Lecture 0: Reivew of some basic material

September 12, 2018

1 Background material on the homotopy category

We begin with the topological category **TOP**, whose objects are topological spaces and whose morphisms are continuous maps. There is a quotient category, the homotopy category (of topological spaces), whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps. There is the obvious natural transformation from **TOP** to the homotopy category which is the identity on objects and associates to a continuous map its homotopy class. Two spaces X and Y are *homotopy equivalent* if they are isomorphic in the homotopy category, which means that there are continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity of X and $f \circ g$ is homotopic to the identity of Y.

As the next lemma shows, in the homotopy category every map is equivalent to an inclusion.

Lemma 1.1. *Given a map $f\colon X \to Y$ there is a space M_f, a homotopy equivalence $p\colon M_f \to Y$ and a map $i\colon X \to M_f$ which is a topological inclusion of X as a closed subspace of M_f such that $p \circ i$ is homotopic to f. That is to say, that given $f\colon X \to Y$ then in the homotopy category Y is isomorphic to an object M_f and contians X as a subspace and the inclusion is mapped to f under this isomorphism.*

Definition 1.2. Recall that a map $p\colon E \to B$ is a *Hurewicz fibration* if it has the homotopy lifting property. That is to say, given $F\colon Y \times I \to B$ and $\widetilde{F}_0\colon Y \to E$ with $p \circ \widetilde{F}_0 = F|_{Y\times\{0\}}$ then there is a map $\widetilde{F}\colon Y \times I \to E$ with $p \circ \widetilde{F} = F$ and $\widetilde{F}|_{Y\times\{0\}} = \widetilde{F}_0$.

As the next lemma shows, in the homotopy category every map is a Hurewicz fibration.

Lemma 1.3. *Let $f\colon X \to Y$ be a morphism in* **TOP**. *Then there is a space $\mathcal{P}(X,Y)$, a homotopy equivalence $g\colon X \to \mathcal{P}(X,Y)$ and a map $q\colon \mathcal{P}(X,Y) \to Y$ which is a Hurewicz fibration such that $q \circ g$ is homotopic to f.That is to say,, that given $f\colon X \to Y$ then in the homotopy category X is isomorphic to a space $\mathcal{P}(X,Y)$ with a map to Y which is the composition of this isomorphism with f and is a Hurewicz fibration.*

The space $\mathcal{P}(X,Y)$ is the space of paths parametrized by the unit interval in the mapping cylinder M_f whose initial point is contained in $i(X) \subset M_f$, given the compact-open topology. The map $g\colon X \to \mathcal{P}(X,Y)$ maps $x \in X$ to the constant path at $i(x) \in M_f$ and the map $q\colon \mathcal{P}(X,Y) \to Y$ sends the path ω to $p(\omega(1))$. Contracting paths to their initial point gives a deformation retraction from $\mathcal{P}(X,Y) \to X$.

2 CW complexes

It turns out that the homotopy category is not the best one for geometric applications. It is better either to restrict the class of spaces under consideration or to take a weaker equivalence relation than homotopy equivalence.

The category of spaces that is reasonable to work with is the category of CW complexes: these are spaces X with an exhaustive filtration:

$$\emptyset = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset \cdots,$$

where each $X^{(n)}$ is adjunction space obtained from $X^{(n-1)}$ be attaching $\coprod D^n$ along the boundary. If the sequence stabilizes at some finite step then we have a *finite dimensional* CW complex. If it does not stabilize at any finite step then we have an infinite dimensional complex and we must add the requirement that the topology on the union is the weak topology, induced by the filtration, meaning that a subset of the union is open if and only if its intersection with each $X^{(n)}$ is open in $X^{(n)}$. A *finite* CW complex is a CW complex with only finitely many cells.

The interiors of the n-disks that are attached are open subsets of $X^{(n)}$ and are called *the open n-cells.*

The category of CW complexes has as objects CW complexes and as morphisms continuous maps. There is the homotopy category of CW complexes which is the homotopy quotient category with morphisms homotopy classes of continuous maps between CW complexes. The main theorem that makes every thing work in this category is Whitehead's Theorem:

Theorem 2.1. *Let $f\colon X \to Y$ be a continuous map between connected CW complexes. Then f is a homotopy equivalence if and only if it induces an isomorphisms on all homotopy groups. That is to say, a map $f\colon X \to Y$ in the category of CW complexes is an isomorphism in the homotopy category if and only if it induces isomorphisms on all homotoopy groups.*

We will prove this in the next lecture once we have developed obstruction theory.

For general spaces we pass to a further quotient category where we force the analogue of Whitehead's theorem to hold.

Definition 2.2. A map $f\colon X \to Y$ between path connected topological spaces is said to be a *weak homotopy equivalence* if it induces an isomorphism on the homotopy groups. This is not an equivalence relation since it is not reflexive, but we let these elementary equivalences generate an equivalence relation, called *weak homotopy equivalence.* There is then the quotient category of **TOP** whose morphisms from X to Y are finite strings of morphisms where the ones that go 'the wrong direction' are weak homotopy equivalences. We call this the *weak homotopy category of topological spaces.*

Theorem 2.3. *The natural map from the homotopy category of CW complexes to the weak homotopy category of topological spaces is an equivalence of categories.*

This means that every topological space is weakly equivalent to a CW complex and that the homotopy classes of maps between two CW complexes is the same as the morphisms between the complexes in the weak homotopy category of **TOP**.

2.1 The homology of a CW complex

Recall that $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is trivial in all degrees except n and the n^{th}-homology is an infinite cyclic group. A generator for this group is equivalent to an orientation fo $\mathbb{R}^n$.

Let $X_0^{(n)}$ be the complement in $X^{(n)}$ of the union of the origins of all the n-disks that are attached. Then $X_0^{(n)}$ deformation retracts onto $X^{(n-1)}$. Hence $H_*(X^{(n)}, X^{(n-1)}) = H_*(X^{(n)}, X_0^{(n)})$. By excision the latter is isomorphic to a direct sum indexed by the n-cells of infinite cyclic groups. To choose a generator for the summand associated with a given n-cell e is to choose an orientation for e. Reversing the orientation multiplies the generator by -1.

Form a chain complex

$$C_n(X) = H_n(X^{(n)}, X^{(n-1)})$$

with $\partial\colon C_n(X) \to C_{n-1}(X)$ being the boundary map for the triple

$$(X^{(n)}, X^{(n-1)}, X^{(n-2)}).$$

One proves directly by induction that the homology of this complex is naturally identified with the singular homology of X. This chain complex is called the complex of *CW chains*. Sometimes this result is called the homology of the homology.

Let us give a more geometric description of the boundary map. Fix an n-cell e. Choose orientations for e and for all the $(n-1)$-cells $\{e'_\alpha\}_\alpha$. Then we have a generator $[e]$ for the summand in $C_n(X)$ associated to e, and we have a basis $\{[e'_\alpha]\}_\alpha$ for $C_{(n-1)}(X)$. Assume that the attaching map f_e for the n-cell e is transverse to the mid-point, m_α, of each $(n-1)$-cell e'_α. Then the preimage of $f_e^{-1}(m_\alpha)$ is a finite set of points in the boundary of e and is empty for all but finitely many α. The orientation for e induces one for its boundary. So we can assign signs (± 1) to each point of $y \in f_e^{-1}(m_\alpha)$ measuring the effect on these orientations of the map f near y. Denote by c_{e,e'_α} sum of these signs over all $y \in f_e^{-1}(m_\alpha)$. Then

$$\partial([e]) = \sum_\alpha c_{e,e'_\alpha}[e'_\alpha],$$

which is a finite sum.

3 Morse Theory

Suppose that M is a compact smooth manifold and that $f\colon M \to \mathbb{R}$ is a Morse function. This means that every critical point of f is non-degenerate. That is to say, denoting by $Hess(f)$ the Hessian of f at p, i.e., the symmetric matrix of second partial derivatives of f at p, up to a linear change of coordinates near p we have $Hess(f)$ is a diagonal matrix with all diagonal entries either $+1$ or -1. The number of -1's is called the *index* of the critical point p. In fact, there is a local coordinate system centered at p so that $f(x_1, \ldots, x_n) = f(p) + \sum_i \pm x_i^2$.

If there is no critical value for f in the interval $[a, b]$, then $f^{-1}([a, b])$ is diffeomorphic to a product $f^{-1}(a) \times [a, b]$. [Proof: Choose a Riemannian metric on M and use the gradient flow for $-\nabla f$.] It follows that the inclusion

$f^{-1}((-\infty, a]) \subset f^{-1}((-\infty, b])$ is a homotopy equivalence and hence induces an isomorphism on homology.

Suppose that p is a critical point of the Morse function f. Fix a Riemannian metric and consider the gradient flow for $-\nabla f$. The *descending manifold of* p is the set of points which under $-\nabla f$ converge as $t \mapsto -\infty$ to p. It is denoted $h^-(p)$. Given the local description of f near p, it is easy to see that the intersection of $h^-(p)$ with a small neighborhood of p is a smoothly embedded disk of dimension equal to the index of f at p. This manifold is tangent to a maximal negative definitive subspace for the Hessian of f at p. Using the flow we see that $h^-(p)$ is a smoothly embedded open disk of dimension equal to the index of the critical point.

Reversing direction of the flow we define the *ascending manifold* $h^+(p)$, for the critial point p to be those points which flow under $-\nabla f$ to p as $t \mapsto \infty$. This is a disk of dimension $n - index(p)$. The ascending and descending manifolds of p meet only at p and meet transversally there. In fact, $h^-(p)$ is contained in $f^{-1}(-\infty, f(p)])$ while $h^+(p)$ is contained in $f^{-1}([f(p), \infty))$.

Suppose that c is a critical value with only one critical point with this value. Choose an interval $[a, b]$ containing c in its interior so that there is only one critical point in $f^{-1}([a, b])$. Then $(f^{-1}([a, b]), f^{-1}(\{a\})$ deformation retracts onto $f^{-1}(\{a\}) \cup h^i(p)$ where $h^i(p)$ is a closed disk of dimension equal to the index of the critical point with boundary in $f^{-1}(a)$. It is the intersection of $f^{-1}([a, b])$ with $h^-(p)$,. [Here again we are using a Riemannian metric.] More generally, if the critical points at which f takes value c are $\{p_1, \ldots, p_k\}$ with indices $\{i_1, \ldots, i_k\}$. Then for an interval $[a, b]$ containing c in its interior and containing no other critical points except for those with critical value c. Let $h_1, \ldots, h_k$ be the intersections of $f^{-1}([a, b])$ with the descending manifolds for these critical points (which are disjoint). Then $f^{-1}([a, b])$ deformation retracts onto $f^{-1}(a) \cup (\cup_i h_i)$. It follows that the relative homology of $(f^{-1}((-\infty, b]), f^{-1}((-\infty, a])$ is the free abelian group generated by the classes of the h_i, with the degree of the class corresponding to h_i being the the index of the critical point p_i.

One can always arrange that if p is a critical point then $f(p)$ is the index of p. In this case we get a description of the homology of M analogous to the homology of the homology described above for CW complexes. We form $C_k = H_k(f^{-1}((-\infty, k + (1/2)], f^{-1}((-\infty, k - (1/2)])$. This is a free abelian group with an infinite cycle summand for each critical point of index k. Choosing an orientation for $h^-(p)$ determines a generator for this summand. The boundary map being the boundary map of the triple. As in the CW complex case, this complex computes the homology of M. Let p be a critical point of index k and $\mathcal{O}_p$ an orientation for $h^-(p)$. Denote by $[p, \mathcal{O}(p)]$ the corresponding element in the chain group C_k. Then, $\partial[p]$ is computed by counting flow lines for $-\nabla f$ that at plus infinity converge to c and at minus infinity converge to a critical point of index one lower. We need to assign signs and this requires choosing orientations for the handles analogous for what we did in the CW case.

If p is a critical point of index k and q is a critical point of index $k-1$, then the flow lines limiting to p as $t \mapsto -\infty$ and limiting to q as $t \mapsto +\infty$ is the intersection of $h(p) \cap M_{k-1/2}$ and $h^+(q) \cap M_{k-1/2}$ where $M_{k-1/2}$ is $f^{-1}(k-1/2)$, which is a smooth codimension-1 submanifold. The factors are spheres of dimension equal to the index of p minus 1 and the index of q minus 1 and they are meeting in a manifold $M_{k-1/2}$ of dimension $n-1$. So under the genericity assumption, the intersection will be a finite set of points of transversal intersection. Using orientations we are able to assign an integer to this intersection and that is the algebraic number of flow lines from p to q and hence the coefficient of $[q, \mathcal{O}_q]$ in $\partial([p, \mathcal{O}_p])$.

4 Simplicial Complexes

One especially nice subcategory of the category of CW complexes is simpicial complexes. The n-simplex $|\Delta^n|$ is the subset of $\mathbb{R}^{n+1}$ which is the convex hull of the $n+1$ unit vectors, which are the vertices of the simplex. Notice that for any non-empty subset of these vertices their span (i.e., their convex hull) is a lower dimensional simplex which is a face of $|\Delta^n|$.

The combinatorial data defining a simplicial complex is a set V, the vertex set, and a collection $\mathcal{S}$ of finite, non-empty, subsets of V which are closed under taking non-empty subsets and which contain subsets of the form $\{v\}$ for $v \in V$. Geometrically, each element of $\mathcal{S}$ is a simplex whose vertices are the elements of the set. Thus, the dimension of the simplex corresponding to $S \in \mathcal{S}$ is of dimension 1 less than the cardinality of S. These simplicies are glued together using the face relations: for $S' \subset S$ the simplex corresponding to S' is identified with the subsimplex of the simplex corresponding to S whose vertices are the elements of S'. A finite simplicial complex is a subspace of the simplex spanned by its vertices. In general, the topology on a simplicial complex is the weak topology induced by its finite subcomplexes, meaning that a subset is open if and only if its intersection with every finite subcomplex is open ore equivalently, a subset is open if and only if its intersection with every closed simplex is an open subset of that closed simplex.

One can consider the category of simplicial complexes and simplicial maps, but this is not a flexible enough category to do homotopy theory. For example, consider $\partial|\Delta^2|$, which is topologically a circle. It has three edges and three vertices. There are continuous maps of arbitrary degree from this space to itself, but simplicial map must have degree $-1, 0, 1$.

To construct all homotopy classes we have to allow rectilinear subdivision. Let K and L be simplicial complexes. A map $K \to L$ is a *subdivision* if (i) the map is a homeomorphism and (ii) each closed simplex of K is mapped into a closed simplex of L by an affine linear map. Thus, the simplicies of K mapping to a given simplex of L divide that simplex into a union of smaller simplicies.

Here is the result that shows allowing subdivision gives us a flexible enough category.

Theorem 4.1. *Let K and L be simplicial complexes and $f\colon K \to L$. Then there is a subdivision $i\colon K' \to K$ and a simplicial map $\varphi\colon K' \to L$ homotopic to $f \circ i$.*

5 Simplical sets

There is another model for (weak) homotopy theory of topological spaces. That is the category of simplicial sets. Let Δ be the category whose objects are the sets $\mathbf{n}$ where $\mathbf{n}$ is the set $\{0, 1, \ldots, n\}$ and whose morphisms are (weakly) order-preserving functions. A *simplicial set* is a contravariant functor from Δ to the category of sets and set functions.

In more down to earth terms this means that for each $n \geq 0$ we have a set X_n and for each order-preserving map $\varphi\colon \mathbf{n} \to \mathbf{m}$ we have a set function $\hat{\varphi}\colon X_m \to X_n$ that respects compositions. Since every order-preserving map is a composition of face maps of the form

$$f_i\colon \mathbf{n} \to \mathbf{n+1}: \quad f_i(j) = \begin{cases} j & \text{if} \;\; j < i \\ j+1 & \text{otherwise.} \end{cases}$$

and the degeneracies of the form

$$s_i\colon \mathbf{n+1} \to \mathbf{n}: \quad s_i(j) = \begin{cases} j & \text{if} \;\; j \leq i \\ j-1 & \text{if} \;\; j > i. \end{cases} \qquad ,$$

to give a simplicial set is to give sets $\{X_n\}_{n\geq 0}$ together with boundary maps which are the images under the functor of the face maps

$$\partial_i\colon X_n \to X_{n-1}$$

and degeneracies $s_i\colon X_n \to X_{n+1}$ satisfying the following relations:

$$\partial_i \partial_j = \partial_{j-1}\partial_i \quad \text{if} \quad i < j$$

$$\partial_i s_j = s_{j-1}\partial_i \quad \text{if} \quad i < j$$

$$\partial_i s_j = \text{id} \ \text{ if } \ i = j \ \text{ or } \ i = j+1$$

$$\partial_i s_j = s_j \partial_{i-1} \ \text{ if } \ i > j+1$$

$$s_i s_j = s_{j+1} s_i \quad \text{if} \quad i \leq j$$

The *geometric realization* of a simplicial set is a simplicial complex. One begins with $\coprod_n X_n \times |\Delta^n|$. Then we introduce an equivalence relation and take the quotient space as follows. If $x \in X_n$ and $\partial_i(x) = y$ then $\{y\} \times |\Delta^{(n-1)|}$ is identified with the i^{th} face of $\{x\} \times |\Delta^n|$. If $s_i(x) = z$, then $\{z\} \times |\Delta^{n+1}|$ is collapsed onto $\{x\} \times |\Delta^n|$ along the edge spanned by i and $i+1$.

Proposition 5.1. *The geometric realization of a simplicial set is a simplicial complex. It has one n-simplex for each non-degenerate element of X_n, i.e., each element of X_n that is not in the image of s_j for any $j \leq n-1$. Furthermore, there is a partial order on the vertices of the geometric realization so that the vertices of any simplex are totally ordered. The elements of X_n are the order-preserving surjections $\{0, \ldots, n\}$ to the vertices of some simplex in the geometric realization.*

5.1 The homology and homotopy groups of a simplicial set

The homology of a simplicial set X is defined by taking $C_n(X)$ equal the free abelian group on X_n and defining the boundary map $C_n(X) \to C_{n-1}(X)$ by

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \partial_i(\sigma).$$

The homotopy groups of a simplicial set are easiest to describe if the simplicial set satisfies the *Kan condition.* This condition says that for every n and for every $0 \leq k \leq n+1$, given elements $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ of X_n that satisfy the consistency condition; $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j$ with $i \neq k$ and $j \neq k$, then there is a $y \in X_{n+1}$ with $\partial_i y = x_i$ for all $0 \leq i \leq n+1$ not equal to k.

There is in each X_n the completely degenerate image of an element $e \in X_0$, namely

$$s_{n-1}s_{n-2}\cdots s_0 e_n.$$

We denote this element by e_n For Kan complexes considers the set $x \in X_n$ such that $\partial_i x = e_{n-1}$ for all i and introduces a relation of homotopy: Elements x and y of this type are *homotopic* if there is $z \in X_{n+1}$ with $\partial_n z = x$, $\partial_{n+1} z = y$ and $\partial_i z = e_n$ for all $i < n$. The equivalence classes are the elements of $\pi_n(K, e)$. For $n > 0$, the multipliciation xy is given by considering $\underbrace{e_n, \ldots, e_n}_{n-1}, x, -, y$. This collection of n-simplicies satisfies the Kan condition and hence there is $z \in X_{n+1}$ with $\partial_i z = e_n$ for $i < n-1$, $\partial_{n-1} z = x$ and $\partial_{n+1} z = y$. Then xy is given by the class of $\partial_n z$.

That is a functor from topolgical spaces to simplicial sets that assigns to each topological space X its simplicial set of continuous maps from geometric simplicies to the sapce, denoted $Sing(X)$. The boundary maps are given by restriction to codimension-1 faces and the degeneracy maps are given by composiion with the degeneracies. There is also a functor from simplicial sets to simpicial complexes and hence topological spaces that assigns to a simpicial set K its geometric realization $|K|$. These functors are adjoint in the sense that $\mathrm{Hom}_{SS}(K, Sing(X))$ is naturally identified with $\mathrm{Hom}_{\mathbf{Top}}(|K|, X)$.

It makes sense to define simplicial objects in any cateogary: The objects are contravariant functors from Δ to that category and morphisms are natural transformations between functors. We will encounter simplicial groups and simplicial Lie algebras for example.

6 Exercises

1. Show the inclusion of a point into the 'sine curve' pseudo-circle induces an isomorphism on all homotopy groups but is not a homotopy equivalence.

2. Show that a compact subset of a CW complex is contained in the union of a finite number of open cells.

3. Show that a function from a CW complex X to a space Y is the same as continuous maps on the $\{\coprod D^n\}_n$ that are compatible with the attaching maps.

4. Show the converse to the proposition about the geometric realization of a simplicial set.

5. Show that the graded subgroup of the chain complex of a simplicial set generated by degenerate simplicies is in fact a subcomplex and that it

has trivial homology. This means that we can define the homology of the simplicial set by taking the chain groups generated by the non-degenerate simplicies.

6. Show that if $f\colon M \to \mathbb{R}$ is a Morse function with only one critical point in $f^{-1}([a,b])$, that critical point being of index k, then $f^{-1}([a,b])$ deformation retracts to $f^{-1}(a) \cup h^k$, where h^k is the intersection of $f^{-1}([a,b])$ with the descending manifold from the critical point.

7. Show that any compact smooth manifold is homotopy equivalent to a finite CW complex with one cell for each critical point, the dimension of the cell being the index of the critical point.

8. Show that the Euler to characteristic of a closed manifold is the alternating sum of $(-1)^i$ times the number of critical points of index i.

9. Verify the relations given for the boundary and degeneracy maps of simplicial sets.

10. Work out the orientation conventions in the case of a Morse function (without assuming that the ambient manifold is orientable).

11. Show that the homotopy sets of a Kan complex with the given multiplication form a group, which is abelian for all $n \geq 2$.