Lecture 5: Simplicial sets and simplicial homotopy theory

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1 The basic definitions

Let Δ be the category whose objects are the sets $\mathbf{n} = \{0, \ldots, n\}$ and whose morphisms are the weakly order-preserving set functions between these sets.

A *simplicial set* is a contravariant functor $\Delta \to \mathit{Sets}$. That is to say a simplicial set consists of objects which are sets $\{K_n\}_{n\geq 0}$ together with morphisms; for each weakly order-preserving map $\mathbf{n} \to \mathbf{m}$ a set function $K_m \to K_n$ with the induced composition properties. All such morphisms are compositions of the following elementary morphisms:

- the boundary maps $\partial_i \colon K_n \to K_{n-1}$ for any $0 \leq i \leq n$, the image under the functor of the strictly order-preserving function

  $$\{0, \ldots, n-1\} \to \{0, \ldots, n\}$$

  whose image misses i, and

- the degeneracies $s_i \colon K_{n-1} \to K_n$, for $0 \leq i \leq n-1$, the image under the functor of the weakly order-preserving surjection

  $$\{0, \ldots, n\} \to \{0, \ldots, n-1\}$$

  with the pre-image of i being i and $i+1$.

Thus, to give a simplicial set is to give the sets $\{K_n\}_{n\geq 0}$ and the face and degeneracy maps between these sets. The face and degeneracy maps are required to satisfy:

- $\partial_i \partial_j = \partial_{j-1} \partial_i$ if $i < j$.
- $s_i s_j = s_{j+1} s_i$ if $i \leq j$.

- $\partial_i s_j = s_{j-1} d_i$ if $i < j$
- $\partial_j s_j = \partial_{j+1} s_j = \mathrm{Id}$
- $\partial_i s_j = s_j \partial_{i-1}$ if $i > j+1$

An element of K_n is said to be *degenerate* if it is the image of a degeneracy map s_i for some i; otherwise it is said to be *non-degenerate*.

Definition 1.1. The first example to consider is the singular complex of a topological space X, denoted $Sing(X)$. Let $|\Delta^n|$ be the geometric n-simplex. Points of $|\Delta^n|$ are given by $(t_0, \dots, t_n)$ with the property that $t_i \geq 0$ for all $i \leq n$ and $\sum_{i=0}^{n} t_i = 1$. The set $Sing_n(X)$ is the set of continuous maps of the geometric n-simplex $|\Delta^n|$ to X. The boundary map $\partial_i \colon Sing_n(X) \to Sing_{n-1}(X)$ is given by sending $f \in Sing_n(X)$ to the restriction of f to the i^{th}-face of $|\Delta^n|$ which is identified with $|\Delta^{n-1}|$ by the simplicial map that is order-preserving on the vertices. The degeneracy $s_i \colon Sing_n(X) \to Sing_{n+1}(X)$ sends f to the composition of f following the collapsing map $|\Delta^{n+1}| \to |\Delta^n|$ given by

$$(t_0, \dots, t_{n+1}) \mapsto (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots t_{n+1}).$$

Example. Let Δ^n be the simplicial set defined by setting $(\Delta^n)_k$ equal to the set of weakly order-preserving maps $\{0, \dots, k\} \to \{0, \dots, n\}$. The boundary and degeneracy maps applied to $f \colon \{0, \dots, k\} \to \{0, \dots, n\}$ are given by pre-compositing f with the face and degeneracy maps of Δ.

Remark 1.2. In the simplicial construction we can replace sets by any category and define for example simplicial groups, simplicial rings, simplicial Lie algebras, simplicial smooth manifolds, etc. We can also define categories of *co-simplicial objects*, which are covariant functors from Δ to another category.

2 The Geometric Realization

The *geometric realization* of a simplicial set K is obtained as follows: Let $|\widetilde{K}| = \coprod_n K_n \times |\Delta^n|$, where each K_n is given the discrete topology. We introduce an equivalence relation on $|\widetilde{K}|$ and geometric realization $|K|$ is the quotient space. The equivalence relation is generated by:

- For $x \in K_n$ with $\partial_i x = y$, the i^{th} face of $\{x\} \times |\Delta^n|$ is glued to $\{y\} \times |\Delta^{n-1}|$ by the simplicial isomorphism that is order-preserving on the vertices, and
- for $z \in K_n$ with $z = s_i y$, $\{z\} \times |\Delta^n|$ is collapsed onto $\{y\} \times |\Delta^{n-1}|$ by the linear projection parallel to the edge with vertices $(i, i+1)$.

It is an easy exercise to see that for each non-degenerate element $x \in K_n$ the projection of the quotient map embeds the interior of $\{x\} \times \Delta^n$ open geometric n-simplex in the geometric realization, that all of these open simplices (which are not usually open subsets of the quotient space) of all dimensions are disjoint, and that their union is $|K|$. Thus, for each n and each non-degenerate $x \in K_n$ we have a continuous map $\{x\} \times |\Delta^n| \to |K|$ which is an embedding on $\mathrm{int}(\{x\} \times |\Delta^n|)$ whose images of the interiors cover $|K|$. Furthermore, since we take the quotient topology, a subset U of $|K|$ is open if and only if for each n and each non-degenerate $x \in K_n$, the pre-image of U is an open subset of $\{x\} \times |\Delta^n|$. This shows that $|K|$ is a CW complex.

Our notation is consistent because the geometric realization of Δ^n is the affine linear simplex $|\Delta^n|$.

Example. Let Y be a simplicial complex whose vertices have a partial order which is a total order on the vertices of every simplex of Y. Let K_Y be the simplicial set with $(K_Y)_n$ being the set of weakly order-preserving maps from $\{0, \ldots, n\}$ to the vertices of a simplex of of Y. The face and degeneracy maps are given by pre-composing an element of $(K_Y)_n$ with the set functions underlying the face or degeneracy in question.

Exercise. Show that the geometric realization $|K_Y|$ in the above example is canonically isomorphic to Y as an ordered simplicial complex.

3 Homology and homotopy groups of simplicial sets

To form the homology groups of a simplicial set K we let $C_n(K)$ be the free abelian group generated by the non-degenerate n-simplices with the boundary map given $\partial([x]) = \sum_{i=0^n} (-1)^i [\partial_{n,i} x]$ where the square brackets of a degenerate simplex is defined to be zero and the square brackets of a non-degenerate simplex is defined to be the generator in the chain group corresponding to that simplex. The homology groups of $C_*(K)$ are the homology groups of K. Notice that these agree with the CW homology groups of $|K|$.

The homotopy groups of a simplicial set are easiest to describe if the simplicial set satisfies the *Kan condition.* Let $\Lambda_{n,k} \subset \Delta^n$ be the sub simplicial set consisting of all weakly order-preserving maps $\{0, \ldots, m\} \to \{0, \ldots, n\}$ whose image does not include the k^{th}-face of Δ^n. A simplicial set K satisfies the Kan condition if for all $n \geq 0$ and for all $0 \leq k \leq n$ any map of $\Lambda_{n,k} \to K$ extends to a map $\Delta^n \to K$. In more down-to-earth terms the condition is any time given $(n-1)$-dimensional simplices $\{x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\}$ with $d_i x_j = d_{j-1} x_i$ for every $i < j$ with $i \neq k$ and $j \neq k$, there is $y \in K_n$ with $d_i y = x_i$ for all $i \neq k$.

More generally, a map of simplicial sets $f \colon K \to L$ is a *Kan fibration* if a relative version of this condition holds, namely given $\alpha \colon \Lambda_{n,k} \to K$ and $\beta \colon \Delta^n \to L$ with $f \circ \alpha = \beta | \Lambda_{n,k}$ there is a map $\widetilde{\alpha} \colon \Delta^n \to K$ extending α with $f \circ \widetilde{\alpha} = \beta$. Then a simplicial set K satisfies the Kan condition if and only the map $K \to \{pt\}$ is a Kan fibration.

For any $e \in K_0$ we denote by $\mathbf{e}$ the sub-simplicial set of K consisting of all iterated degeneracies of e. [Check that this is a sub-simplicial set.]

Definition 3.1. Suppose that K is a Kan complex and $e \in K_0$. Consider the set $P_n(\mathbf{e})$ of $y \in K_n$ such that $\partial_i y \in \mathbf{e}$ for all $i \leq n$. We define an equivalence relation on this set by setting $y \cong z$ if there is $w \in K_{n+1}$ with

- $\partial_i w \in \mathbf{e}$ for $i \leq n-1$,
- $\partial_n w = y$, and
- $\partial_{n+1} w = z$.

Lemma 3.2. *This is an equivalence relation on the set $P_n(\mathbf{e})$.*

Proof. For each k we denote by e_k the unique k-simplex in $\mathbf{e}$. Suppose that $y, y', y'' \in P_n$ and $y \cong y'$ and $y' \cong y''$. Then there are elements $v, v' \in K_{n+1}$ with

- $\partial_i v = \partial_i v' = e_n$ for all $0 \leq i \leq n-1$
- $\partial_n v = y$ and $\partial_{n+1} v = y'$
- $\partial_n v' = y'$ and $\partial_{n+1} v' = y''$.

The Kan condition implies that there is $w \in K_{n+1}$ with

- $\partial_i w = e_n$ for $0 \leq i \leq n-1$
- $\partial_n w = v$

- $\partial_{n+2}w = v'$.

We set $v'' = \partial_{n+1}w$. Then we see that

- $\partial_i v'' = e_n$ for $0 \leq i \leq (n-1)$
- $\partial_n v'' = \partial_n v = y$
- $\partial_{n+1} v'' = \partial_{n+1} v' = y''$.

This shows that $y \cong y''$, proving transitivity of the relation.

Fix $y \in P_n$. Then $s_n y$ has the following boundaries:

$$\underbrace{e_n, \ldots, e_n}_{n\text{times}} y, y,$$

showing that y is equivalent to itself, i.e., proving reflexivity of the relation.

Lastly, we establish symmetry. Let $y, y' \in P_n$ with $y \cong y'$. Thus there is $v \in K_{n+1}$ with $\partial_i v = e_n$ for $i \leq (n-1)$, $\partial_n v = y$ and $\partial_{n+1} v = y'$. The Kan condition applies to $(\underbrace{e_{n+1}, \ldots, e_{n+1}}_{n-\text{times}}, v, s_n(y),)$ producing v'' with $\partial_i v'' = e_n$ for $i \leq n-1$, $\partial_n v'' = \partial_{n+1} v = y'$, and $\partial_{n+1} v = \partial_{n+1} s_n(y) = y$. This proves that $y' \cong y$. □

Definition 3.3. We define the n^{th} homotopy set of K, denoted $\pi_n(K, \mathbf{e})$, to be the set of equivalence classes of P_n under this equivalence relation.

Definition 3.4. For every $n \geq 1$, we define a product on $\pi_n(K, \mathbf{e})$ as follows. Given $[y], [z]$ in $\pi_n(K, \mathbf{e})$ we define $[y] * [z] = [u]$ where $u = \partial_n w$ where $w \in K_{n+1}$ is any element given by the Kan condition applied to

$$(\underbrace{e_n, \ldots, e_n}_{(n-1)\text{times}}, y, -, z),$$

with e_n being the unique element of degree n in $\mathbf{e}$.

4 Comparison of $|Sing(X)|$ and X

We have functors: *Sing* from the topological category to the category of simplicial sets; and $|\cdot|$ from the category of simplicial sets to topological spaces. These functors are *adjoint* in the sense that

$$\mathrm{Hom}(K, Sing(X)) = \mathrm{Hom}(|K|, X).$$

We also have the following comparison result.

Proposition 4.1. *For any topological space X there is a natural continuous map $|Sing(X)| \to X$. For any point $e \in X$ the simplicial set $Sing(e)$ is naturally a sub-simplicial set of $Sing(X)$. Its only non-degenerate simplex is a 0-simplex so that the geometric realization $|Sing(e)|$ is a point. The map of pairs $(|Sing(X)|, |Sing(e)|) \to (X, e)$ induces an isomorphism $\pi_i(|Sing(X)|, |Sing(e)|) \to \pi_i(X, e)$ for all i. In particular if X is a CW complex, $|Sing(X)| \to X$ is a homotopy equivalence.*

Proof. For each simplex $\{x_n\}|\times|\Delta^n$ in $Sing_n(X)\times|\Delta^n|$ we define the map on this simplex using the map $x_n \colon |\Delta^n| \to X$. These maps are compatible with the equivalence relation and hence define a continuous map $|Sing(X)| \to X$.

First (as a warm-up) let us show that the map induces an isomorphism on homology. The reason is that this map identifies the simplicial chains on $|Sing(X)|$ with the non-degenerate singular chain complex of X. We know that the subcomplex of degenerate singular simplices has trivial homology so that the non-degenerate singular chains compute the usual singular homology of a space. And we know that the simplicial chains on an ordered simplicial complex give the singular homology of the simplicial complex. The claimed homology isomorphism follows.

Fix a point $e \in X$ Clearly, we have the induced relative map

$$(|Sing(X)|, |Sing(e)|) \to (X, e).$$

We show that for every $e \in X$ and for all n the induced maps

$$\pi_n(|Sing(X)|, |Sing(e)|) \to \pi_n(X, e)$$

are isomorphisms. Any element in $\pi_n(X, e)$ is represented by a map $(|\Delta^n|, \partial|\Delta^n|) \to (X, e)$. Thus, there is an element in $x \in Sing_n(X)$ such that for all $\partial_j x$ is the unique element of $Sing_{n-1}(e)$. This element produces a map $(|\Delta^n, \partial(|\Delta^n|)) \to (|Sing(X)|, |Sing(e)|)$ representing an element in $\pi_n(|Sing(X)|, |Sing(e)|)$ mapping to x. This shows that the map on π_n is onto. Since $Sing(X)$ is a Kan complex, any element in $\pi_n(|Sing(X)|, |Sing(e)|)$ is represented an element in $Sing_n(X)$ whose boundary lies in $Sing(e)$. If such an element represents the trivial element in $\pi_n(X, e)$ then there is an extension of the given map $|\Delta^n| \to X$ to a map $|\Delta^{n+1}| \to X$ sending all the faces except the first one to e. This shows that the map on π_n is one-to-one.

If X is a CW complex, then by Whitehead's theorem $|Sing(X)| \to X$ is a homotopy equivalence. $\square$

4.1 Exercises

Exercise 1. Show that the product is well-defined and determines a group structure on the set of equivalence classes provided that $n > 0$. Show that this group structure is abelian if $n \geq 2$. For all $n > 0$, the symbol $\pi_n(K, |\mathbf{e}|)$ refers to this group structure.
Exercise 2. Show that if K is a Kan complex that for every ordered triangulation T of S^n, denoting $S(T)$ by the associated simplicial set and $a_0 \in S(T)_0$ a base point, any map $(S(T), \mathbf{a_0}) \to (K, \mathbf{e})$ determines an element of $\pi_n(K, \mathbf{e})$. Furthermore, if $\widetilde{T}$ is an ordered triangulation of $S^n \times I$ with $a_0 \times I$ as a 1-simplex let $S(\widetilde{T})$ be the associated simplicial set. If $F\colon S(\widetilde{T}), \mathbf{a_0} \times \mathbf{I}) \to (K, \mathbf{e}$ is a map the the restrictions f_1 and f_0 to the ends determine the same element in $\pi_n(K, \mathbf{e})$.
Exercise 3. Show that $Sing(X)$ satisfies the Kan condition.
Exercise 4. Show that for a Kan complex K the group $\pi_n(K, \mathbf{e})$ is identified with $\pi_n(|K|, |\mathbf{e}|)$.

5 Homotopy Category of simplicial sets

Let K and L be simplicial sets. The product $K \times L$ is defined by:

- for every $n \geq 0$ we define $(K \times L)_n = K_n \times L_n$
- for every $n \geq 0$ and every $0 \leq i \leq n$ we define $\partial_i(k, \ell) = (\partial_i(k), \partial_i(\ell))$
- for every $n \geq 0$ and every $0 \leq i < n$ we define $s_i(k, \ell) = (s_i(k), s_i(\ell))$.

One checks directly that all the required relations hold.

At first glance this definition seems strange in that the product of two simplices should have dimension equal to the sum of the dimensions. So in the product, Where are the higher dimensional simplices? But this first glance overlooks the degenerate simplices which contribute the required higher dimensional faces.
Example. Let us consider $\Delta^1 \times \Delta^1$. The product has 4 vertices as expected. It has nine 1-simplices. There are the four degenerate 1-simplices associated with the vertices, and there are four products of a degenerate 1-simplex in one factor times the non-degenerate 1-simplex in the other factor: these are the four edges of the square. There is also the 1-simplex whose projection to each factor is non-degenerate. This represents the diagonal of the square. There are exactly two non-degenerate 2-simplices: each projects to a degenerate 2-simplex in each factor mapping onto the non-degenerate 1-simplex

and these projections are distinct. These are the two triangles in the square cut off by the diagonal. There are no higher dimensional non-degenerate simplices. Thus, the geometric realization of $\Delta^1 \times \Delta^1$ is the square triangulated with two triangles meeting along the diagonal.

Exercise: 1. Analyze the product $\Delta^n \times \Delta^1$ and show that its geometric realization is the standard triangulation of $|\Delta^n| \times |\Delta^1|$.

2. Show that $|K \times \Delta^1|$ is the natural triangulation (using the ordering of the vertices of each simplex of K) of $|K| \times I$.

3. Question 1 for $\Delta^n \times \Delta^m$.

Definition 5.1. Let K and L be simplicial sets. A homotopy of maps from K to L is a map $H\colon K \times \Delta^1 \to L$.

There are two embeddings $i_0, i_1\colon K \subset K \times \Delta^1$. First sends $k \in K_n$ to $(k, s_0^n(0))$ and the second sends k to $(k, s_0^n(1))$. We say that H is a homotopy from $H \circ i_0\colon K \to L$ to $H \circ i_1\colon K \to L$. We let this relation generate an equivalence relation on the category of simplicial sets. The quotient category is the homotopy category of simiplicial sets.

Let us give a simplicial $SK(\pi, n)$ set whose geometric realization is the Eilenberg-MacLane space $K(\pi, n)$. Its k simplices are the ordered simplicial cocycles of degree n on $|\Delta^k|$ with values in π. The face and degeneracy maps are induced by pulling back cocylcles using the boundary and degereneracy maps.

Lemma 5.2. *$SK(\pi, n)$ is a Kan complex and its homotopy groups are trivial in all degrees except n. The n^{th} homotopy group is identified with π.*

Proof. First we check that $SK(\pi, n)$ is a Kan complex. Given an n-cocycle on $\Lambda_{m,k}$ we use the fact that the includsion $|\Lambda_{m,k}| \subset |\Delta^m|$ is a homotopy equivalence to find an n-coclcye on Δ^m extending the given cocycle on $\Lambda_{m,k}$. Let α be an n-cocycle on Δ^m vanishing on the boundary. If $m \neq n$ or if $m = n$ and α is a coboundary, then there is an extension of α to an n-cocycle on Δ^{n+1} which agrees with α on the last face and is trivial on all other faces. This shows that under these conditions $[\alpha]$ is trivial in $\pi_m(SK(\pi, n))$. That is to say all homotopy groups of this simplicial set are trivial except for π_n and the integration of α over the n-simplex gives a function from $\pi_n(SK(\pi, n)$ to π. Suppose given cocycles α and β on $|\Delta^n|$. Consider the cocycle on $\Lambda_{(n+1),n}$ that is trivial on all faces except the $(n-1)^{st}$, where it is α and the $(n+1)^{st}$, where it is β. This cocycle extends to a cocycle on $|\Delta^{n+1})$ that is trivial on the n^{th}-face if and only if the integrals of α and β add to zero.. This proves that integration is an injective homomorphism

$\pi_n(SK(\pi, n) \to \pi$. Since there is a cocycle on $|\Delta^n|$ taking any given element of π as value, we see that this map is onto as well. $\square$

Suppose that we have a two-stage Postnikov tower $K(\pi', m) \to X \to K(\pi, n)$ with k-invariant $\alpha \in H^{m+1}(K(\pi, n); \pi')$. We choose a cocycle represntative $\widetilde{\alpha}$ for this cohomology class. In $SK(\pi, n)$ it is represented by an $(m+1)$-cocycle on $SK(\pi, n)$ with values in π'. Then a simplicial model for X has as k-simplices all pairs (a, b) where a is an n-cocycle with values in π on $|\Delta^k|$ and b is a m-cochain on $|\Delta^k|$with values in π' and $dm = \widetilde{\alpha}(a)$.