Lecture 2: Spectral Sequences

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1 Definition of the spectral sequence associated with a filtered cochain complex

Let C^* be a cochain complex and F^* a decreasing filtration on C^* preserved by d, meaning that $d(F^k(C^n)) \subset F^k(C^{n+1})$. Such a filtration induces a decreasing filtration on cohomology, also denoted F^*: a class in $H^n(C^*)$ is in $F^k(H^n(C^*))$ if and only if there is a cocycle representative for this class contained in $F^k(C^n)$. Thus, the associated gradeds of this filtation, $F^p(H^*(C^*))/F^{p+1}(H^*(C^*))$ are the quotient of the group of n-cocycles in F^p divided out by the sum of the group of coboundaries in F^p and the cocycles in F^{p+1}:

$$F^p(H^n(C^*))/F^{p+1}(H^n(C^*)) = \frac{\mathrm{Ker}(d\colon F^p(C^n) \to C^{n+1})}{d(C^{n-1}) \cap F^p + \mathrm{Ker}(d\colon F^{p+1}(C^n) \to C^{n+1})}.$$

To avoid technical difficulties with convergence of the spectral sequence to what we are interested in computing, we assume that for each n the filtration $F^*(C^n)$ is a finite filtration meaning that for each n there are integers k, ℓ with $F^k(C^n) = C^n$ for $F^\ell(C^n) = 0$. In fact, all the examples we shall see are **first quadrant spectral sequences**, meaning that (i) $C^* = 0$ for $* < 0$, (ii), $F^0(C^*) = C^*$, and (iii) for every n we have $F^{n+1}(C^n) = 0$. These conditions are not needed but they are very common and easily imply the the finiteness assumption, which is crucial for convergence results.

A spectral sequence produces a sequence of approximations, eventually stabilizing in each degree under the above finiteness condition, to the associated graded of the filtration on $H^*(C^*)$. The approximations are obtained by taking better and better approximations to cocycles and coboundaries. At each iterative stage, coycles in a given degree and filtration level are replaced by cochains of that degree and filtration level whose differential is contained in a higher indexed filtrant. Instead of dividing out by all coboundaries, i.e., the image of d, we divide out by the image of d restricted to smaller and smaller filtration levels. We also divide out by the cochains in the next higher filtration level whose coboundary satisfies the same filtration condition used for the cocycle replacement. Because of the finiteness, eventually we arrive at all cocycles of a given degree and filtration level modulo all coboundaries in this filtration level plus all cocycles of the next higher filtration. This is exactly the associated graded of the induced filtration on cohomology. The only tricky part is deciding how to match up the indices indicating which filtration level the coboundary is required to lie in and which filtration level we take coundaries from.

The result of this process is a triply-graded sequence of groups $E_r^{p,q}$. The index p represents the filtration level; the sum $p+q$ is the total degree (degree in the chain complex); $r \geq 0$ is the stage in the iterative process. Often, the bigraded group $E_r^{*,*}$ is referred to as **the r^{th}-page** of the spectral sequence. One obtains $E_{r+1}^{*,*}$ from $E_r^{*,*}$ by taking the cohomology of a differential d_r that d of the cochain complex induces on $E_r^{*,*}$.

The initial step, the 0^{th}-page of the spectral sequence is defined to be the associated graded of the cochain complex:

$$E_0^{p,q} = F^p(C^{p+q})/F^{p+1}(C^{p+q}).$$

The subscript 0 indicates that we are at the 0^{th}, or beginning stage of the process.

This term fits the model I described above: cocycles of degree $p+q$ and filtration level p are replaced by cochains of this degree and filtration level whose differential is contained in $F^{p+0}(C^{n+1})$. [This condition is automatically satisfied since d preserves filtration level.] The allowed coboundaries are $d(F^{p+1}(C^{n-1}))$, and we divide out by $F^{p+1}(C^n)$. But again since d preserves the filtation level, the sum of these two groups is $F^{p+1}(C^n)$.

Since the differential d of C^* preserves the filtration it induces a differential $d_0\colon E_0^{p,q} \to E_0^{p,q+1}$, i.e., an operator of bidegree $(0,1)$, which is of square zero. One definition of $E_1^{*,*}$ is that it is the cohomology of $(E_0^{*,*}, d_0)$. Let us express it in another way, along the lines we indicated above.

$$E_1^{p,q} = \frac{\{x \in F^p(C^{p+q}) \big| dx \in F^{p+1}(C^{p+q+1})\}}{F^{p+1}(C^{p+q}) + dF^p(C^{p+q-1})}.$$

Analogously to what happened before, d induces a differential $d_1\colon E_1^{p,q} \to E_1^{p+1,q}$, i.e., an operator of bidegree $(1,0)$, which is of square 0. By definition $E_2^{*,*}$ is the cohomology of $(E_1^{*,*}, d_1)$. The description in terms of quotient groups of subgroups of the original cochain complex is:

$$E_2^{p,q} = \frac{\{x \in F^p(C^{p+q}) \big| dx \in F^{p+2}(C^{p+q+1})\}}{\{y \in F^{p+1}(C^{p+q}) \big| dy \in F^{p+2}(C^{p+q+1})\} + dF^{p-1}(C^{p+q-1}) \cap F^p(C^{p+q})}.$$

From these three examples we can now see the general pattern. The general description in terms of quotients of subgroups of the original cochain complex is:

$$E_r^{p,q} = \frac{\{x \in F^p(C^{p+q}) \big| dx \in F^{p+r}(C^{p+q+1})\}}{\{y \in F^{p+1}(C^{p+q}) \big| dy \in F^{p+r}(C^{p+q+1})\} + dF^{p+1-r}(C^{p+q-1}) \cap F^p(C^{p+q})}.$$

Lemma 1.1. *d induces a differential $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$, i.e. an operator of bidegree $(r, 1-r)$, of square zero whose cohomology is $E_{r+1}^{p,q}$.*

First, a claim:

Claim 1.2. *Suppose $B \subset A$ are abelian groups and every element of A/B has a representative in a subgroup $A' \subset A$. Then the map induced by the inclusion of $A' \subset A$*

$$A'/(A' \cap B) \to A/B$$

is an isomorphism.

Proof. Certainly, the map $A'/(A' \cap B) \to A/B$ induced by the inclusion is an injection and the statement that every element in A/B has a representative in A' means that this map is onto as well. □

Now we turn to the proof of the lemma.

Proof. Let $[x] \in \mathrm{Ker}(d_r\colon E_r^{p,q} \to E_r^{p+r,q+1-r})$. Then $dx = d\alpha + \beta$ where $\alpha \in F^{p+1}$ and $d\alpha \in F^{p+r}$ and $\beta \in F^{p+r+1}$. It follows that $[x] = [x - \alpha] \in E_r^{p,q}$. This shows that every element in $\mathrm{Ker} d_r$ has a representative in the subgroup $\{x \in F^p \big| dx \in F^{p+r+1}\} \subset \{x \in F^p \big| dx \in F^{p+r}\}$. The intersection of this subgroup with the expression in the denominator of $E_r^{p,q}$ is

$$dF^{p+1-r} \cap F^p + \{y \in F^{p+1} \big| dy \in F^{p+r+1}\}.$$

Using the above claim, we see that

$$\mathrm{Ker}(d_r) = \frac{\{x \in F^p \big| dx \in F^{p+r+1}\}}{dF^{p+1-r} \cap F^p + \{y \in F^{p+1} \big| dy \in F^{p+r+1}\}}.$$

On the other hand,

$$\begin{aligned}\operatorname{Im}(d_r) &= \frac{dF^{p-r}\cap F^p + dF^{p+1-r}\cap F^p + \{y\in F^{p+1}\big| dy\in F^{p+r+1}\}}{dF^{p+1-r}\cap F^p + \{y\in F^{p+1}\big| dy\in F^{p+r+1}\}}\\ &= \frac{dF^{p-r}\cap F^p + \{y\in F^{p+1}\big| dy\in F^{p+r+1}\}}{dF^{p+1-r}\cap F^p + \{y\in F^{p+1}\big| dy\in F^{p+r+1}\}}.\end{aligned}$$

Putting these two forrmlae together we see that

$$H^*(E_r^{p,*}, d_r) = \frac{\{x\in F^p\big| dx\in F^{p+r+1}\}}{dF^{p-r}\cap F^p + \{y\in F^{p+1}\big| dy\in F^{p+r+1}\}} = E_{r+1}^{p,*}.$$

□

The finiteness assumption implies that, given n, for all r sufficiently large

$$\begin{aligned}E_r^{p,q} &= \frac{\operatorname{Ker}(d\colon F^p(C^{p+q})\to C^{p+q+1})}{\operatorname{Ker}(d\colon F^{p+1}(C^{p+q})\to C^{p+q+1}) + d(C^{p+q-1})\cap F^p(C^{p+q})}\\ &= \frac{F^p(H^{p+q}(C^*))}{F^{p+1}(H^{p+q}(C^*))}.\end{aligned}$$

This stable value is denoted $E_\infty^{p,q}$. This relationship is captured in the terminology: *$E_\infty^{p,q}$ is a composition series for $F^*(H^*(C^*))$* or *the spectral sequence converges to the cohomology of the complex.*

A spectral sequence is said to *collapse at E_r*, if $E_r^{p,q} = E_\infty^{p,q}$ for all p, q. A filtration-preserving map between filtered cochain complexes $\varphi\colon (C^*, F^*)\to (C^*, F^*)$ is said to *strictly preserve the filtration* or *be strictly compatible with the filtration* if for every p we have $\varphi(C^*)\cap F^p(C^*) = \varphi(F^p(C^*))$.

Lemma 1.3. *Let F^* be a decreasing filtration on a cochain complex C^* satisfying the finiteness condition above.*

- *The spectral sequence for this filtration collapses at E_0 if and only if the differential of the cochain complex is zero.*
- *The spectral sequence collapses at E_1 if and only if d is strictly compatible with the filtration, meaning that for all n*

$$d(C^{n-1})\cap F^r(C^n) = d(F^r(C^{n-1}))$$

Proof. Let us prove the first statement in the lemma. If $d = 0$ then the maps d_r are zero for all r and hence $E_{r+1}^{*,*} = E_r^{*,*}$ for all $r\geq 0$. Hence, the spectral sequence collapses at E_0. Conversely, suppose $d_i = 0$ for all i. The condition $d_0 = 0$ means $d(F^p) \subset F^{p+1}$. Suppose, by induction, that for some $r \geq 1$, for all p we have $d(F^p) \subset F^{p+r}$. Since the $d_i = 0$ for all i $E_r^{p,q} = E_0^{p,q} = F^p(C^{p+q})/F^{p+1}(C^{p+q})$. Thus, $d_r \colon E_r^{p,q} \to E_r^{p+r,q+1-r}$ is the map induced by d

$$F^p(C^{p+q})/F^{p+1}(C^{p+q}) \to F^{p+r}(C^{p+q+1})/F^{p+r+1}(C^{p+q+1}).$$

Since $d_r = 0$, this implies that $d(F^p) \subset F^{p+r+1}$. We conclude that we have $d(F^p) \subset F^{p+r}$ for every r. By the finiteness assumption, this implies that $d = 0$.

Now, let us consider the second statement. Suppose that d is strictly compatible with the filtration. Then, by strictness we have

$$\begin{aligned} E_1^{p,q} &= \frac{\{x \in F^p(C^{p+q}) \big| dx \in F^{p+1}\}}{dF^p(C^{p+q-1}) + F^{p+1}(C^{p+q}}) \\ &= \frac{\{x \in F^p(C^{p+q}) \big| dx \in F^{p+1}\}}{d(C^{p+q-1}) \cap F^p + F^{p+1}(C^{p+q})} \end{aligned}$$

By strictness, for any $x \in F^p$ with $dx \in F^{p+1}$, there is $y \in F^{p+1}$ with $dy = dx$. This means that every element in the second expression for $E_1^{p,q}$ has a representative $x \in F^p(C^{p+q})$ with $dx = 0$. Applying Claim 1.2 we see that

$$E_1^{p,q} = \frac{\{x \in F^p(C^{p+q}) \big| dx = 0\}}{d(C^{p+q-1}) + \{y \in F^{p+1}(C^{p+q)} \big| dy = 0\}} = E_\infty^{p,q}.$$

This proves that the spectral sequence collapses at E_1.

Conversely, suppose that $d_i = 0$ for all $i \geq 1$. We show by downward induction on p that $\mathrm{Im} d \cap F^p = dF^p$. This is clearly true in each degree for p sufficiently large. Given that in some degree it is true for $p+1$ we establish it for p. We show by (upward) induction on ℓ that $d(F^{p-\ell}) \cap F^p = dF^p$. This is clearly true for $\ell = 0$ since d is compatible with the filtration. Suppose for some $\ell \geq 1$ the statement is true for $\ell - 1$. Since the map $d_\ell \colon E_\ell^{p-\ell,q} \to E_\ell^{p,q-\ell+1}$ is trivial we see that if $x \in F^{p-\ell}$ and $dx \in F^p$, then

$$dx = d\alpha + y^{p+1},$$

where $\alpha \in F^{p-\ell+1}$ and $y \in F^{p+1}$. It follows that $y \in \mathrm{Im}\, d$ and by the (downward) inductive assumption on p, we know that $y = dz$ with $z \in F^{p+1}$. [Notice that y and x are in the same degree.] Thus, $dx \in d(F^{p-\ell+1}) \cap F^p$, and hence by the inductive hypothesis on ℓ we see that $dx \in dF^p$. This completes the induction on ℓ and hence the induction on p. □

Remark 1.4. We have been dealing with spectral sequences for decreasing filtrations on cochain complexes. There are also spectral sequences for increasing filtrations on chain complexes. Suppose that C_* is a (free abelian) chain complex with an increasing filtration F_* preserved by the boundary operator. Then there is a dual decreasing filtration F^* on the dual cochain complex C^* compatible with the dual differential defined as follows: $F^r(C^*)$ consists of those elements that vanish on $F_{r-1}(C_*)$. These spectral sequences are dual in an appropriate sense.

2 Some examples

2.1 The spectral sequence of a double complex

By a double complex we mean a bigraded group $\oplus C^{p,q}$ with two differentials, δ of bi-degree $(1,0)$ and δ' of bi-degree $(0,1)$, which anti-commute: i.e., $\delta\delta' = -\delta'\delta$. We define the associated graded total complex by $TC^n = \oplus_{p+q=n} C^{p,q}$ and $d = \delta + \delta' \colon TC^n \to TC^{n+1}$. [Check that $d^2 = 0$.]

We define a decreasing filtration on TC^* by $F^p = \oplus_{\{p',q' | p' \geq p\}} C^{p',q'}$. Clearly, d preserves this filtration. Hence there is a spectral sequence. It is clear that $E_0^{p,q} = C^{p,q}$ and $d_0 = \delta'$ so that $E_1^{p,q} = H^p(TC, \delta')$. The differential d_1 is the map induced on $H^*(TC^*, \delta')$ by δ. [The differential δ induces a map on the δ'-cohomology since it anti-commutes with δ.] The higher differentials do not have such a direct description in general. Nevertheless, under appropriate finiteness assumptions, the E_∞ term of the spectral sequence is a composition series for (i.e., the associated gradeds of a filtration of) $H^*(TC, d)$.

2.2 The Hodge-to-deRham spectral sequence

One natural geometric example of a double complex comes from the complex-valued differential forms on a complex manifold M. The complex structure determines a splitting of the cotangent bundle: $T^*M \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ where $T^{1,0}$ is the complex linear maps $TM \to \mathbb{C}$ and $T^{0,1}$ are the complex anti-linear maps to $\mathbb{C}$. In local holomorphic coordinates $(z_1, \ldots, z_n)$ with $z_j = x_j + iy_j$, a section of $T^{1,0}$ is written as $\sum_i f_i(x_1, y_1, \ldots, x_n, y_n) dz_i$ with the f_i being C^∞ complex-valued functions. Similarly, sections of $T^{0,1}$ are written in local coordinates as $\sum_i g_i d\overline{z}_i$ for C^∞-functions g_i. Taking exterior powers we have a decomposition of the complex-valued k-forms $\Omega^k(M; \mathbb{C})$ as $\oplus_{p+q=k} \Omega^{p,q}$. A term is $\Omega^{p,q}$ is a sum of C^∞-functions times the wedge product of p of the dz_i and q of the $d\overline{z}_j$; such elements are referred to *as being of type* (p,q).

The differential is $\partial + \overline{\partial}$, where ∂ is differentiation with respect to the holomorphic coordinates and $\overline{\partial}$ is differentiation with respect to the anti-holomorphic coordinates.

This then is a double complex with $\delta' = \overline{\partial}$ and $\delta = \partial$. The complex $(E_0^{p,*}, d_0)$

$$\Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \to \cdots .$$

Since this complex is a soft resolution of the sheaf of holomorphic differentials Ω^p [for the sheaf Ω^p the local coefficients are required to be holomorphic functions]. We have

$$E_1^{p,q} = H^q(\Omega^p).$$

The spectral sequence converges to the complex-valued deRham cohomology of the manifold M.

This spectral sequence is called the *Hodge-to-deRham spectral sequence.* There is an amazing theorem, which is the beginning of Hodge theory, that we will discuss in a later lecture which says:

Theorem 2.1. *Let M be a compact Kähler manifold, e.g., a smooth complex projective variety. The the Hodge-to-deRham spectral sequence collapses at E_1. Furthermore, there is a natural (for holomorphic maps) splitting $E_1^{p,q} = E_\infty^{p,q}$ back into $H^{p+1}(M;\mathbb{C})$ given by the cohomology classes represented by closed forms of type (p,q). In particular, $H^n(M;\mathbb{C}) = \oplus_{p,q} H^{p,q}(M)$ where $H^{p,q}(M)$ are the classes with cocycle representatives of type (p,q).*

2.3 The Serre spectral sequence of a fibration

In dealing with the singular (co)-homology of fibrations it is convenient to use singular cubes instead of singular simplices. At first glance, it seems that everything should work the same. You have natural boundary maps for cubes so one can form a chain complex generated by singular cubes in a topological space. The exact sequence of a pair works as before, and since you can subdivide cubes into arbitrarily small subcubes, the Meyer-Vietoris axiom (or equivalently excision) holds. The twist is that the dimension axiom does not hold: The complex of singular cubes mapping to a point has a $\mathbb{Z}$ in each degree (for every $n \geq 0$ there is exactly one singular n-cube mapping to a point) but the boundary map is zero! Thus, the homology of a point is not correct. To remedy this one considers singular cubes that are non-degenerate. A *degenerate* singular cube is a map $I^n \to X$ which factors through the projection $I^n \to I^{n-1}$ given by $(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_{n-1})$ (projecting out the last coordinate). It is easy to see that the boundary of a degenerate cube is a sum of degenerate cubes. [Proof: All faces are degenerate except possibly the two faces obtained by setting the n^{th}-coordinate equal to zero and 1. Because the original cube is degenerate, these singular cubes agree and since they occur with opposite sign is the expression for the boundary, they cancel out.]

We denote by $\mathrm{Cube}_*(X)$ the chain complex generated by singular cubes, non-degenerate or not, (with the obvious boundary). By what I just said, $\mathrm{Cube}_*(X)$ does not compute the singular homology of X, but the graded subgroup $\mathrm{DCube}_*(X)$ generated by the degenerate cubes is a subcomplex and the quotient $CC_*(X) = \mathrm{Cube}_*(X)/\mathrm{DCube}C_*(X)$, called the *cubical chain complex*, does compute the singular homology of X.

The cubical cochain complex is the dual complex. It can be thought of as $\mathbb{Z}$-valued functions on singular cubes that vanish on degenerate cubics. The cubical cohomology is the cohomology of the dual cochain complex of X and is identified with the singular cohomology of X.

Fix a Serre fibration $\pi\colon E \to B$. We make the following assumption:
Assumption: We assume that the base of the fibration, B, is path connected and that the action of $\pi_1(B, b_0)$ on the homology and cohomology of the fiber $F_{b_0} = \pi^{-1}(b_0)$ is trivial, so that the homologies, resp, cohomologies, of all fibers of π are canonically identified.

Now we turn to the increasing filtration.

Definition 2.2. For each $0 \le p \le n$ denote by $\mathrm{proj}_{p,n}\colon I^n \to I^p$ be the map given by $(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_p)$. A singular $f\colon I^{p+q} \to E$ is in *filtration level* p if $\pi \circ f$ factors through $\mathrm{proj}_{p,p+q}$, i.e., if there is a commutative diagram

$$\begin{array}{ccc} I^{p+q} & \longrightarrow & E \\ {\scriptstyle \mathrm{proj}_{p,p+q}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ I^p & \longrightarrow & B. \end{array}$$

This defines increasing filtations:

$$0 = F_{-1}(\mathrm{Cube}_n(E)) \subset F_0(\mathrm{Cube}_n(E)) \subset \cdots \subset F_n(\mathrm{Cube}_n(E)) = \mathrm{Cube}_n(E).$$

The boundary map is easily seen to be compatible with this filtration, so that F_* induces an increasing filtration on the chain complex $\mathrm{Cube}_*(E)$. There is an induced increasing filtration, also denoted F_*, of the cubical chain complex $CC_*(E)$ and dually a decreasing filtration

$$CC^n(E) = F^0(CC^n(E)) \supset \cdots \supset F^n(CC^n(E)) \supset F^{n+1}(CC^n(E)) = 0.$$

Associated to any singular cube $\sigma\colon I^{p+q} \to E$ of filtration level p there is the unique map $\overline{\sigma}_p\colon I^p \to B$ satisfying

$$\pi \circ \sigma = \overline{\sigma}_p \circ \mathrm{proj}_{p,p+q}\colon I^{p+q} \to I^p \to B.$$

Claim 2.3. *A singular cube $\sigma\colon I^{p+q} \to E$ in filtration level p is contained in filtration level $p-1$ if and only if $\overline{\sigma}_p$ is degenerate.*

Proof. Let σ be a singular cube of filtration level p. Then σ is in filtration level $p-1$ if and only if $\pi \circ \sigma$ can be written as $\overline{\sigma}_{p-1} \circ \mathrm{proj}_{p-1,p+q}$ for some map $\overline{\sigma}_{p-1}\colon I^{p-1} \to B$. This is true if and only if $\overline{\sigma}_p = \overline{\sigma}_{p-1} \circ \mathrm{proj}_{p-1,p}$ for some map $\overline{\sigma}_{p-1}\colon I^{p-1} \to B$, which is true if and only if $\overline{\sigma}_p$ is degenerate. □

The map π induces a function from the set of singular cubes in E of filtration level p to the set of singular p-cubes in B. The singular cubes of filtration level p whose image is a degenerate singular p-cube in B are exactly those of filtration level $p-1$. Thus, we have a decomposition

$$F_p(\mathrm{Cube}_*(E))/F_{p-1}(\mathrm{Cube}_*(E)) = \oplus_{\tau_p}\big(F_p(\mathrm{Cube}_*(E))/F_{p-1}(\mathrm{Cube}_*(E))\big)_{\tau_p},$$

where τ_p ranges over the non-degenerate p-cubes in B and the summand indexed by τ_p is the subgroup generated by singular cubes σ in E of filtration level p with $\overline{\sigma}_p = \tau_p$.

Claim 2.4. *In the associated spectral sequence for $(\mathrm{Cube}_*(E), F_*)$ the differential d_0 preserves this decomposition.*

Proof. Consider a singular n-cube $\sigma\colon I^n \to E$ of filtration level p with $\overline{\sigma}_p\colon I^p \to B$ non-degenerate. We have

$$\partial\sigma = \sum_{i=0}^{n}(-1)^i\big(\{t_1,\ldots,t_{i-1},1,t_{i+1},\ldots,t_n\} - \{t_1,\ldots,t_{i-1},0,t_{i+1},\ldots,t_n\}\big).$$

The terms indexed by $0 \le i \le p$ are in filtration level $p-1$ and hence are zero in $E^0_{p,q-1}$. Those indexed by $p+1 \le i \le n$ are of filtration level p and their projection to B factors through $\overline{\sigma}_p$. The claim follows. □

Now for each non-degenerate map $\tau_p\colon I^p \to B$ let $F_{\tau_p} = \pi^{-1}(\tau_p(0,\ldots,0))$ be the fiber over the initial corner of τ_p. For each such τ_p we define a map

$$R_{\tau_p}\colon \big(F_p(\mathrm{Cube}_{p+q}(E))/F_{p-1}(\mathrm{Cube}_{p+q}(E))\big)_{\tau_p} \to \mathrm{Cube}_q(F_{\tau_p})$$

by sending any σ with $\overline{\sigma}_p = \tau_p$ to

$$\sigma|_{\underbrace{\{0,\ldots,0\}}_{p-\text{times}}\times I^q}.$$

(Since $\overline{\sigma}_p = \tau_p$ the restriction of σ to $\underbrace{\{0,\ldots,0\}}_{p-\text{times}}\times I^q$ maps to the fiber F_{τ_p}.)

Now it is time to see how to divide out by degenerate cubes.

Claim 2.5. *Suppose that σ^n is degenerate and that $\pi\circ\sigma^n$ factors through τ_p for a non-degenerate map $\tau_p\colon I^p \to B$. Then $R_{\tau_p}(\sigma)$ is a degenerate cube in F_{τ_p}*

Proof. First of all, since τ_p is non-degenerate and σ^n is degenerate, $p < n$. Thus, $R_{\tau_p}(\sigma)$ is the restriction of σ to a positive dimensional cube $\underbrace{\{0,\ldots,0\}}_{r-\text{times}}\times I^{n-p}$. Since σ is degenerate, so is its restriction to $\underbrace{\{0,\ldots,0\}}_{r-\text{times}}\times I^{n-p}$. ☐

Corollary 2.6. *For each non-degenerate map $\tau_p\colon I^p \to B$ the map*

$$R_{\tau_p}\colon \big(F_p(CC_*(E))/F_{p-1}(CC_*(E))\big)_{\tau_p} \to CC_{*-p}(F_{\tau_p})$$

sends d_0 to $(-1)^p\partial$.

Proof. The previous claim shows that degenerate cubes map to degenerate cubes under R_{τ_p} so that there is a well-defined map on the quotient groups. We computed that

$$d_0(\sigma^n) = (-1)^p\sum_{i=0}^{n-p}\big((t_1,\ldots,t_{p+i},1,t_{p+i+2},\ldots,t_n)-(t_1,\ldots,t_{p+i},0,t_{p+i+2},\ldots,t_n)\big),$$

so that $R_{\tau_p}(d_0(\sigma^n)) = (-1)^p\partial R_{\tau_p}(\sigma)$. ☐

Since d_0 preserves the decomposition $\oplus_{\tau_p}\big(F_p(CC_*(E))/F_{p-1}(CC_*(E))\big)_{\tau_p}$ of $E^0_{p,*}$, it follows that there is a decomposition

$$E^1_{p,q} = \oplus_{\tau_p}\big(E^1_{p,q}\big)_{\tau_p}$$

and that for each non-degenerate cup $\tau_p\colon I^p \to B$ we have a map

$$(R_{\tau_p})_*\bigl(E^1_{p,q}\bigr)_{\tau_p} \to H_q(F_{\tau_p}).$$

Using the Serre homotopy extension property we shall define a map

$$J_{\tau_p}\colon CC_*(F_{\tau_p}) \to \bigl(F_p(CC_*(E))/F_{p-1}(CC_*(E))\bigr)_{\tau_p}$$

which is a chain map when we use the differentials $(-1)^p\partial$ on $CC_*(F_{\tau_p})$ and d_0 on $\bigl(F_p(CC_*(E))/F_{p-1}(CC_*(E))\bigr)_{\tau_p}$. We shall do this in a way such that $R_{\tau_p}\circ J_{\tau_p}$ is the identity on $CC_*(F_{\tau_p})$. We construct J_{τ_p} by induction on the dimension of non-degenerate cubes in F_{τ_p}. The Serre homotopy extension applied inductively to $I^r\times\{\underbrace{0,\dots,0}_{p-r}\}$, allows us to extend any map $I^0\to F_{\tau_p}$ to a map $I^p\to E$ projecting to τ_p. Suppose by induction for all $s<q$ we have extended all maps of $I^s\to F_{\tau_p}$ to maps $I^p\times I^s\to E$ covering τ_p compatible with the boundary maps. Let $I^q\to F_{\tau_p}$ be given. The inductive hypothesis tells us that we have an extension

$$\bigl((\underbrace{0,\dots,0}_{p-\text{terms}})\times I^q\bigr)\cup\bigl(I^p\times\partial I^q\bigr)\to E$$

projecting to τ_p. Suppose, by induction on r, that we have an extension of this map over $I^r\times I^q$ so that on this subset the map projects to $\tau_p|I^r$. Then we have a map defined on

$$(I^r\times I^q)\cup(I^{r+1}\times\partial I^q)$$

projecting to τ_p and we wish to extend it to a map on $I^{r+1}\times I^q$ projecting to $\tau_p|_{I^{r+1}}$. This uses the Serre homotopy extension property and the fact that there is a homeomorphism

$$\bigl((I^r\times I^q)\cup(I^{r+1}\times\partial I^q)\bigr)\times I\to I^{r+1}\times I^q$$

which is the inclusion on

$$\bigl((I^r\times I^q)\cup(I^{r+1}\times\partial I^q)\bigr)\times\{0\}.$$

This completes the induction and constructs the chain map J_{τ_p} as required. (The sign $(-1)^p$ is required since the factor I^p comes in front of the chain in F_{τ_p}.)

Clearly, $R_{\tau_p}\circ J_{\tau_p}$ is the identity map of $CC_*(F_{\tau_p})$. Now we show that $J_{\tau_p}\circ R_{\tau_p}$ is chain homotopic to the identity.

By induction on q suppose that for any $r < q$ and any $\varphi\colon I^{p+r} \to E$ projecting to τ_p we have a map

$$K_{\tau_p}\colon I^{p+r} \times I \to E$$

with the following properties:

- $K_{\tau_p} | I^{p+r} \times \{0\} = \varphi$,
- $K_{\tau_p} | I^{p+r} \times \{1\} = J_{\tau_p} \circ R_{\tau_p}(\varphi)$
- K_{τ_p} followed by the projection to B is the map $I^{p+r} \times I \to I^p \xrightarrow{\tau_p} B$,
- $K_{\tau_p} \big| \big((\underbrace{0, \ldots, 0}_{p-\text{times}}) \times I^r \times I \big)$ is the constant homotopy from $R(\varphi)$ to itself, and
- the maps K_{τ_p} are compatible with all the boundary maps on I^r.

For $r = 0$, we have to show that given two maps $I^p \to E$ that agree on $(\underbrace{0, \ldots, 0}_{p-\text{times}})$ and that cover τ_p there is a homotopy between them that projects to the constant homotopy in B. This is a direct application of the Serre homotopy extension property. Now suppose we have maps K_{τ_p} as above for all $r < q$. We construct the map K_{τ_p} as required for each $\varphi\colon I^{p+q} \to E$ covering τ_p. The inductive hypothesis implies that we have

$$K_{\tau_p}\colon I^p \times \partial I^q \times I \to E$$

satisfying all the properties (for $I^p \times \partial I^q$ replacing I^{p+q}). By induction on r we extend the restriction of this map to $I^r \times \partial I^q \times I$ over $I^r \times I^q \times I$ satisfying all the properties above when $I^r \times I^q \times I$ replaces $I^p \times I^q \times I$. For $r = 0$ we already have the extension since the map is required to be the constant homotopy on $I^0 \times I^q$. Now suppose that for some $r \geq 1$ we have the extension over $I^{r-1} \times I^q \times I$ as required. Now we have a map as required defined on

$$(I^{r-1} \times I^q \times I) \cup (I^r \times \partial^q \times I) \cup (I^r \times I^q \times \partial I) = \big(I^{r-1} \times (I^q \times I)\big) \cup \big(I^r \times \partial (I^q \times I)\big).$$

Since this subset of $I^r \times I^p \times I$ deformation retracts to $I^{r-1} \times I^q \times \{0\}$, applying the Serre homotopy extension property again, we see that the required extension over $I^r \times I^q \times I$ exists. This completes the inductive step and establishes the existence of the homotopy K_{τ_p} as required.

It follows that R_{τ_p} and J_{τ_p} induce inverse isomorphisms from $(E^1_{p,q})_{\tau_p}$ to $H_q(F_{\tau_p})$. Because the action of the fundamental group of B on the homology of the fiber is trivial, there is a canonical identification ι_{τ_p} of $H_*(F_{\tau_p})$ with $H_*(F_0)$ where F_0 is the fiber over some chosen basepoint. For each non-degenerate $\tau_p \colon I^p \to B$, sending $\sigma \in \big(F_p(CC_*(E))/F_{p-1}(CC_*(E))\big)_{\tau_p}$ to $\tau_p \otimes \iota_{\tau_p} R_{\tau_p}(\sigma) \in CC_p(B) \otimes CC_{*-p}(F_0)$ produces an isomorphism

$$E^1_{p,q} \to CC_p(B) \otimes H_q(F_0)$$

Next, we identify d_1. Let ζ be a q-cycle in F_{τ_p} and consider $J_{\tau_p}(\zeta)$. Let us denote by $[\zeta]_0 = \iota_{\tau_p}([\zeta]) \in H_*(F_0)$. Then $[J_{\tau_p}(\zeta)] \in E^1_{p,q}$ corresponds to $\tau_p \otimes [\zeta]_0$ in $CC_*(B) \otimes H_*(F_0)$ under the isomorphism just constructed.

We denote $\partial \tau_p = \sum_a (-1)^{\epsilon(a)} (\tau_p)_a$ where a indexes the codimension-1 faces of I^r. Since ζ is a cycle $\partial J_{\tau_p}(\zeta)$ is a cycle projecting to $\partial \tau_p$ and

$$d_1([J_{\tau_p}\zeta)]) = \sum_a (-1)^{\epsilon(a)} (\tau_p)_a \otimes [J_{\tau_p}(\zeta)|_{F_{(\tau_p)_a}}].$$

For any corner c on I^p let γ be a path consisting of edges of I^p connecting $\underbrace{(0,\ldots,0)}_{p-\text{times}}$ to c. The restriction of $J_{\tau_p}(\zeta)$ to the pre-image under the projection mapping of the image under τ_p of this 1-chain produces a $(q+1)$-chain whose boundary is the difference of the cycle ζ in F_{τ_p} (which is, recall, the intersection of $J_{\tau_p}(\zeta)$ with F_{τ_p}) and the cycle in $F_c = p^{-1}(c)$ that is the intersection of $J_{\tau_p}(\zeta)$ with F_c. This proves that the resulting cycles in F_{τ_p} and F_c represent the same homology class under the canonical identifications of the homology of each of these fibers with of $H_*(F_0)$. Thus, under these canonical identifications the above equation for $d_1([J_{\tau_p}(\zeta)])$ yields

$$d_1(\tau_p \otimes [\zeta]_0) = d_1([J_{\tau_p}(\zeta)]) = \sum_a (-1)^{\epsilon(a)} (\tau_p)_a \otimes [\zeta]_0 = \partial \tau_p \otimes [\zeta]_0$$

where ∂ is the boundary map for $CC_*(B)$. Of course, this is exactly the boundary map of the chain complex $CC_*(B) \otimes H_*(F_0)$, giving us the following corollary.

Corollary 2.7. *Let $p \colon E \to B$ be a Serre fibration with B path connected and with trivial action of $\pi_1(B, b_0)$ on the homology of the fiber $F_0 = p^{-1}(b_0)$. The E^2-term of the Serre spectral sequence for this Serre fibration is identified with*

$$E^2_{p,q} = H_p(B; H_q(F_0)).$$

There are similar arguments for the dual cohomology spectral sequence yielding:

Proposition 2.8. *With the hypotheses in the previous corollary, the dual spectral sequence for $CC^*(E)$ has*

$$E_1^{p,q}(CC^*(E)) = \mathrm{Hom}(CC^p(B), H^q(F_0)),$$

and

$$E_2^{p,q} = H^p(B; H^q(F)).$$

These two spectral sequences are called the *homology, resp. cohomology, Serre spectral sequence for the fibration.* Each is a first quadrant spectral sequence and converges to the $H_*(E)$, resp $H^*(E)$.

Exercise 1: Let $\pi\colon E \to B$ be a Serre fibration and suppose that B is a CW complex. Filter the singular chains of E by setting $F_p(Sing_*(E))$ equal to the image of $Sing_*(\pi^{-1}B^{(p)})$ in $Sing_*(E)$. Take the dual filtration on singular cochains. Show that this filtration also leads to the Serre spectral sequence.

Exercise 2: Show that in general for path connected base that the E_2-term of the Serre spectral sequence is $H^p(B;\mathcal{H}^q(F))$, where $\mathcal{H}^*(F)$ is the locally constrant system of cohomologies of the fibers.

Exercise 3: Suppose that $\pi\colon M \to N$ is a smooth map between smooth manifolds that is a locally trivial smooth fibration. Define a filtration on the differential forms of M by setting $F^P(\Omega^n(M))$ equal to the subspace of forms that vanish pointwise on any collection of tangent vectors at least $n-p+1$ of which are vertical, i.e., in the kernel of $d\pi$. Show that the spectral sequence for this fibration is Serre spectral sequence with real coefficients.

2.4 The Atiyah-Hirzebruch spectral sequence

This spectral sequence is for extraordinary cohomology theories h^*, such as K-theory and cobodism theory. We have $E_2^{p,q}(X) = H^p(X; h^q(\{pt\}))$ and it converges to $h^*(X)$. Let me consider the homology version and bordism.

Notice that in general there is no functorial chain complex whose homology is the extraordinary homology, so the basic construction of the spectral sequence is different from all those we have encountered so far. We use a more general method, due to William Massey, called **exact couples.**

We consider the bordism of a space, denote $\Omega_*(X)$. By definition $\Omega_n(X)$ is the equivalence classes of closed, oriented, smooth n-manifolds mapping to X, where $f_0\colon M_0 \to X$ and $f_1\colon M_1 \to X$ are equivalent if there is a map

$F\colon W^{n+1} \to X$ where W^{n+1} is a compact, oriented, smooth manifold with $\partial W = M_1 - M_0$ and $F|_{\partial W} = f_0 \coprod f_1$. (Here, the minus sign means reverse the orientation of the manifold.) The group structure is given by disjoint union of manifolds and maps. The identity element is the unique map of the empty manifold into the space. The inverse (i.e., negative) of an element is obtained by reversing the orientation.

Form the triangle

$$\begin{array}{ccc} \oplus_p \Omega_*(X^{(p)}) & \xrightarrow{\quad i \quad} & \oplus_p \Omega_*(X^{(p)}) \ . \\ & {}_{\partial}\nwarrow \qquad \swarrow_{j} & \\ & \oplus_p \Omega_*(X^{(p)}, X^{(p-1)}) & \end{array}$$

The map i is induced by the inclusions $X^{(p)} \subset X^{(p+1)}$ the map j is induced by the inclusions $X^{(p)} \subset (X^{(p)}, X^{(p-1)})$, and ∂ is the sum of the boundary maps $\Omega_*(X^{(p)}, X^{(p-1)}) \to \Omega_{*-1}(X^{(p-1)})$.

Notice that each of the three terms is bigraded by p and $q = * - p$, so that the total degree $p + q = *$. The bidegrees of the maps are as follows:

- i has bi-degree $(+1, 0)$
- j has bi-degree $(0, 0)$
- ∂ has bi-degree $(-1, 0)$.

Notice that what makes this triangle *exact* is that it is exact in the usual sense at each vertex.

The term in the bottom apex of bi-degree (p, q) is $E^1_{p,q}$. By the suspension isomorphism this term is identified with $C_p(X) \otimes \Omega_q(\{pt\})$ where $C_p(X)$ is the group of CW chains on X. One an also see this as follows: Any map $(M^{p+q}, \partial M^{p+q}) \to (D^p, \partial D^p)$ can be deformed slightly to be transverse to the central point of the disk. The pre-image is a closed manifold Y^q. This association gives the identification of $E^1_{p,q}$ with $C_p(X) \otimes \Omega_q(\{pt\})$.

The differential $d_1\colon E^1_{p,q} \to E^1_{p-1,q}$ is the composition $j \circ \partial$ (notice that the bi-degree of the differential is $(-1, 0)$). Thus,

$$E^2_{p,q} = \frac{\mathrm{Ker}(j \circ \partial)\colon \Omega_{p+q}(X^{(p)}, X^{(p-1)}) \to \Omega_{p+q-1}(X^{(p-1)}, X^{(p-2)})}{\mathrm{Im}(j \circ \partial)\colon \Omega_{p+q+1}(X^{(p+1)}, X^{(p)}) \to \Omega_{p+q}(X^{(p)}, X^{(p-1)})}.$$

By the suspension isomorphism in extraordinary homology we have identifications

$$\Omega_{p+q}(X^{(p)}, X^{(p-1)}) = \oplus_e \mathbb{Z}[e] \otimes \Omega_q(\{pt\}),$$

where e ranges over the p-cells of X and $[e]$ is the generator of the relative homology group determined by e with a fixed orientation. Under these identifications the map

$$\Omega_{p+q}(X^{(p)}, X^{(p-1)}) \to \Omega_{p+q-1}(X^{(p-1)}, X^{(p-2)})$$

is identified with $\partial \otimes \mathrm{Id}$ where ∂ is the boundary map in the CW chain complex of X. Hence, we identify $E^2_{p,q} = H_p(X; \Omega_q(\{pt\}))$.

The $E^2_{p,q}$-term is also naturally identified with the image of

$$\Omega_{p+q}(X^{(p)}, X^{(p-2)}) \to \Omega_{p+q}(X^{(p)}, X^{(p-1)})$$

modulo the image of

$$\partial \colon \Omega_{p+q+1}(X^{(p+1)}, X^{(p)}) \to \Omega_{p+q}(X^{(p)}, X^{(p-1)}).$$

That is to say, $E^2_{p,q}$ is the image in $E^1_{p,q}$ of compact $(p+q)$-manifolds mapping to $X^{(p)}$ whose boundary maps to $X^{(p-2)}$ modulo closed $(p+q)$-manifolds mapping to $X^{(p)}$ that are boundaries of compact $(p+q+1)$-manifolds mapping to $X^{(p+1)}$.

At this point the construction iterates. Let us simplify the notation by relabeling the diagram

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & A \\ & {\scriptstyle\partial}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & C & \end{array}.$$

We replace this exact couple by another one, the derived exact couple:

$$\begin{array}{ccc} A' & \xrightarrow{\;i'\;} & A' \\ & {\scriptstyle\partial'}\nwarrow \quad \swarrow{\scriptstyle j'} & \\ & C' & \end{array}$$

where $A' \subset A$ is the image of i, the map i' is $i\big|_{A'}$. The term C' is the homology of $d_1 = j \circ \partial$. The map ∂' is induced from ∂. [Proof: For any $x \in \mathrm{Ker}(j \circ \partial)$ exactness implies that $\partial(x) \in \mathrm{Ker}(j) = \mathrm{Im}(i)$, so that $\partial(x) \in A'$. If $x \in \mathrm{Im}(j \circ \partial)$ then $\partial(x) = 0$.] Lastly, j' is defined as follows. For $x \in A'$ we write $x = i(y)$ and define $j'(x) = j(y)$. First notice that $j(y) \in \mathrm{Ker}(\partial) \subset \mathrm{Ker}(j \circ \partial)$. Secondly, notice that the indeterminacy of y is $\mathrm{Im}(\partial)$ and hence modulo $\mathrm{Im}(j \circ \partial)$ the class of $j(y)$ depends only on x not the lift y of x. This shows that $j' \colon A' \to C'$ is well-defined. One continues inductively in this manner replacing an exactly couple by its derived exact couple.

It is fairly easy to work out what the higher terms are analogously to what we did for E^2. Namely $E^k_{p,q}$ is the quotient of two subgroups of $E^1_{p,q}$. The first is the image in this group of compact $(p+q)$-manifolds mapping to $X^{(p)}$ whose boundary maps to $X^{(p-k)}$ modulo closed manifolds mapping to $X^{(p)}$ which are boundaries of compact $(p+q+1)$-manifolds mapping into X^{p+k-1}. Clearly, for $k > p$ the first group consists of the image in $E^1_{p,q}$ of all closed $(p+q)$-manifolds mapping to X^p and for $k > q$, the second group consists of closed manifolds in $X^{(p)}$ that are boundaries of compact manifolds mapping into X. This is exactly the associated graded of the filtration on $\Omega_{p+q}(X)$ defined by

$$F_p(\Omega_{p+q}(X)) = \mathrm{Im}(\Omega_{p+q}(X^{(p)} \to \Omega_{p+q}(X)).$$

Exercise: Show how a filtered complex defines an exact couple whose spectral sequence agrees (from E_1 onward) with the spectral sequence of a filtered complex defined previously in this lecture.