Lecture 3: Products in Cohomology

October 22, 2018

The singular cohomology of a space has extra structure. Namely, it is a graded ring with a signed-symmetric multiplication called *cup product*. Interestingly, unlike the case of real cohomology and differential forms where there is a nice multiplication on forms that leads to a multiplication on deRham cohomology, there is no canonical cochain formula for multiplication (we will see why next lecture). Nevertheless, there is a fairly standard formula on cochains, called the *Whitney product*, that does lead to the cup product on cohomology, but this cochain formula is not signed-symmetric. As we shall see in the next lecture, there can be no such signed-symmetric formula on the cochain level.

For this lecture we fix a topological space X. All chains (resp., cochains) are singular chains (resp., cochains) of X. Likewise, all singular simplices are singular simplices of X.

Definition 0.1. Given cochains α^p and β^q their *Whitney cup product* is given by the following formula for any $(p+q)$ singular simplex σ:

$$\langle \alpha^p \cup \beta^q, \sigma_{p+q} \rangle = \langle \alpha^p, fr_p(\sigma) \rangle \cdot \langle \beta^q, bk_q(\sigma) \rangle,$$

where $fr_p(\sigma)$ is the front p-face of σ, i.e., the restriction of σ to the face spanned by the first $p+1$ vertices of the standard simplex, and $bk_q(\sigma)$ is the back q-face of σ, i.e., the restriction of σ to the face spanned by the last $q+1$ vertices of the standard simplex. Notice that these two vertex sets have one vertex in common – the last vertex of the front p-face is the first vertex of the back q-face.

Lemma 0.2. *The Whitney cup product is bilinear, associative and satisfies:*

$$d(\alpha^p \cup \beta^q) = (d\alpha^p) \cup \beta^q + (-1)^p \alpha^p \cup (d\beta^q).$$

The 0*-cocycle that evaluates* 1 *on each singular* 0*-simplex is a two-sided unit for this product.*

Proof. It is clear from the definition that the formula for cup product is bilinear, associative and has the claimed element as a two-sided unit.

To prove the formula for $d(\alpha\cup\beta)$ let us view the formula for cup product in a slightly different way. The tensor product $Sing_*(X)\otimes Sing_*(X)$ is a chain complex with boundary map given by

$$\partial(a_p\otimes b_q)=\partial a_p\otimes b_q+(-1)^p a_p\otimes\partial b_q.$$

$Sing^*(X)\otimes Sing^*(X)$ is a cochain complex with $d(\alpha^p\otimes\beta^q)=d\alpha^p\otimes\beta^q+(-1)^p\alpha^p\otimes d\beta^q$. The duality pairing between these is given by

$$\langle\alpha\otimes\beta,a\otimes b\rangle=\langle\alpha,a\rangle\langle\beta,b\rangle.$$

We define the *Whitney co-product* as follows: for each n simplex $\sigma\colon\Delta^n\to X$ we define

$$Wh(\sigma)=\sum_k fr_k(\sigma)\otimes bk_{n-k}(\sigma)\in Sing_*(X)\otimes Sing_*(X).$$

We extend linearly to define $Wh\colon Sing_*(X)\to Sing_*(X)\otimes Sing_*(X)$.

The formula for the duality pairing yields:

$$\alpha\cup\beta=Wh^*(\alpha\otimes\beta), \tag{0.1}$$

i.e., for every singular simplex σ we have

$$\langle\alpha\otimes\beta,Wh(\sigma)\rangle=\langle\alpha\cup\beta,\sigma\rangle.$$

Claim 0.3. *Wh is a map of chain complexes, i.e., $\partial Wh(\sigma)=Wh(\partial\sigma)$.*

Proof. Let σ be a singular n-simplex. Then

$$\begin{aligned}\partial Wh(\sigma)&=\sum_{k=0}^{n}\partial\big(fr_k(\sigma)\otimes bk_{n-k}(\sigma)\big)\\&=\sum_{k=0}^{n}\Big(\sum_{\ell=0}^{k}(-1)^\ell\partial_\ell fr_k(\sigma)\otimes bk_{n-k}(\sigma)+(-1)^k\sum_{\ell=0}^{n-k}(-1)^\ell fr_k(\sigma)\otimes\partial_\ell bk_{n-k}(\sigma)\Big)\end{aligned}$$

In these expressions most terms have exactly one repeated index and omit one index, but there are some terms that have no repeated index and omit no indices. The latter are the terms

$$\sum_k\big[(-1)^k\partial_k fr_k(\sigma)\otimes bk_{n-k}(\sigma)+(-1)^k fr_k(\sigma)\otimes\partial_0 bk_{n-k}(\sigma)\big],$$

and these terms cancel in (telescoping) pairs. We are left with:

$$\partial Wh(\sigma) = \sum_{k=0}^{n} \Big(\sum_{\ell=0}^{k-1} (-1)^{\ell} \partial_{\ell} fr_k(\sigma) \otimes bk_{n-k}(\sigma) + (-1)^k \sum_{\ell=1}^{n-k} (-1)^{\ell} fr_k(\sigma) \otimes \partial_{\ell} bk_{n-k}(\sigma) \Big). \tag{0.2}$$

This expression consists of a linear combination with signs ± 1 of all terms that have an omitted index and a repeated index with the first factor being the simplex spanned by all the unomitted vertices less than or equal to the repeated vertex and the second factor being the simplex spanned by all the unomitted vertices greater than or equal to the repeated vertex. The sign of the term with k as repeated vertex and ℓ as deleted vertex is $(-1)^{\ell}$.

We claim that $Wh(\partial\sigma)$ is equal to the Expression 0.2. We have

$$Wh(\partial\sigma) = \sum_{\ell} (-1)^{\ell} Wh(\partial_{\ell}\sigma)$$

where $\partial_{\ell}\sigma$ is the ℓ^{th}-face of σ. We also have

$$Wh(\partial_{\ell}\sigma) = \sum_{k} fr_k(\partial_{\ell}\sigma) \otimes bk_{n-k-1}(\partial_{\ell}\sigma).$$

This gives all terms with ℓ as omitted vertex and any repeated vertex other than k. Thus, summing over ℓ and k we have the same set of terms as in $\partial Wh(\sigma)$. Since the sign of all the terms that have ℓ as the omitted vertex in both expressions is $(-1)^{\ell}$, we see that $Wh(\partial\sigma) = \partial Wh(\sigma)$.

□

Claim 0.4. *For cochains α^p and β^q and for a singular $p+q$-simplex σ we have*

$$\langle d\alpha \otimes \beta + (-1)^p \alpha \otimes d\beta, Wh(\sigma) \rangle = \langle \alpha \otimes \beta, Wh(\partial\sigma) \rangle.$$

Proof. It is a direct computation to show that the value of $\alpha \otimes \beta$ on the Expression 0.2 is equal to

$$\langle d\alpha \otimes \beta + (-1)^p \alpha \otimes d\beta, Wh(\sigma) \rangle.$$

□

From these claims, we have

$$\begin{aligned}\langle d\alpha \cup \beta + (-1)^p \alpha \cup \beta, \sigma\rangle &= \langle \alpha \otimes \beta, Wh(\partial\sigma)\rangle \\ &= \langle \alpha \cup \beta, \partial\sigma\rangle \\ &= \langle d(\alpha \cup \beta), \sigma\rangle\end{aligned}$$

This completes the proof of Lemma 0.2. □

Corollary 0.5. *The Whiney cup product induces a well-defined product on cohomology*

$$H^p(X) \otimes H^q(X) \to H^{p+q}(X)$$

which defines an associative ring structure. This ring has a two-sided unit which is the cohomology class of the 0*-cocycle evaluating* 1 *on each singular* 0*-simplex. The cohomology ring structure is natural for continuous maps between topological spaces. That is to say the cohomology ring is a functor from the topological category (indeed from the homotopy category) to the category of graded rings with unit.*

It is not clear at all from the definition of the Whitney cup product how $\alpha \cup \beta$ and $\beta \cup \alpha$ are related. At the cochain level there is no obvious relationship. Nevertheless, we have:

Proposition 0.6. *The cup product on cohomology is graded commutative; namely, for cohomology classes* $[\alpha^p]$ *and* $[\beta^q]$ *we have:*

$$[\alpha^p] \cup [\beta^q] = (-1)^{pq} [\beta^q] \cup [\alpha^p].$$

The proof of this result will lead naturally in the next lecture to the construction of the Steenrod squares, so I want to examine it somewhat carefully.

Proof. Define $T\colon Sing_*(X) \otimes Sing_*(X) \to Sing_*(X) \otimes Sing_*(X)$ by

$$T(\sigma_p \otimes \tau_q) = (-1)^{pq} \tau_q \otimes \sigma_p.$$

It is easy to see that T is a map of chain complexes.

According to Equation 0.1 we have

$$\langle \alpha \cup \beta, \sigma\rangle = \langle \alpha \otimes \beta, Wh(\sigma)$$

$$(-1)^{pq} \langle \beta \cup \alpha, \sigma\rangle = \langle \alpha \otimes \beta, T(Wh(\sigma))\rangle.$$

Claim 0.7. *$T \circ Wh$ is chain homotopic to Wh. That is to say there is a map $H\colon Sing_*(X) \to Sing_*(X) \otimes Sing_*(X)$ that raises degree by 1 and satisfies:*

$$\partial H + H\partial = T \circ Wh - Wh.$$

Proof. The is a standard "acyclic carriers" result. We have the elements $Wh(|\Delta^n|) \in Sing_*(|\Delta^n|) \otimes Sing_*(|\Delta^n|)$ which are the sum over k of the tensor product of the front k-face of $|\Delta^n|$ with the back $(n-k)$-face of $|\Delta^n|$. Similarly, we have $T \circ Wh(|\Delta^n|) \in Sing_*(|\Delta^n|) \otimes Sing_*(|\Delta^n|)$. Inductively, we define elements $H_n \in (Sing_*(|\Delta^n|) \otimes Sing_*(|\Delta^n|))_{n+1}$ satisfying

$$\partial H_n + H_{n-1}(\partial|\Delta^n|) = T \circ Wh(|\Delta^n|) - Wh(|\Delta^n|). \tag{0.3}$$

(Here $H_{n-1}(\partial|\Delta^n|)$ means the sum

$$\sum_{r=0}^{n}(-1)^r(f_r \otimes f_r)_* H_{n-1}$$

where f_r is the inclusion of $|\Delta^{n-1}| \subset |\Delta^n|$ as the r^{th}-face.) We begin with $H_0 = 0$. Suppose inductively we have defined H_k for all $k < n$ satisfying Equation 0.3. We must find $H_n \in Sing_*(|\Delta^n|) \otimes Sing_*(|\Delta^n|)$ solving the equation

$$\partial H_n = -H_{n-1}(\partial|\Delta^n|) + T \circ Wh(|\Delta^n|) - Wh(|\Delta^n|). \tag{0.4}$$

The inductive hypothesis shows that

$$\partial H_{n-1}(\partial|\Delta^n|) + H_{n-2}(\partial\partial|\Delta^n|) = T \circ Wh(\partial|\Delta^n|) - Wh(\partial|\Delta^n|).$$

Of course $H_{n-2}(\partial\partial|\Delta^n|) = 0$. Since T and Wh are chain maps, it follows that the right-hand side of Equation 0.4 is a cycle. Since $|\Delta^n|$ is contractible, $Sing_*(|\Delta^n|) \otimes Sing_*(|\Delta^n|)$ is acyclic. It follows that there is an element H_n as required. This completes the inductive proof that H_n as required exist for all $n \geq 0$.

For each $n \geq 0$ and for each singular n-simplex σ we define $H(\sigma) = \sigma_*(H_n)$. This defines a linear map

$$H\colon Sing_*(X) \to Sing_*(X) \otimes Sing_*(X)$$

Clearly, for every $n \geq 0$ and every singular n-simplex σ, we have

$$\partial H(\sigma) + H(\partial\sigma) = T \circ Wh(\sigma) - Wh(\sigma). \tag{0.5}$$

□

Now let α^p and β^q be cocycles. By Equation 0.5 we have

$$\begin{aligned}(-1)^{pq}\langle\beta\cup\alpha,\sigma\rangle-\langle\alpha\cup\beta,\sigma\rangle&=\langle\alpha\otimes\beta,T\circ Wh(\sigma)-Wh(\sigma)\rangle\\&=\langle\alpha\otimes\beta,\partial H(\sigma)+H(\partial\sigma)\rangle\end{aligned}$$

Since $\alpha\otimes\beta$ is a cocycle in $Sing^*(X)\otimes Sing^*(X)$,

$$\langle\alpha\otimes\beta,\partial H(\sigma)\rangle=0.$$

Thus, we have

$$(-1)^{pq}\langle\beta\cup\alpha,\sigma\rangle-\langle\alpha\cup\beta,\sigma\rangle=\langle d(H^*(\alpha\otimes\beta)),\sigma\rangle.$$

Since this is true for every singular simplex in X, it follows that

$$(-1)^{pq}\beta\cup\alpha-\alpha\cup\beta=d(H^*(\alpha\otimes\beta)),$$

and hence $\alpha\cup\beta$ and $(-1)^{pq}\beta\cup\alpha$ represent the same cohomology class.

This completes the proof of the signed-commutivity of the cup product cohomology. □

Remark 0.8. In the above I chose to work with the singular chain complex of $|\Delta^n|$ but I could have just as well worked with the simplicial chain complex of $|\Delta^n|$.

Similar arguments show the following:

Proposition 0.9. *If α and β are cocycles on X, then $\alpha\otimes\beta\in Sing^*(X)\otimes Sing^*(X))$ defines a singular cohomology class on $X\times X$. Letting $\Delta_X\colon X\times X$ be the diagonal map, $\Delta_X^*([\alpha\otimes\beta])=[\alpha]\cup[\beta]$ in $H^*(X)$.*

Proof. (Sketch): The geometric realization $|Sing(X)|$ has a natural map to X which is a weak homotopy equivalence. It follows that $|Sing(X)|\times|Sing(X)|$ has a map to $X\times X$ which is a weak homotopy equivalence. The product of the geometric realizations has the product cell structure from the celll structure on the factors. $Sing_*(X)\otimes Sing_*(X)$ is identified with the cellular chains on this cell complex and hence computes the homology of $X\times X$.

The geometric realization $|Sing(X)\times Sing(X)|$ produces a subdivision of the cell complex structure on $|Sing(X)|\times|Sing(X)|$. Hence there is a map from the cellular chains on $|Sing(X)|\times|Sing(X)|$, which is $Sing_*(X)\otimes Sing_*(X)$, to the cellular chains on $|Sing(X)\times Sing(X)|$. This map assigns to a cell in $|Sing(X)|\times|Sing(X)|$, a linear combination of all the cells in $|Sing(X) \times Sing(X)|$ that contain an open subset of the original cell with coefficients which are signs determined by the relativie orientations. This map induces an isomorphism on cohomology. Thus, there is a cellular cocycle $\widetilde{\alpha \otimes \beta}$ on $|Sing(X) \times Sing(X)|$ that restricts to a cocycle cochomologous to $\alpha \otimes \beta \in Sing_*(X) \otimes Sing_*(X)$. The diagonal map $|Sing(X)| \to |Sing(X)| \times |Sing(X)$ is a cellular map when the range is given the cell structure coming from $|Sing(X) \times Sing(X)|$. It is weakly homotopy equivalent to the diagonal map $X \to X \times X$. The pullback by the Δ_X^* of $\widetilde{\alpha \otimes \beta}$ is a cocycle representing the pullback of the exterior product. The pullback by the Whitney map of the restriction of $\widetilde{\alpha \otimes \beta}$ is cohomologous to the pullback by the Whitney map of $\alpha \otimes \beta$, which is $\alpha \cup \beta$.

The same acyclic carrier argument applied to the simplicial cochain complex on $|Sing(X) \times Sing(X)|$ shows that $(\Delta_X)_*$ and Wh are chain homotopic. This gives the result. □

Notice that if X and Y are topological spaces, then an analogous construction produces a map

$$H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$

which is the exterior product of cohomology classes. the exterior product is related to the cup product. Let $p_1, p_2 \colon X \times X \to X$ be the projections onto the two factors. Given α, β in $H^*(X)$ the exterior product $\alpha \otimes \beta$ is equal to $p_1^*(\alpha) \cup p_2^*(\beta)$.

0.1 Higher Order Products

Let me give an indication of higher-order products.

Suppose that we have 3 cohomology classes $a, b, c \in H^*(X)$ with $a \cup b = 0$ and $b \cup c = 0$. We define a higher-order product $\langle a, b, c \rangle$. It lies in

$$H^{|a|+|b|+|c|-1}(X) / \big(a \cdot H^{|b|+|c|-1}|(X) + c \cdot H^{|a|+|b|-1}(X)\big).$$

Choose cocycle representatives α, β, γ for these three classes. We know that $\alpha \cup \beta$ and $\beta \cup \gamma$ are exact, so choose cochains η and μ with $\delta\eta = \alpha \cup \beta$ and $\delta\mu = \beta \cup \gamma$. Now we form

$$M(\alpha, \beta, \gamma) = (-1)^{|a|-1} \alpha \cup \mu + \eta \cup \gamma.$$

Direct computation shows that this cochain is a cocycle. Fixing the cocycles α, β, γ we can vary η and μ by a cocyles. This will change $M(\alpha, \beta, \gamma)$ by the sum of a product of α with a cocycle plus γ with a cocycle. It is easy to see that this is the complete indeterminacy of the construction.

This triple product is call the *Massey triple product of* a, b, c.

One can repeat this construction producing higher order products which will be defined under more and more vanishing conditions and will have greater and greater indeterminacy.

Exercise. Define a co-product on $H_*(X)$. Show that it is a co-associative, signed co-commutative co-ring with co-unit.