definition of a map of degree $k$

$$H_k \colon Sing_*(X) \to Sing_*(X) \otimes Sing_*(X).$$

**Lemma 1.2.**

$$\partial H_k - (-1)^k H_k \partial = T \circ H_{k-1} - (-1)^{k-1} H_{k-1}.$$

*Proof.* (Sketch) We work modulo two. One computes $\partial H_k$ by summing over all possible ways of deleting a single vertex from one of the diagrams $P(a_0, \ldots, a_k : n)$. Those terms where the deleted vertex is not one of the repeated vertices are exactly the terms that appear in $H_k \partial$. Thus, $\partial H_k - (-1)^k H_k \partial$ is the sum of the terms obtained by deleting a repeated vertex from one of the diagrams $P(a_0, \ldots, a_k; n)$. There are two types of terms as before – non-degenerate and degenerate which cancel out in pairs except for the extremal members which in turn yield $T \circ H_{k-1} - (-1)^{k-1} H_{k-1}$. $\square$

### 1.3 The $\cup_i$-products and Steenrod squares

Let $\alpha \in Sing^p(X; \mathbb{Z}/2\mathbb{Z})$ and $\beta \in Sing^q(X; \mathbb{Z}/2\mathbb{Z})$. For each $k$ we define

$$\alpha \cup_k \beta \in Sing^{p+q-k}(X; \mathbb{Z}/2\mathbb{Z})$$

as follows.

We define

$$\langle \alpha \cup_k \beta, \sigma^{p+q-k} \rangle = \langle \alpha \otimes \beta, H_k(\sigma) \rangle.$$

Clearly, $\cup_0$ is the usual Whitney cup product.

**Claim 1.3.** *The higher cup products satisfy (mod 2 formula):*

$$\delta(\alpha \cup_k \beta) = \delta(\alpha) \cup_k \beta + \alpha \cup_k \delta(\beta) + \alpha \cup_{k-1} \beta + \beta \cup_{k-1} \alpha.$$

*Proof.* We work modulo 2. The basic formula for $H$ tells us

$$\langle \alpha \otimes \beta, \partial H_k(\sigma) + H_k(\partial\sigma) + T \circ H_{k-1}(\sigma) + H_{k-1}(\sigma) \rangle = 0.$$

Thus,

$$\begin{aligned} &\langle \alpha \otimes \beta, \partial H_k(\sigma) \rangle \\ &\qquad = \langle \alpha \cup_k \beta, \partial\sigma \rangle + \langle \beta \cup_{k-1} \alpha, \sigma \rangle + \langle \alpha \cup_{k-1} \beta, \sigma \rangle \\ &\qquad\qquad = \langle \delta(\alpha \cup_k \beta) + \beta \cup_{k-1} \alpha + \alpha \cup_{k-1} \beta, \sigma \rangle \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle \alpha \otimes \beta, \partial H_k(\sigma) \rangle &= \langle \delta(\alpha \otimes \beta), H_k(\sigma) \rangle \\ &= \langle \delta(\alpha) \cup_k \beta + \alpha \cup_k \delta(\beta), \sigma \rangle. \end{aligned}$$

From these two equations the claim follows. □

We define the Steenrod square $Sq^k(\alpha^p) = \alpha \cup_{p-k} \alpha$. It follows immediately from the above formula that if $\alpha$ is closed then so is $Sq^k(\alpha)$ and if we vary $\alpha$ by a coboundary then we also vary $Sq^k\alpha$ by a coboundary. Thus, $Sq^k$ passes to a map $H^p(X; \mathbb{Z}2\mathbb{Z}) \to H^{p+k}(X; \mathbb{Z}/2\mathbb{Z})$. This cohomology operator is natural in $X$; that is to say it is a natural transformation from the functor $H^p(-; \mathbb{Z}/2\mathbb{Z})$ to the functor $H^{p+k}(-; \mathbb{Z}/2\mathbb{Z})$, both functors going from the homotopy category to the category of $(\mathbb{Z}/2\mathbb{Z})$-vector spaces.

### 1.4 A Cohomological Re-interpretation

There is a more cohomological way to view the Steenrod squares. Let $S^\infty$ be the direct limit (with the weak topology) of the system of inclusions $S^n \subset S^{n+1}$ associated with the natural linear inclusions $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ as the first $(n+1)$-coordinate hyperplane. Then $S^\infty$ can also be viewed as the unit sphere in $\mathbb{R}^\infty$, with the subspace topology.

We have the action of the group of two elements on $S^\infty$ with the non-trivial element acting by the antipodal map. There is an equivariant cell decomposition of $S^\infty$ with two cells in each degree: the upper and lower hemispheres of $S^n$, denoted $e^n_\pm$with respect to its last coordinate. The $n$-skeleton of this CW structure is $S^n \subset S^\infty$. One checks that

$$\partial e^n_+ = e^{n-1}_- + (-1)^n e^{n-1}_+.$$

Let $C_*(S^\infty)$ be the CW chains for this cell structure. The chain complex is a chain complex of $Z[\mathbb{Z}/2\mathbb{Z}]$-modules where the $\mathbb{Z}/2\mathbb{Z}$-action is induced by the antipodal map, Each chain group is a free module, and we choose the generator to be the upper hemisphere, $e^n_+$ of $S^n$ with its usual orientation. With this choice of basis, the boundary map $C_n \to C_{n-1}$ is given by $e^n_+ \mapsto e^{n-1}_- + (-1)^n e^{n=1}_+$ for $n > 0$. This chain complex is a projective (indeed free) resolution of the $\mathbb{Z}[\mathbb{Z}2\mathbb{Z}]$-module $\mathbb{Z}$ (where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $\mathbb{Z}$).

We use the $H_k$ to define a chain map

$$H\colon C_*(S^\infty) \otimes Sing_*(X) \to Sing_*(X) \otimes Sing_*(X)$$

by

$$H(e^k_+ \otimes \sigma) = H_k(\sigma)$$
$$H(e^k_- \otimes \sigma) = T(H_k(\sigma)).$$

The content of Lemma 1.2 is that $H$ is a chain map. By construction this map is a homomorphism of graded $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$-modules when we use the action on the domain that is the tensor product of the given action on $C_*(S^\infty)$ and the trivial action on $Sing_*(X)$ and use the action on the range which is induced by $T$ (the signed flip of factors). Consequently, $H$ is a $\mathbb{Z}/2\mathbb{Z}$-equivariant chain map.

We have identifications of $H^*(Sing_*(X))$ with $H^*(X)$ and also of $H^*(Sing_*(X)\otimes Sing_*(X))$ with $H^*(X \times X)$, the latter being equivariant with respect to the maps induced by the flip (with appropriate signs). The map $H$ induces a map of equivariant cohomology

$$H^*_{\mathbb{Z}/2\mathbb{Z}}(X \times X; \mathbb{Z}/2\mathbb{Z}) \to H^*_{\mathbb{Z}/2\mathbb{Z}}(S^\infty \times X; \mathbb{Z}/2\mathbb{Z}) = H^*(\mathbb{R}P^\infty \times X; \mathbb{Z}/2\mathbb{Z}).$$

(The last inequality follows from the fact that the $\mathbb{Z}/2\mathbb{Z}$-action on $S^\infty \times X$ is free.) Since $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$ is a polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ on a generator $x$ of degree 1, by the Künneth Theorem we have

$$H^*(\mathbb{R}P^\infty \times X; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x] \otimes H^*(X).$$

Thus, the map $H$ induces a map

$$H^*_{\mathbb{Z}/2\mathbb{Z}}(Sing_*(X) \otimes Sing_*(X); \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}][x] \otimes H^*(X).$$

Fix a $(\mathbb{Z}/2\mathbb{Z})$-cocycle $\alpha \in Sing^p(X)$. Then $\alpha \otimes \alpha$ is an equivaraint cocycle in the cochain complex $\mathrm{Hom}(Sing_*(X) \otimes Sing_*(X); \mathbb{Z}/2\mathbb{Z})$ whose equivariant cohomology class depends only on the cohomology class of $\alpha$. Then $H^*([\alpha \otimes \alpha])$ can be written uniquely as $\sum_{k=0}^{2p} x^k Sq_{p-k}([\alpha])$ for cohomology classes $Sq_{p-k}([\alpha]) \in H^{2p-k}(X; \mathbb{Z}/2\mathbb{Z})$. As the notation indicates, the classes $Sq_{p-k}([\alpha])$ depend only on the cohomology class $[\alpha] \in H^p(X; \mathbb{Z}2\mathbb{Z})$. In fact, the only possibly non-zero coefficients in this expression are $Sq_{p-k}([\alpha])$ for $0 \leq k \leq p$.

The classical definition of the Steenrod squares is $Sq^k[\alpha] = Sq_{2p-k}([\alpha])$ for any $p$-dimensional class $[\alpha]$. This defines cohomology operations that are natural under continuous maps and do not change as we vary the map by homotopy. The resulting cohomology operations are independent of the choice of higher chain homotopies (again using acyclic carriers to find a homotopy between any two choices) and are natural operations (since they are defined on the standard simplices and pushed forward by singular simplices). They are also invariant under suspension. Clearly, $Sq^n([\alpha]) = \alpha^2$. It is not too hard to prove $Sq^0([\alpha]) = [\alpha]$. This corresponds to the statement that $H_p(\sigma^p) = \sigma^p \otimes \sigma^p$ and modulo 2 we have $\langle \alpha, \sigma \rangle^2 = \langle \alpha, \sigma \rangle$.

## 2 The Acyclic Carriers Approach

Just as we constructed $H_1$ using acyclic carriers, it is possible to construct maps $H_k \colon Sing_*(X) \to Sing_*(X) \otimes Sing_*(X)$ such that

$$\partial H_k - (-1)^k H_k \partial = T \circ H_{k-1} - (-1)^{k-1} H_{k-1}.$$

Suppose inductively we have constructed maps $H_i$ for $i < k$ as required. Now by induction on $n$, suppose for all $n' < n$ for $|\Delta^{n'}|$ we have elements $H_k(|\Delta^{n'}|)$ such that

$$\partial H_k(|\Delta^{n'}|) = (-1)^k H_k(\partial |\Delta^{n'}|) + T \circ H_{k-1}(|\Delta^{n'}|) - (-1)^{k-1} H_{k-1}(|\Delta^{n'}|).$$

We construct $H_k(|\Delta^n|)$ such that the above formula holds for $n$ instead of $n'$.

The point is that the inductive hypotheses tell us:

**Claim 2.1.**

$$(-1)^k H_k(\partial |\Delta^n|) + T \circ H_{k-1}(|\Delta^n|) - (-1)^{k-1} H_{k-1}(|\Delta^n|).$$

*is a closed element in* $Sing^*(|\Delta^n|) \otimes Sing_*(|\Delta^n|)$.

*Proof.*

$$\partial(-1)^k H_k(\partial|\Delta^n|) = H_k(\partial\partial|\Delta^n|)+(-1)^k[T\circ H_{k-1}(\partial|\Delta^n|)-(-1)^{k-1}H_{k-1}(\partial\Delta^n|)].$$

The first term on the right-hand side vanishes and the boundary of the second two terms yields

$$T \circ \partial H_{k-1}(|\Delta^n|) - (-1)^{k-1}\partial H_{k-1}(|\Delta^n|).$$

By the inductive hypothesis on $H_{k-1}$ we have

$$\begin{aligned}(-1)^k T \circ H_{k-1}(\partial|\Delta^n|) + T \circ \partial H_{k-1}(|\Delta^n|) &= T^2 \circ H_{k-2}(|\Delta^n|) - (-1)^{k-2}T \circ H_{k-2}(|\Delta^n|) \\ &= (-1)^{k-1}[T \circ H_{k-2}(|\Delta^n|) - (-1)^{k-2}H_{k-2}(|\Delta^n|)]\end{aligned}$$

Similarly, we have

$$\begin{aligned}&(-1)^k(-1)(-1)^{k-1}H_{k-1}(\partial|\Delta^n|) - (-1)^{k-1}\partial H_{k-1}(|\Delta^n|) \\ &\quad = (-1)^k[T \circ H_{k-2}(|\Delta^n|) - (-1)^{k-2}H_{k-2}(|\Delta^n|)].\end{aligned}$$

These two expressions cancel, completing the proof of the claim. □

Since this complex is acyclic, it follows that there is an element $H_k(|\Delta^n|)$ with

$$\partial H_k(|\Delta^n|) = (-1)^k H_k(\partial|\Delta^n|) + T \circ H_{k-1}(|\Delta^n|) - (-1)^{k-1}H_{k-1}(|\Delta^n|).$$

This completes the induction on $n$ showing that the $H_k(|\Delta^n|)$ exist for all $n$

Once we have the elements $H_k(|\Delta^n|)$ for all $n$, for any singular $n$-simplex $\sigma$ we define

$$H_k(\sigma) = (\sigma_*) \otimes (\sigma_*)H_k(|\Delta^n|).$$

This produces the next step in the system of higher homotopies and completes the induction on $k$.