Cohomology of Manifolds

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1 Cohomology with compact support and locally finite homology

For this lecture X is a locally compact Hausdorff space and all manifolds are second countable (meaning that they have a countable base for the topology).

1.1 Cohomology with compact supports

We define the cohomology of X with compact supports as follows: A singular cochain $\alpha \in Sing^*(X)$ has *compact supports* if there is a compact subset $K \subset X$ (depending on α) such that α vanishes on any singular simplex $\sigma\colon |\Delta^n| \to X$ whose image is disjoint from K. The singular cochains with compact support, denoted $Sing_c^*(X)$, form a subcomplex of the singular cochain complex. The cohomology of this subcomplex is the *cohomology of X with compact supports*, and is denoted $H_c^*(X)$.

Suppose that $U \subset X$ is an open subset. Let α be a cocycle with compact support on U and denote by $K \subset U$ a compact set containing the support of α. Let $V = X \setminus K$. Then $\{U, V\}$ is an open cover of X. Let $Sing_*^{\{U,V\}}(X) \subset Sing_*(X)$ denote the subcomplex generated by all singular simplices whose image is contained either in U or in V. The inclusion of this subcomplex into $Sing_*(X)$ induces an isomorphism on homology, and consequently the dual map between the algebraic duals induces an isomorphism on cohomology. We define $\widetilde{\alpha}$ as a dual element of $Sing_*^{\{U,V\}}(X)$ whose value on a singular simplex σ is given by:

$$\langle \widetilde{\alpha}_K, \sigma\rangle = \begin{cases} \langle \alpha, \sigma\rangle & \text{if} \quad \sigma \subset U \\ 0 & \text{if} \quad \sigma \subset V \end{cases} \qquad .$$

Notice that if $\sigma \subset U \cap V$ then $\langle \alpha, \sigma \rangle = 0$ so that the definition makes sense. Clearly $\widetilde{\alpha}_K$ is a cocycle and hence its cohomology class $[\widetilde{\alpha}_K] \in H^*(Sing_*^{\{U,V|\}}(X))$ determines a class, denoted $\mathrm{ext}([\widetilde{\alpha}_K])$, in $H_c^*(X)$.

Claim 1.1. *Associating to a compact supported cocycle α on U the class $\mathrm{ext}[\widetilde{\alpha}_K] \in H_c^*(X)$ gives a well-defined map $H_c^*(U) \to H_c^*(X)$.*

Proof. First let us show that the class $\mathrm{ext}[\widetilde{\alpha}_k]$ is independent of the choice of compact subset $K \subset U$ containing the support of α. Given any two such, K_0 and K_1, there is a third $K' \subset U$ containing both K_0 and K_1. Thus, to show the independence of the choice of K it suffices to consider the case when $K \subset K'$. Let $V = X \setminus K$ and $V' = X \setminus K'$. We have

$$Sing_*^{\{U,V'\}}(X) \subset Sing_*^{\{U,V\}}(X) \subset Sing_*(X)$$

with all inclusions inducing isomorphisms on cohomology and hence their algebraic duals induce isomorphisms on cohomology. It is clear that the restriction of $\widetilde{\alpha}_K$ to $Sing_*^{\{U,V'\}}(X)$ is $\widetilde{\alpha}_{K'}$. From this the independence of K follows.

Now suppose that there is a compactly supported cochain β in U with $d\beta = \alpha$. Let $K \subset U$ be a compact subset contaning the support of β. Then $\widetilde{\alpha}_K = d(\widetilde{\beta}_K)$, showing that $\mathrm{ext}[\widetilde{\alpha}_K] = 0$. Since the map is clearly linear, this completes the proof that it is well-defined $\square$

Notice that there is *not* a natural contravariant map $H_c^*(X) \to H_c^*(U)$.

1.2 Locally finite homology

This homology is also called *Borel-Moore homology.*

Consider arbitrary formal expressions $\sum_\sigma \lambda_\sigma \sigma$ (where σ varies over the singular simplices of X and each λ_σ is an integer) subject to the condition that for each compact subset $K \subset X$ only finitely many σ have the property that $\lambda_\sigma \neq 0$ and the image of σ meets K. The collection of such expressions forms an abelian group under coordinate-wise addition. Furthermore, the boundary, ∂, of such an expression is well-defined and is also an expression of the same form. Clearly, $\partial^2 = 0$. Thus, such expressions form a chain complex, denoted $Sing_*^{lf}(X)$. The homology of this chain complex is the locally finite homology, denoted $H_*^{lf}(X)$.

Claim 1.2. *The locally finite chain complex is*

$$\lim_{\leftarrow} Sing_*(X, X \setminus K)$$

as K ranges over the directed set of compact subsets of X.

Proof. Let $\sum_\sigma \lambda_\sigma \sigma$ be a locally finite chain. For any compact subset K all but finitely many of the terms with non-zero coefficient are contained in $X \setminus K$. Thus, the image of the formal sum is a finite sum in $Sing_*(X, X \setminus K)$. This determines a chain map $p_K\colon Sing_*^{lf}(X) \to Sing_*(X, X \setminus K)$. If $c = \sum_\sigma \lambda_\sigma$, then

$$p_K(c) = \sum_{\sigma | \mathrm{Im}(\sigma) \cap K \neq \emptyset\}} \lambda_\sigma \sigma.$$

These maps are compatible as we vary K; i.e., if $K \subset K'$ then for any locally finite chain c image of $p_{K'}(c)$ under the map $Sing_*(X, X \setminus K') \to Sing_*(X, X \setminus K)$ induced by inclusion of pairs is $p_K(c)$. Hence, these maps define a map of chain complexes:

$$Sing_*^{lf}(X) \to \lim_{\leftarrow} Sing_*(X, X \setminus K).$$

This map is clearly an injection on the chain groups. We need only see that it is onto. Suppose $\{c_K\}$ is an element of the projective limit. We represent each c_K uniquely as a finite linear combination of singular simplices none of which is disjoint from K. If $K \subset K'$, then c_K is obtained from $c_{K'}$ by deleting all the terms in the expression for $c_{K'}$ that are singular simplices disjoint from K. The formal expression for the locally finite chain is the sum of all the terms that appear in at least one c_K. It is easy to see that this is a locally finite expression that restricts to $Sing_*(X, X \setminus K)$ to give c_K.

□

Now let us specialize to second countable, locally compact spaces.

Claim 1.3. *Any second countable locally compact Hausdorff space X is a countable union of increasing compact sets.*

Proof. Second countable means that there is a countable basis for the topology $\{U_n\}_{n=1}^\infty$. Each point $x \in X$ has a neighborhood with a compact closure.This neighborhood contains an element of the countable base U_x, a basis element containing x, The closure $\overline{U}_x$ is compact. Thus, we have $X \subset \cup_{x \in X} \overline{U}_x$, where for each $x \in X$, the open set U_x is an element of the countable base. This shows that X is a countable union of compact sets: $\overline{U}_1, \ldots, \overline{U}_k \ldots$. We then set $K_k = \cup_{i=1}^k \overline{U}_i$. This is the required increasing sequence of compact sets whose union is X. □

In the case when X is second countable there is a simpler way to express locally finite chains. We can write it as an increasing union of compact sets

$K_1 \subset K_2 \subset \cdots$. In this case every locally finite chain can be written as a countable sum

$$\sum_n \lambda_n \sigma_n$$

where for each $k \geq 1$ there is $n_k \geq 0$ such that $\sigma_n \cap K_k = \emptyset$ for all $n \geq n_k$.

1.3 Restriction of locally finite chains to an open subset

Let $U \subset X$ be an open set. Then there is a map $Sing_*^{lf}(X) \to Sing_*^{lf}(U)$ which we now describe. Let $c = \sum \lambda_\sigma \sigma$ be a locally finite chain in X. (We remove all terms with zero coefficient, so we can assume that $\lambda_\sigma \neq 0$ for every σ in the formal expression.) For each σ we add to the vertices of σ a vertex at the barycenter of each of this faces that is not contained in U. The vertices of each of the new simplices are ordered so that the vertices of the original simplex come before all the new vertices, and have the same order, and the new vertices are ordered by the dimension of the open faces that contain them. Then we form the linear subdivision of σ with this collection of vertices and replace σ by the sum of the newly created simplices with coefficients $\pm\lambda_\sigma$, the sign begin determined by the relative orientation of the subsimplex to the original simplex. (Thus, if σ is entirely contained in U then σ is unchanged, and if no positive dimensional face of σ is contained in U then σ is replaced by its barycentric subdivision.) After this process we remove all simplices that are not faces of top dimensional simplices that meet U. All simplices whose closures are contained in U will remain unchanged in the later steps. Then we repeat the process, ad infinitum and take the union over all the steps of all simplices whose closure is contained in U. The result gives us a locally finite chain in U and this operation is compatible with the boundary map so that it induces a map $H_*^{lf}(X) \to H_*^{lf}(U)$.

1.4 Relationship between cohomology with compact supports and locally finite homology

Suppose that α is a cochain with compact support K and suppose that A is a locally finite chain in X. Since all but finitely many of the terms in the formal linear combination giving A are disjoint from K, we can evaluate α on A. Thus, there is an evaluation pairing

$$Sing_c^*(X) \otimes Sing_*^{lf}(X) \to \mathbb{Z},$$

whose adjoint is a map $Sing_c^*(X) \to \mathrm{Hom}(Sing_*^{lf}(X), \mathbb{Z})$. This map is clearly injective, but it is not onto. Its image is the set of continuous homomorphism from $Sing_*^{lf}(X) \to \mathbb{Z}$ when $Sing_*^{lf}(X)$ is given the topology of coming from its description as an inverse limit. Unraveling the definition, one sees that a homomorphism is continuous if and only if there is a compact set K such that the map $Sing_*^{lf}(X) \to \mathbb{Z}$ factors through the projection $Sing_*^{lf}(X) \to Sing_*(X, X \setminus K)$.

This map satisfies

$$\langle d\alpha, \beta \rangle = \langle \alpha, \partial\beta \rangle$$

for compactly supported cochains α and locally finite chains β. Thus, there is an induced map

$$H_c^*(X) \otimes H_*^{lf}(X) \to \mathbb{Z}.$$

Similarly, there is a map

$$Sing^*(X) \otimes Sing_*^{lf}(X) \to Sing_*^{lf}(X)$$

satisfying the analogous equation and consequently inducing a map

$$H^*(X) \otimes H_*^{lf}(X) \to H_*^{lf}(X).$$

1.5 Cap Product

There is a pairing closely related to the evaluation map called *cap product*. For a cochain α of degree k and a singular chain γ of degree r we define

$$\langle \beta, \alpha \cap \gamma \rangle = \langle \beta \cup \alpha, \gamma \rangle$$

for any cochain β. Extending linearly gives a pairing, the *cap product*

$$\cap \colon Sing^k(X) \otimes Sing_r(X) \to Sing_{r-k}(X).$$

One proves that if α is a k-cochain and γ is an r-chain, then

$$\partial(\alpha \cap \gamma) = \alpha \cap \partial\gamma + (-1)^{(r-k)}\delta\alpha \cap \gamma.$$

Thus, if α is a cocycle and γ is a cycle, then $\alpha \cap \gamma$ is a cycle. Furthermore, if we vary α by a coboundary and γ by a boundary, their cap product changes by a boundary. Thus, the cap product at the chain and cochain level induces a product

$$\cap \colon H^k(X) \otimes H_r(X) \to H_{r-k}(X)$$

natural with respect to continuous maps. There is a natural map $H_0(X) \to \mathbb{Z}$ which assigns to a zero cycle the sum of its coefficients. When $k = r$ cap product followed by this natural map is the evaluation of a cohomology class of degree k on a homology class of degree k.

Cap product also induces analogues

$$Sing_c^k(X) \otimes Sing_n^{lf}(X) \to Sing_{n-k}(X)$$

and

$$Sing^k(X) \otimes Sing_n^{lf}(X) \to Sing_{n-k}^{lf}(X)$$

leading to cap product pairings

$$H_c^k(X) \otimes H_n^{lf}(X) \to H_{n-k}(X)$$

and

$$H^k(X) \otimes H_n^{lf}(X) \to H_{n-k}^{lf}(X).$$

1.6 The Fundamental Class

Lemma 1.4. *Let M be an n-dimensional topological manifold. For any $x \in M$ we have*

$$H_*(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By excision we can replace M be a ball U in a coordinate chart contained in M. Then $H_*(U, U \setminus \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. Since $\mathbb{R}^n$ is contractible and $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}, the result is immediate from the long exact sequence of a pair. $\square$

Definition 1.5. A *local orientation* for M at x is a choice of a generator for $H_n(M, M \setminus \{x\})$.

Notice that a local orientation for M determines a local orientation for every point $y \in M$ in some neighborhood of M. The reason is that any relative cycle in $(M, M \setminus \{x\})$ has boundary missing a small neighborhood of $\{x\}$ and hence determines a relative class in $H_n(M, M \setminus \{y\})$ for all y in this neighborhood. This proves that pairs $\{x \in M, o_x\}$ where o_x is an orientation for M at x naturally form a double covering of M. The manifold M is said to be *orientable* if this covering is a trivial double covering and in this case an *orientation* for M is a choice of a section of this covering. An *oriented* manifold is a manifold together with an orientation. Thus, an oriented manifold has an induced local orientation at each point.

Definition 1.6. Let M be an oriented n-manifold. A *fundamental class* for M is an element $[M] \in H_n^{lf}(M)$ with the property that for each $x \in M$ the image of $[M] \in H_n(M, M \setminus \{x\})$ is the local orientation for M at x compatible with the global orientation of M.

Theorem 1.7. *Let M be an oriented topological n-manifold. Then M has a unique locally finite fundamental class, denoted $[M]$. Furthermore, $H_*^{lf}(M) = 0$ for $* > n$,*

First we show:

Proposition 1.8. *Let M be an oriented n-manifold and let $K \subset M$ be a compact set. Then there is a unique class $\alpha_K \in H_n(M, M \setminus K)$ with the property that for every $x \in K$ the image of α_K in $H_n(M, M \setminus \{x\})$ is the local orientation of M. Also, $H_*(M, M \setminus K) = 0$ for $* > n$.*

We begin the proof of the proposition with a series of lemmas.

Lemma 1.9. *The proposition holds when M is a round, open ball in $\mathbb{R}^n$ and K is a compact convex subset of the open ball.*

Proof. Since M is contractible, $H_n(M, M \setminus K)$ is identified with $H_{n-1}(M \setminus K)$. Radial projection from any point of $x \in K$ gives a homotopy equivalence between $M \setminus K$ and S^{n-1}. Thus, $H_n(M, M \setminus K) = \mathbb{Z}$ and the higher homology groups vanish. It is immediate that the image of a generator maps to a generator in $H_n(M, M \setminus \{x\})$, so there is a unique class mapping to the local orientation for every $x \in K$. □

Lemma 1.10. *The proposition holds for M the round ball and K a union of r compact convex subsets.*

Proof. The proof is by induction on the number r of compact convex subsets whose union is K. We just established the result when $r = 1$. Suppose that it is true for $r - 1$ and let $K = C_1 \cup \cdots \cup C_r$. Set $K' = C_1 \cup \cdots \cup C_{r-1}$. We argue by induction using the Meyer-Vietoris sequence for

$$(M, M \setminus K) \to \big((M, M \setminus K') \coprod (M, M \setminus C_r)\big) \to (M, M \setminus K' \cap C_r).$$

The middle two terms are covered by the inductive hypothesis and the fact that we have established the result when K is a single compact convex set. Since K' is a union of $r - 1$ compact, convex sets, the last term is also covered by the inductive hypothesis. In particular there are classes $\alpha_{K'} \in H_n(M, M \setminus K')$ and $\alpha_{C_r} \in H_n(M, M \setminus C_r)$ as in the proposition. It follows easily from the homology long exact sequence that $H_*(M, M \setminus K) = 0$ or $* > n$ and that there is a unique element $\alpha_K \in H_n(M, M \setminus K)$ with image $(\alpha_{K'}, \alpha_{C_r})$. This class satisfies the condition given in the proposition. □

Lemma 1.11. *The proposition holds for M the round ball and K an arbitrary compact set.*

Proof. We find a finite union $B = \cup_i B_i$ of closed round balls contained in M and containing K. By the previous lemma the proposition holds for $H_*(M, M \setminus \cup_{i=2}^k B_i)$. Since taking homology commutes with colimits we see that the theorem holds for $(M, M \setminus B)$ and *a fortiori* it holds for $(M, M \setminus K)$. $\square$

Proof. Now we are ready to prove the proposition in complete generality. Now we consider an arbitrary manifold M and an arbitrary compact subset K. We cover K by finitely many open balls in coordinate patches $K \subset U_1 \cup \cdots \cup U_k$. Let $U = U_1 \cup \cdots \cup U_k$ and we write K as $K_1 \cup \cdots \cup K_k$ where each K_i is compact and $K_i \subset U_i$. We set $U' = U_1 \cup \cdots \cup U_{k-1}$ and $K' = K_1 \cup \cdots \cup K_{r-1}$

We consider the Meyer-Vietoris sequence associated to

$$(M, M \setminus K) \to \big((M, M \setminus K') \coprod (M, M \setminus K_r)\big) \to (M, M \setminus K' \cap K_r).$$

By excision we can rewrite these terms as

$$(U, U \setminus K) \to (U', U' \setminus K') \coprod (U_k, U_k \setminus K_k) \to (U_k, U_k \setminus K' \cap K_k).$$

The inductive hypothesis and the case of a compact set in a single open ball show that each of the middle terms satisfies the lemma. That the last term satisfies the lemma is covered by the case of a compact set in single open ball. From the homology long exact sequence we see that $H_*(M, M \setminus K) = 0$ for $* > n$ and also, that there is a unique class $\alpha_K \in H_n(M, M \setminus K)$ that maps to $(\alpha_{K'}, \alpha_{K_r})$. Clearly, for any $x \in K$, the class α_K restricts to $H_n(M, M \setminus \{x\})$ to give the local orientation.

This completes the proof of the proposition. $\square$

Proof. (of the theorem) Now we establish the relationship between these classes and a locally finite fundamental class for M. Let M be an oriented n-manifold. Take an increasing sequence of compact sets $K_1 \subset K_2 \subset \cdots$ whose union in M. The proposition shows that that there is a unique element $\{\alpha_{K_n}\} \in \lim_{\leftarrow} H_n(M, M \setminus K_k)$ with the property that for each $x \in K_k$ the image in $H_n(M, M \setminus \{x\})$ of the class α_{K_k} is the local orientation. We need to compare this projective limit with the locally finite homology.

As we have seen the locally finite chains are the projective limit of the chains on $(M, M \setminus K_k)$. The issue is that in general taking homology does not commute with taking projective limits. In the case these operations commute.

Lemma 1.12.

$$H_n^{lf}(X) = H_n(\lim_{\leftarrow}(M, M \setminus K_k)) = \lim_{\leftarrow} H_n(M, M \setminus K_k).$$

Proof. First let us show that the map given in the lemma is surjective. Suppose that we have an element

$$\{[\zeta_k]\} \in \lim_{\leftarrow} H_*(M, M \setminus K_k).$$

We show that it is possible to choose different representative cycles ζ'_k homologous to the ζ_k such that under the map $(M, M \setminus K_k) \to (M, M \setminus K_{k-1})$ the cycle ζ'_k maps to the cycle ζ'_{k-1}. That will show that $H_*^{lf}(M)$ maps onto $\lim_{\leftarrow} H_*(M, M \setminus K_k)$. Suppose we have fixed $\zeta'_1, \ldots, \zeta'_k$ as required. Then the image of ζ_{k+1} differs from ζ'_k by $\partial\beta_k$ for some $\beta_k \in Sing_*(M, M \setminus K_k)$. Since the inclusion map of relative cochains is onto, we can find a cochain $\beta_{k+1} \in Sing_*(M, M \setminus K_{k+1})$ mapping to β_k. We set $\zeta'_{k+1} = \zeta_{k+1} - \partial\beta_{k+1}$. This is the required cycle, completing the inductive step.

Now we show that the map given in the lemma is injective. Suppose that $\{\zeta_k\}$ is a cycle in the projective system of chain complexes whose image in the projective system of homology groups is trivial. This means that each ζ_k is a boundary in $Sing_*(M, M \setminus K_k)$: say $\zeta_k = \partial\beta_k$. To show that the element is a boundary we must show that we can choose the β_k to be compatible. Suppose that we have arranged the the image of β_i in $Sing_*(M, M \setminus K_{i-1})$ is β_{i-1} for all $i \leq k$. Consider the difference of the image of β_{k+1} and β_k. This is a cycle in $(M, M \setminus K_k)$ since the boundaries of the β_i are compatible. But $H_{n+1}(M, M \setminus K_k) = 0$ so that this difference is a boundary; that is to say there is $\gamma_k \in Sing_*(M, M \setminus K_k)$ with $\partial\gamma_k = \beta_{k+1} - \beta_k$. Lift γ_k to an element γ_{k+1} and replace β_{k+1} by $\beta_{k+1} - \partial\gamma_{k+1}$. This constructs β_{k+1} as required and completes the induction. $\square$

Now we are in a position to prove the theorem. Let M^n be an oriented n-manifold, and choose an increasing sequence $K_1 \subset K_2 \subset \cdots$ of compact subsets whose union is M. By the previous lemma $H_n^{lf}(M) = \lim_{\leftarrow} H_n(M, M \setminus K_k)$. The unique classes $\alpha_{k_k} \in H_n(M, M \setminus K_k)$ that restrict to the local orientation at any point of K_k are compatible under inclusions and hence form an element in $\lim_{\leftarrow} H_n(M, M \setminus K_k)$. Let $[M] \in H_n^{lf}(M)$ be the corresponding locally finite homology class. For any $x \in M$ there is k such that $x \in K_k$. By construction the image of α_{K_k} in $H_n(M, M \setminus \{x\})$ is the local orientation. Hence it is also true that the image of $[M]$ in $H_n(M, M \setminus \{x\})$ is the local orientation. This proves that $[M]$ is a fundamental class for M. It uniqueness follows from the fact that the map from the n^{th} locally finite homology to the inverse limit of $H_n(M, M \setminus K_k)$ is an isomorphism and the uniqueness of the α_{K_k}.

The fact that $H_*^{lf}(M) = 0$ for $* > n$ follows from the analogue of Lemma1 1.12 for the higher homology groups. This result follows from the same argument as in the lemma, suing the fact that the higher homology groups of $(M, M \setminus K_k)$ vanish. $\square$

2 Poincaré Duality and Lefschetz Duality

2.1 Poincaré Duality

Theorem 2.1. *(Poincaré Duality) Suppose that M is an oriented n-manifold let $[M] \in H_n(M)$ be its fundamental class Then the map*

$$\cap[M]\colon H_c^*(M) \to H_{n-*}(M)$$

given by cap product with the fundamental class is an isomorphism for all $*$.

First, we show:

Lemma 2.2. *Let U be an open convex subset of a open unit ball in $\mathbb{R}^n$. Then $H_c^*(U)$ is trivial for all $* \neq n$ and $H_c^n(U) = \mathbb{Z}$. Furthermore, the map*

$$\cap[U]\colon H_c^n(U) \to H_0(U)$$

is an isomorphism.

Proof. We can assume that $0 \in U$. For any $0 < t \leq 1$ the map $x \mapsto tx$ defines a diffeomorphism form U to an open subset U_t of U.

Claim 2.3. *For any $0 < t < 1$, the inclusion of $U_t \subset U$ iinduces an isomorphism on compactly supported cohomology.*

Proof. Fix $0 < t < 1$. First we show that the inclusion $U_t \to U$ induces a surjective map on compactly supported cohomology. The point is that there is a one parameter family of diffeomorphsims $J_s\colon U \to U$ such that J_0 is the identity and $J_1(U_t)$ contains the support of α. Thus, $J_1^*\alpha$ is compactly supported in U_t and $J_1^*\alpha$ and $J_0^*\alpha = \alpha$ represent the same compactly supported cohomology class in U.

Now we show that the map is injective. Suppose that α' is a compactly supported cocycle in U_t and $\alpha' = \partial\beta$ for a compactly supported cochain β in U. Then we can assume that the one parameter family J_s as in the first paragraph is in fact the trivial family on the support of α; i.e.,, for all s the restriction $J_s|_{\mathrm{supp}(\alpha')}$ is the identity. Then $\delta J_1^*\beta = J_1^*(\alpha') = \alpha'$ and $J_1^*\beta$ is compactly supported in U'. This proves the claim. $\square$

Returning to the proof of the lemma, we now prove the result for an open ball B. Denote by $\overline{B}$ its closure. The extension by zero determines a map $H^*_c(B) \to H^*(\overline{B}, \partial\overline{B})$ Let $B' \subset B$ be a smaller ball. Thus, we have

$$H^*_c(B') \to H^*(\overline{B}'\partial\overline{B}') \to H^*_c(B) \to H^*(\overline{B}, \partial\overline{B}).$$

By the claim, the map $H^*_c(B') \to H^*_c(B)$ is an isomorphism so that $H^*(\overline{B}', \partial\overline{B}')$ maps onto $H^*_c(B)$. On the other hand, the map $H^*(\overline{B}', \partial\overline{B}') \to H^*(\overline{B}, \partial\overline{B})$ is an isomorphism implying that the map $H^*(\overline{B}', \partial\overline{B}') \to H^*_c(B)$ is injective. Thus proves that for any U as in the statement of the lemma $H^*_c(U) = 0$ for all $* \neq n$ and $H^n_c(U) = \mathbb{Z}$. Lastly, we need to see that $\cap[B] \colon H^n_c(B) \to H_0(B)$ is an isomorphism. When we identity $H_0(B)$ with $\mathbb{Z}$ in the natural way, the map $\cap[B] \colon H^n_c(B) \to \mathbb{Z}$ is the evalaution of $H^n_c(B)$ on $[B]$. The restriction of $[B]$ to $H_n(B, B \setminus \{0\}$ gives the local orientation at the origin and the map $H^n_c(B) \to H^n(B, B \setminus \{x\})$ is an isomorphism. It follows in this case that $\cap[B]$ is an isomorphism.

The lemma for any open convex subset U of an open ball follows since there is a ball $B_0 \subset U$ and $H^*_c(B) \to H^*_c(U)$ and $H_*(B) \to H_*(U)$ are both isomorphisms and the restriction of $[U]$ to B_0 is $[B_0]$. This proves the lemma. $\square$

Now we prove the result for any open subset U in an open ball.

Proposition 2.4. *For any open subset U of an open ball in $\mathbb{R}^n$*

$$\cap[U] \colon H^*_c(U) \to H_{n-*}(U)$$

is an isomorphism.

Proof. We have already established this result for an open convex subset of an open ball in $\mathbb{R}^n$. Now we prove, by induction on r, that the result holds for any open subset of an open ball that is the union of $\leq r$ open convex sets. Given that the result holds for r we establish it for $r+1$. Suppose $U = B_1 \cup \cdots \cup B_{r+1}$, where the B_i are open balls. Let $U' = B_1 \cup \cdots \cup B_r$ and $V = (B_1 \cup \cdots \cup B_r) \cap B_{r+1}$ and we consider the Meyer-Vietoris sequence for

$$\begin{array}{ccccccccc}
H^*_c(V) & \longrightarrow & H^*_c(U') \oplus H^*_c(B_{r+1}) & \longrightarrow & H^*_c(U) & \longrightarrow & H^{*+1}_c(V) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
H_{n-*}(V) & \longrightarrow & H_{n-*}(U') \oplus H_{n-*}(V) & \longrightarrow & H_{n-*}(U) & \longrightarrow & H_{n-*-1}(V) & \longrightarrow
\end{array}$$

where the vertical arrows are cap product with the fundamental classes. Since the restriction of the fundamental class of U to U' and B_{r+1} induces their fundamental classes and the fundamental class of U' and B_{r+1} each restrict to V to give its fundamental class, this diagram is commutative.

Since both U' and V are the union of r open convex sets, the result follows for any finite union of open balls by the five lemma and the inductive hypothesis.

Any open subset of an open ball in $\mathbb{R}^n$ is a countable union of open balls. Both homology and cohomology with compact supports commute with taking colimits over such increasing unions. Also, the fundamental class of the union restricts to give the fundamental class of any finite union of open balls, we see that the result for any open subset of an open ball in $\mathbb{R}^n$ follows by taking colimits of the result for finite unions. □

Now we are in a position to prove the theorem.

Proof. Let M be an oriented n-manifold that is the union of a finite number of open n-balls. We shall show by induction on the number of balls that

$$\cap[M] \colon H_c^*(M) \to H_{n-*}(M)$$

is an isomorphism. We already know the result when M is a single ball. Suppose that we know the result as along an M is a union of at most r open balls and suppose that M is a union of $r+1$ open balls. We write M' for the union of the first r balls, B for the $(r+1)^{st}$ ball, and $V = M' \cap B$. The argument now proceeds using the Meyer-Vietoris sequences for compactly supported cohomology and for ordinary homology for $V \to M' \coprod B \to U$ with the comparison between the long exact sequences given by cap product with the fundamental classes. Since the fundamental classes restrict to fundamental classes on open sets, the diagram is commutative. The inductive hypothesis shows that cap product with the fundamental class is an isomorphism for M' and B. Since V is an open subset of an open ball, we also know the result for V. The five lemma now applies to prove the result for any manifold M that is a union of a finite number of open n-balls.

Any topological manifold is a countable union of open balls. Taking colimits over a countable increasing union of the result just established for finite unions of open balls gives the result for an arbitrary oriented topolgical manifold M. This completes the proof of Poincaré Duality for topological manifolds. □

Corollary 2.5. *If M is a compact, oriented n-manifold, then*

$$\cap[M]\colon H^*(M) \to H_{n-*}(M)$$

is an isomorphism.

2.2 Lefschetz Duality

Let M be a compact manifold with boundary.

Claim 2.6. *The map $H^*_c(\mathrm{int}(M)) \to H^*(M, \partial M)$ induced by inclusion is an isomorphism.*

Proof. Let $\partial M \times I \subset M$ be a collar neighborhood of the boundary, with $\partial M \times \{0\}$ being the boundary of M. For each $0 < t \leq 1$ and let M_t be the open submanifold that is the complement of $\partial M \times [0, t]$. The argument in Claim 2.3 shows that the inclusion $M_t \subset \mathrm{int}(M)$ induces an isomorphism on compactly supported cohomology. Also, for any t the inclusion $\overline{M}_t \subset M$ induces isomorphisms

$$H^*(\overline{M}_t, \partial\overline{M}_t) \cong H^*(M, \partial M \times [0, t]) \cong H^*(M, \partial M).$$

We have the maps induced by the inclusions

$$H^*_c(M_1) \to H^*(\overline{M}_1, \partial\overline{M}_1) \to H^*_c(\mathrm{int}(M)) \to H^*(M, \partial M).$$

Since $H^(\overline{M}_1, \partial\overline{M}_1) \to H^*(M, \partial M)$ is an isomorphism we see that $H^*_c(\mathrm{int}(M)) \to H^*(M, \partial M)$ is surjective. Since $H^*_c(M_1) \to H^*_c(\mathrm{int}(M))$ is an isomorphsim it follows that $H^*_c(M_1) \to H^*(\overline{M}_1, \partial\overline{M}_1)$ is injective. Since there is a diffeomorphism of $\overline{M}_1 \to M$, it follows that $H^*_c(\mathrm{int}(M)) \to H^*(M, \partial M)$ is an isomorphism.

□

Claim 2.7. *Let M be a compact, oriented n-manifold with boundary. The restriction map $H_n(M, \partial M) \to H_n^{lf}(\mathrm{int}(M))$ is an isomorphism.*

Proof. Given a class in $H_k(M, \partial M)$ we represent it by a relative k-cycle. The restriction of this cycle to $\mathrm{int}(M)$ defines a locally finite k-cycle. This gives a well defined map $H_*(M, \partial M) \to H_*^{lf}(M)$. We know that $H_n^{lf}(\mathrm{int}(M)) = \lim_{\leftarrow} H_n(M, M \setminus K)$ as K ranges over the compact sets of $\mathrm{int}(M)$. Choose a collar neighborhood $\partial M \times [0, 1]$ of ∂M with $\partial M \times \{0\}$ being the boundary. The for each $0 < t \leq 1$ the complement of $\partial M \times [0, t)$ is a compact set, denoted K_t, in M. The $K_{1/n}$ form a cofinal system of compact sets in $\mathrm{int}(M)$ so that

$$H_n^{lf}(M) = \lim_{\leftarrow,n} H_n(M, M \setminus K_{1/n}).$$

On the other hand, the inclusion $(K_{1/n}, \partial K_{1/n}) \subset (M, M \setminus K_{1/n})$ induces an isomorphism on relative homology as does the inclusion $(M, \partial M), (M, M \setminus K_{1/n})$. Thus, the inclusion induces an isomorphism $H_n(M, \partial M) \to H_n(M, M \setminus K_{1/n})$. It now follows that $H_n^{lf}(\mathrm{int}(M))$ is identified with $H_n(M, \partial M)$. This identification is clearly induced by the restriction mapping desribed above.

Thus, there is a unique class $[M, \partial M] \in H_n(M, \partial M)$ whose restriction to $H_n^{lf}(\mathrm{int}(M))$ is the fundamental class of the open oriented manifold. □

Definition 2.8. For a compact, oriented n-manifold M the unique class $[M, \partial M] \in H_n(M, \partial M)$ whose restriction to $\mathrm{int}(M)$ is $[M] \in H_n^{lf}(\mathrm{int}(M))$ is the *relative fundamental class of M*.

Corollary 2.9. *(Lefschetz duality) Let M be a compact, oriented manifold with boundary. Then for every k the map*

$$\cap[M, \partial M] \colon H^k(M, \partial M) \to H_{n-k}(M)$$

is an isomorphism.

Proof. Since the restriction of $[M, \partial M]$ to $\mathrm{int}(M)$ is $[\mathrm{int}(M)]$ there is a commutative diagram

$$\begin{array}{ccc} H_c^k(\mathrm{int}(M)) & \xrightarrow{\cap[\mathrm{int}(M)]} & H_{n-k}(M) \\ \downarrow & & \downarrow \mathrm{Id} \\ H^k(M, \partial M) & \xrightarrow{\cap[M,\partial M]} & H_{n-k}(M). \end{array}$$

By Claim 2.6 the first vertical arrow is an isomorphism. The result follows since Poincaré duality tells us that the upper arrow is an isomorphism. □

Notice that for M a compact, oriented n-manifold with boundary there is a map $\cap[M, \partial M] \colon H^*(M) \to H_{n-*}(M, \partial M)$. We shall also see that it is also true that

$$\cap[M, \partial M] \colon H^k(M) \to H_{n-k}(M, \partial M)$$

is an isomorphism.

The last fact that we need is the following.

Lemma 2.10. *Let M be a compact manifold, possibly with boundary. Then $H^*(M)$ and $H^*(M, \partial M)$ are finitely generated.*

Claim 2.11. *(Wilder's Theorem) Let K and L be compact subsets of M and $K \subset \mathrm{int}L \subset L$. Then the restriction mapping $H^i(L) \to H^i(K)$ has finitely generated image.*

Proof. M is locally compact and every point has a compact contractible neighborhood, which clearly has finitely generated cohomology.

We prove by induction on i that Wilder's Theorem holds for H^i. It is obviously true for $i = -1$.

Suppose that we know the result for some $i - 1$ and let us establish it for i. Fix a compact set L and consider the set of all compact subsets $A \subset \mathrm{int}(L)$ for which there is a compact subset A' with

$$A \subset \mathrm{int}(A') \subset A' \subset \mathrm{int}(L)$$

with the restriction mapping $H^i(L) \to H^i(A')$ has finitely generated image. Every point in $\mathrm{int}(L)$ has such a neighborhood.

We claim that if A_0 and A_1 are compact subsets of $\mathrm{int}(L)$ with the stated property on H^i then $A_0 \cup A_1$ also has this property. We fix compact subsets A'_i associated with A_i. Fix compact subsets A''_0 and A''_1 with A''_i contained in the interior of A'_i and with A_i contained in the interior of A''_i. We consider the Meyer-Vietoris sequences

$$\begin{array}{ccccc} H^{i-1}(\mathrm{int}(A'_0) \cap \mathrm{int}(A'_1)) & \longrightarrow & H^i(\mathrm{int}(A'_0) \cup \mathrm{int}(A'_1)) & \longrightarrow & H^i(\mathrm{int}(A'_0)) \oplus H^i(\mathrm{int}(A'_1)) \\ \downarrow & & \downarrow & & \\ H^{i-1}(\mathrm{int}(A''_0) \cap \mathrm{int}(A''_1)) & \longrightarrow & H^i(\mathrm{int}(A''_0) \cup \mathrm{int}(A''_1)) & & \end{array}$$

There is a compact subset between $A''_0 \cup A''_1$ and $\mathrm{int}(A'_0) \cup \mathrm{int}(A'_1)$. Thus, it follows by the inductive hypothesis that

$$H^{i-1}(\mathrm{int}(A'_0) \cup \mathrm{int}A'_1) \to H^{i-1}(\mathrm{int}(A''_0) \cup \mathrm{int}(A''_1))$$

has finitely generated image. It also follows from the hypothesis on A_0 and A_1, and the definition of A'_0 and A'_1 that the sum of the restriction maps $H^i(L) \to H^i(\mathrm{int}(A'_0) \oplus H^i(\mathrm{int}(A'_1)$ has finitely generated image.

It is now an easy diagram chase to show that the image of $H^i(L) \to H^i(\mathrm{int}A''_0) \cup \mathrm{int}(A''_1))$ has finitely generated image. Since there is a compact subset containing $A_0 \cup A_1$ in its interior and contained in $\mathrm{int}(A'''_0) \cup \mathrm{int}(A''_1)$, it follows that $A_0 \cup A_1$ has the required property. Thus, all compact subsets of $\mathrm{int}(L)$ have the required property. Since L was an arbitrary compact subset this completes the induction and establishes the result.

□

Now we return to the proof of the theorem. Let M be a compact manifold. Then we see that $H^*(M) \to H^*(M)$ has finitely generated image, meaning that $H^*(M)$ is finitely generated. It then follows from Poincaré duality that the same is true for $H_*(M, \partial M)$. But the homology of ∂M is also finitely generated, so that from the long exact sequence of the pair we see that $H_*(M)$ is finitely generated.

3 Exercises

1) Show that if $X = U \cup V$ with U an V open subsets of a locally compact space X, then there is a Meyer-Vietoris sequence of cohomology with compact supports

$$\to H_c^*(U \cap V) \to H_c^*(U) \oplus H_c^*(V) \to H_c^*(X) \to H_c^{8+1}(U \cap V) \to .$$

2) Show that if an oriented n-manfiold $M = U \cup V$ with U and V open then the fundamental class of M restricts to give the fundamental class of U and V.
3) With notation from 2) show that the diagram whose upper row is the Meyer-Vietoris sequence for cohomology with compact supports and whose lower row is the Meyer-Vietris sequence for ordinary homology and whose vertical maps are cup product with the fundamental class forms a commutative diagram.
4) Suppose that $U_1 \subset U_2 \subset U_3 \subset \cdots$ with $X = \cup_k U_k$ and the U_k being open subsets of X. Show that $H_c^*(X) = \lim_{\to} H_c^*(U_k)$
5) Fill in the details of the argument sketched above that there is a restriction mapping $H_*^{lf}(X) \to H_*^{lf}(U)$ when U is an open subset of a locally compact Hausdorff space X.