Lecture 7: Consequences of Poincaré Duality

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Let M be a closed (compact, without boundary) oriented n-manifold, and let $[M]$ be its fundamental class.

For all k we have that the map $\alpha \mapsto \alpha \cap [M]$ induces an isomorphism

$$\cap[M] \colon H^k(M) \to H_{n-k}(M).$$

We also know that both the homology and cohomology of M are finitely generated. Thus, the Universal Coefficient Theorem gives a short exact sequence.

$$0 \to \mathrm{Hom}(\mathrm{Tor}(H_{k-1}(M)), \mathbb{Q}/\mathbb{Z}) \to H^k(M) \to \mathrm{Hom}(H_k(M)/\mathrm{Tor}, \mathbb{Z}) \to 0.$$

The first term is the torsion subgroup of $H^k(M)$ and the last is the free abelian group which is the quotient of $H^k(M)$ by its torsion subgroup.

Thus, we deduce isomorphisms

$$\mathrm{Hom}(H_k(M)/\mathrm{Tor}, \mathbb{Z}) \xrightarrow{\cong} H_{n-k}(M)/\mathrm{Tor}$$

$$\mathrm{Hom}(\mathrm{Tor}H_{k-1}(M), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathrm{Tor}(H_{n-k}(M)).$$

These isomorphisms are equivalent to pairings

$$H_k(M)/\mathrm{Tor} \otimes H_{n-k}(M)/\mathrm{Tor} \to \mathbb{Z},$$

called *the intersection pairing*, and

$$\mathrm{Tor}(H_{k-1}(M) \otimes \mathrm{Tor}(H_{n-k}(M)) \to \mathbb{Q}/\mathbb{Z},$$

called the *linking pairing*. Each of these pairings is perfect in the sense that the adjoint of these pairings are the above isomorphisms.

1 Relationship to the Thom Isomorphism

Suppose that $E \to X$ is an n-dimensional vector bundle. We give this bundle a metric (a positive definite pairing on each fiber varying continuously as we change fibers). This is equivalent to reducing the structure group of the bundle from $GL(n, \mathbb{R})$ to $O(n)$. Then we denote by $D(E) \to X$ the unit disk bundle and $S(E) \to X$ the unit sphere bundle.

Theorem 1.1. *(Thom Isomorphism) Suppose that the bundle $\pi \colon E \to X$ is orientable n-dimensional vector bundle (meaning that the structure group has been reduced from $O(n)$ to $SO(n)$). Then there is a unique cohomology class $U \in H^n(D(E), S(E))$ whose restriction to any fiber is the relative fundamental class of that fiber. Furhermore, the map*

$$H^*(X) \to H^{*+n}(D(E), S(E))$$

given by

$$a \mapsto \pi^* a \cup U$$

is an isomorphism

Proof. This follows directly from a relative version of the Serre Spectral Sequence. □

Definition 1.2. The class U in the above theorem is called the *Thom Class* and the isomorphism is called the *Thom Isomorphism.*

Now suppose that M is an n-dimensional manifold and $R \subset M$ is a closed k-dimensional submanifold. Then R has a normal bundle in M; that is to say there is a vector bundle $\nu \to R$ and a diffeomorphism ψ_R from $D(\nu)$ onto a neighborhood of R in M, the map sending the zero section of $D(\nu)$ by the identity to R. Such a pair consisting of a bundle and a diffeomorphism is called a tubular neighborhood

Now suppose that both M and R are oriented. Then the normal bundle ν inherits an orientation. Let U be the Thom class of this orientation. Under the diffeomorphism and excision we can view U as a class in $H^{n-k}(M, M \setminus D(E))$.

Proposition 1.3. *The image of U in $H^{n-k}(M)$ is Poincaré dual to the image of the fundamental class $[X]$ in $H_k(M)$ under the embedding $X \to M$.*

Proof. Since $H_k(D(\nu)) = \mathbb{Z}$ generated by $[X]$ embedded as the zero section it follows that $U \in H^{n-k}(D(\nu), S(\nu))$ is the Poincaré dual class to $[X]$ in

$(D(\nu), S(\nu)$. Thus, $U \cap [D(\nu), S(\nu)] = [X]$. Since the fundamental class of M restricts to the relative fundamental class of $(D(\nu), S(\nu))$, it follows that $U \cap [M] = \pm[X]$. The choice of U from the orientations of M and R leads to a sign of $+1$. $\square$

Notice that if S is an oriented submanifold of dimension $n-k$ meeting R transversally, then the algebraic intersection of R with S is $\langle [U], [S] \rangle$.

2 The Intersection Pairing

Unraveling the definitions we see that the pairing

$$\varphi\colon H_k(M)/\mathrm{Tor} \otimes H_{n-k}(M)/\mathrm{Tor} \to \mathbb{Z}$$

is given as follows. For homology classes $a \in H_k(M)$ and $b \in H_{n-k}(M)$, the pairing $\varphi(a,b)$ is obtained (by taking Poincaré dual cohomology classes $\alpha \in H^{n-k}(M)$ and $\beta \in H^k(M)$ and forming

$$\langle \alpha \cup \beta, [M] \rangle.$$

Equivalently, the intersection $a \cdot b$ is given by the evaluation on b of the Poincaré dual cohomology class to a. From the first description it follows that the intersection pairings are signed symmetric:

$$\varphi(a_k, b_{n-k}) = (-1)^{k(n-k)} \varphi(b_{n-k}, a_k).$$

Let us give a geometric description of the intersection pairing in the case when the ambient manifold is smooth.

Given homology classes $a \in H_k(M)$ and $b \in H_{n-k}(M)$ we choose cycle representatives $\widetilde{a}$ and $\widetilde{b}$. We can assume that every singular simplex appearing each of these cycles is a smooth map and also that any two simplices meet transversally. This means that the only points of intersection are where the interior of a k-simplex in $\widetilde{a}$ meets the interiior of an $(n-k)$-simplex in $\widetilde{b}$. At every such point x of intersection both $\widetilde{a}$ and $\widetilde{b}$ are local embeddings and their tangent spaces are complementary in T_xM. We assign a sign to each point of intersection by comparing the direct sum of orientations on the tangent space of $\widetilde{a}$ and of $\widetilde{b}$ with the ambient orientation of the tangent space of M. The sum of the signs over the (finitely many) points of intersection gives the intersection pairing applied to (a,b).

To see that the pairing is well-defined suppose we have $\widetilde{a}$ homologous to $\widetilde{a}'$ both being transverse to $\widetilde{b}$. Then there is a $(k+1)$-chain $\widetilde{c}$ with

$\partial\widetilde{c} = \widetilde{a}' - \widetilde{a}$. we can suppose that $\widetilde{c}$ is also transverse to $\widetilde{b}$. Then the top dimensional simplices meet in a 1-manifold whose boundary is either in a codimension-1 face of $\widetilde{c}$ or of $\widetilde{b}$. Since $\widetilde{b}$ is a cycle its codimension-1 faces cancel out in pairs. This means that the intersection 1-manifold continues across such faces without introducing a boundary.. A similar argument works for the codimension-1 faces of $\widetilde{c}$ that are interior to $\widetilde{c}$. Thus, we see that the boundary of the intersection of $\widetilde{c}$ with $\widetilde{b}$ is $\widetilde{a}' \cdot \widetilde{b} - \widetilde{a} \cdot \widetilde{b}$. But the algebraic boundary of a 1-manifold is zero. This shows that varying $\widetilde{a}$ by a boundary does not change the algebraic intersection with $\widetilde{b}$. Symmetrically, varying $\widetilde{b}$ by a boundary does not changes its algebraic intersection with $\widetilde{a}$. This shows that the algebriac intersection is well-defined on homology. It is exactly the pairing produced by Poincaré Duality.

It is also clear from this description that the pairing is signed symmetric.

2.1 Middle dimensional intersection pairings: $4k+2$ case

Now suppose that M is closed, oriented and of dimension $2n$. Then we have a pairing

$$H_n(M)/\mathrm{Tor} \otimes H_n(M)/\mathrm{Tor} \to \mathbb{Z}$$

that is $(-1)^n$ symmetric and unimodular, meaning that if we choose a basis then the pairing is represented by a $(-1)^n$ symmetric matrix of determinant ± 1..

Claim 2.1. *If n is odd, then there is a basis in which the pairing is an orthogonal sum of 2×2 matrices*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. First note that by skew symmetry we have $x \cdot x = 0$ for all $x \in H_n(M)$. Let x be an indivisible element, i.e., part of a basis. Then there is a homomorphism $H_n(M) \to \mathbb{Z}$ sending x to 1. Hencem there is y such that $x \cdot y = 1$

The 2×2 matrix giving the pairing on the span of x, y is exactly the 2×2 matrix given in the statement of the claim. Since this matrix is unimodular, x, y generate an orthogonal direct summand of $H_n(M)$ and we continue by induction. □

Corollary 2.2. *If M^{4k+2} is a closed, orientable manifold, the the Euler characteristic of M is even*

Notice that this is not true without the orientability assumption: The Euler characteristic of $\mathbb{R}P^2$ is 1.

2.2 Middle Dimensional Intersection Pairings: $4k$ case

If M^{4k} is a closed, oriented manifold then the intersection pairing

$$H_{2k}(M) \otimes H_{2k}(M) \to \mathbb{Z}$$

is symmetric. Choosing a basis it is given by a symmetric matrix of determinant ± 1.

The most elementary of such pairings are $\langle 1 \rangle$ and $\langle -1 \rangle$: one dimensional pairings with generator x with $x \cdot x = \pm 1$. Of course, we can take orthogonal direct sums of these. Pairings represented by diagonal matrices with ± 1's down the diagonal. But there are other pairings. There is the hyperbolic pairing given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We know that this pairing is not diagonalizable since $x \cdot x$ is even for all x. A form with this property is called *an even form*. It is an easy exercise to show that a form is even if and only if any matrix representative for it has only even entries down the diagonal. The *parity* of a pairing is even if $x \cdot x \equiv 0 \pmod 2$ for all x, and otherwise the parity is odd.

Another example of an even pairing is given by the matrix associated with the Dynkin diagram of E_8. It is an 8×8 matrix with basis identified with the vertices in the Dynkin diagram for the Lie algebra E_8. This matrix has all diagonal entries $+2$. All off diagonal elements are either 0 or 1, and an off diagonal entry at position (i, j) is 1 if and only if there is a bond in the Dynkin diagram connecting the i^{th} and j^{th} vertex. It turns out that this matrix has determinant ± 1 (which turns out to be equivalent to the fact that the center of the simply connected form of E_8 is the trivial group, or equivalently that E_8 has no non-simply connected form). The form is an even, positive definite form.

Classifying non-degenerate symmetric forms over $\mathbb{R}$ is easy:

Claim 2.3. *Let V be a finite dimensional vector space with a non-degenerate symmetric (real linear) pairing*

$$A \colon V \otimes V \to \mathbb{R}.$$

(Non-degernate means that if $A(v, w) = 0$ for all $w \in V$ then $v = 0$.) Then there is a basis $\{e_1, \ldots, e_k\}$ for V such that $A(e_i, e_i) = \pm 1$ for all i, and $A(a_i, a_j) = 0$ for all $i \neq j$. That is to say, the symmetric matrix of the pairing is diagonal with ± 1s down the diagonal. The number of $+1$'s and the number of -1's that appear are invariants of the isomorphism class of the pairing.

Proof. Suppose that $V \neq 0$ and choose $x \in V$. Suppose that$A(x,x) \neq 0$. Then $e_1 = x/\sqrt{|A(x,x)|}$ has $A(e_1,e_1) = \pm 1$. If $A(x,x) = 0$, then there is $y \in V$ with $A(x,y) = 1/2$. If $A(y,y) = 0$, then $A(x+y,x+y) = 1$. Thus, we can always fine $x \in V$ with $A(x,x) \neq 0$, and hence there is an element $e_1 \in V$ with $A(e_1,e_1) = \pm 1$. Extend e_1 to a basis $\{e_1,\ldots,e_k\}$ and for every $i > 1$ replace e_i with $e_i - A(e_1,e_i)e_1$. After this replacement $A(e_1,e_i) = 0$ for all $i > 1$. This means that V is an orthogonal sum of $\langle e_1 \rangle$ and the subspace V' spanned by $\{e_2,\ldots,e_k\}$. We then go by induction to find a basis as required.

Arrange that $A(e_i,e_i) = +1$ for $1 \leq i \leq k^+$ and equal -1 for $k^+ + 1 \leq i \leq k$ and let V^+ be the subspace of V spanned by the $\{e_i\}_{i=1} Y k^+$ and V^- be the subspace spanned by $\{e_i\}_{i=k^+ +1}^{k}$. Then the pairing is positive definite on V^+. Suppose that $V' \subset V$ is subspace on which the pairing is positive definite. Then $V' \cap V^- = \{0\}$ and hence the projection of V' to V^+ is an injection, meaning the $\dim(V^+) \leq k^+$. Thus, the number of $+1$'s down the diagonal is the the maximal dimension of any subspace on which the pairing is positive definite. □

Definition 2.4. We define the *signature* of a pairing to be the number of $+1$'s minus the number of -1's in any diagonalization of the pairing as in the previous claim. The pairing is *positive definite* if and only if the signature equals the rank and is *negative definite* if and only if the signature is equal to minus the rank. Otherwise, the pairing is said to be *indefinite*. Notice that the signature of a pairing is between minus the rank of the pairing and plus the rank of the pairing and is congruent to the rank modulo 2.

If L is a lattice (a finitely generated free abelian group) and if $A\colon L \otimes L \to \mathbb{Z}$ is a non-degenerate symmetric pairing (meaning the determinant of a matrix representative is non-zero) on L, then the *signature* of the pairing is the signature of the extension of A to a real-linear non-degenerate symmetric pairing on $L \otimes \mathbb{R}$.

We gave an example, E_8, of an even, unimodular, symmetric, positive-definite pairing of rank 8, and hence of signature 8. In fact we have:

Lemma 2.5. *If L is a lattice with an even symmeric, unimodular pairing, then the signature of L is congruent to 0 modulo 8.*

There is a nice classification result for indefinite, unimodular pairings. It is quite intricate to prove and we shall not discuss the proof.

Theorem 2.6. *Two indefinite unimodular pairings are isomorphic (over $\mathbb{Z}$) if and only if they have the same rank, signature and parity.*

This result does not extend to definite pairings.

Claim 2.7. *$E_8 \oplus \langle 1\rangle$ and $\oplus_{i=1}^{p}\langle 1\rangle$ are both odd pairings of rank 9 and signature 9. They are not isomorphic,*

Proof. The only thing that needs establishing to prove the claim is that the pairings are not isomorphic. Let us consider the x in each pairing with $x \cdot x = 1$. The only solutions in $E_8 \oplus \langle 1\rangle$ are the two generators of the second factor, whereas in $\oplus_{i=1}^{9}\langle 1\rangle$ there are the nine basis elements and their negatives. □

For every rank n there are only finitely many isomorphism classes of definite forms of rank n. For example, there are two even definite forms of rank 16:

3 The linking pairing

Let M be a closed, oriented n-manifold. The l*inking pairing* is the pairing $\mathrm{Tor}H_{k-1}(M) \otimes \mathrm{Tor}H_{n-k}(M) \to \mathbb{Q}/\mathbb{Z}$ produced by Poincaré duality. Let us give a geometric description along the lines of the intersection pairing. Let $a \in H_{k-1}(M)$ and $b \in H_{n-k}(M)$ be torsion classes. Choose representative cycles $\widetilde{a}$ and $\widetilde{b}$ which we can assume are smooth and in general position. The latter means that the cycles are disjoint. Then for some $N \geq 1$ there is a chain $\widetilde{c}$ of degree k with $\partial\widetilde{c} = N\widetilde{a}$. We can assume that $\widetilde{c}$ is smooth and transverse to $\widetilde{b}$. Thus, as in the definition of the intersection pairing we have the algebraic intersection $\widetilde{c}\cdot\widetilde{b} \in \mathbb{Z}$. As in the case of the intersection pairing, if we replace $\widetilde{c}$ by $\widetilde{c}'$, a chain with the same boundary with the property that $\widetilde{c}' - \widetilde{c}$ is itself a boundary we do not change the algebraic intersection with $\widetilde{b}$. More generally, if $\widetilde{c}'$ and $\widetilde{c}$ have the same boundary, then their difference is a cycle and the difference of their algebraic intersections with $\widetilde{b}$ is the homological intersection of the homology class represented by $\widetilde{c}' - \widetilde{c}$ with b. Since b is a torsion class, this homological intersection is zero. This proves that $\widetilde{c}\cdot\widetilde{b}$ is independent of the choice of $\widetilde{c}$ with boundary $N\widetilde{a}$. If we consider $\widetilde{c}'$ with $\partial\widetilde{c}' = N'\widetilde{a}$ we see that

$$\frac{1}{N}\widetilde{c}\cdot\widetilde{b} = \frac{1}{N'}\widetilde{c}'\cdot\widetilde{b}.$$

Thus, given the disjoint cycles $\widetilde{a}$ and $\widetilde{b}$ representing torsion classes we have a well-defined rational number defined by choosing $\widetilde{c}$ with $\partial\widetilde{c}$ being a multiple of $\widetilde{a}$ and intersecting $\widetilde{c}$ with $\widetilde{b}$ and dividing by the multiple in question. This rational number is the *linking number* of the disjoint cycles $\widetilde{a}$ and $\widetilde{b}$.

If we vary $\widetilde{a}$ by a homology, to another cycle $\widetilde{a}'$ disjoint from $\widetilde{b}$, then this homology will have an algebraic intersection number with $\widetilde{b}$ which is an integer and it is easy to see that the difference linking number of the cycles $\widetilde{a}'$ with $\widetilde{b}$ and the linking number of the cycles $\widetilde{a}'$ with $\widetilde{b}$ is exactly that integer. Similarly, if we vary $\widetilde{b}$ by a homology to $\widetilde{b}'$ disjoint from $\widetilde{a}$ this homology has intersection number with $\widetilde{a}$, which is an integer and the linking number of the cycles changes by this integer. Consequently, the homology classes a and b have a well-defined linking pairing in $\mathbb{Q}/\mathbb{Z}$. This pairing is denoted $\mathrm{lk}(a, b)$. It is the pairing produced by Poincaré duality.

The linking pairing is also signed symmetric: If a has degree $k-1$ and b has degree $n-k$, then

$$\mathrm{lk}(a, b) = (-1)^{k(n-k+1)}\mathrm{lk}(b, a).$$

4 Homotopy Type of simply connected 4-manifolds

Let X be a simply connected space whose homology satisfies Poincaré duality in dimension 4, meaning that there is a classes $[X] \in H_4(X)$ such that $\cap[X]$ induces an isomorphism $H^*(X) \to H_{4-*}(X)$. Then $H_2(X)$ is a free abelian group (since $\mathrm{Tor}(H_2(X))$ is isomorphic to $\mathrm{Tor}(H^2(X))$ which in turn is dual to $\mathrm{Tor}(H_1(X))$ which vanishes since X is simply connected). Thus, there is a map $\vee S^2 \to X$ from a wedge of 2-spheres to X inducing an isomorphism on H_2. Since $H_3(X) = 0$ it follows that $H_*(X, \vee S^2) = 0$ for $* < 4$ and $H_4(X) \to H_4(X, \vee S^2)$ is an isomorphism, so that $H_4(X, \vee S^2) = \mathbb{Z}$. This means that there is an map $f\colon (D^4, S^3) \to (X, \vee S^2)$ inducing an isomorphism on H_4. Thus, we form $\vee S^2 \cup D^4 \to X$ where the map to X which is given by f on D^4 and the attaching map $S^3 \to \vee S^2$ is the restriction of f to S^3. The resulting map is an isomorphism on homology and hence is a homotopy equivalence.

The group $\pi_3(\vee_{i=1}^k S^2)$ is isomorphic to the group of $k \times k$ symmetric matrices over $\mathbb{Z}$. One way to see this is deform a map $S^3 \to \vee S^2$ transverse to a point in the interior of each S^2. Then the preimage of each point gives us a framed link L in S^3 and the linking number of these (and the self-linking number using the framing) gives the symmetric matrix. The condition that X satisfies Poincaré duality is the condition that this matrix is unimodular. Two such X are homotopy equivalent if and only if the pairings are isomorphic.

We have sketched a proof of the following:

Claim 4.1. *Simply connected CW complexes satisfying 4-dimensional Poincaré duality up to homotopy equivalence are classified by the isomorphism type* *of the intersection pairing on H_2. All unimodluar, symmetric pairing come from such CW complexes.*

This leads naturally to a question:

Question: Which unimodular pairings are realized as the intersection pairing on H_2 of a closed, oriented 4-manifold

We have some examples $\mathbb{C}P^2$ with the orientation induced from its natural complex structure represents $\langle 1\rangle$; $\mathbb{C}P^2$ with the opposite orientation represents $\langle -1\rangle$. $S^2 \times S^2$ represents

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have:

Theorem 4.2. *(M. Freedman) Every symmetric unimodular form is the intersection form on H_2 of a simply connected, oriented, closed topological 4-manifold. If the form is even it is the form of a unique 4-manifold up to homeomorphism. If it is an odd form there are exactly two non-homeomorphic 4-manifolds realizing this form. For each odd form exactly one of these 4-manifolds with that intersection form has the property that its product with a circle has a smooth structure.*

Theorem 4.3. *(S. Donaldson) If a positive definite form is realized as the intersection form on H_2 a smooth, simply connected 4-manifold, then that form is diagonal with +1's down the diagonal. Thus, for example, $E_8 \oplus \langle 1\rangle$ is not the form of a smooth 4-manifold.*

The contrast of these two theorems shows that the theory of topologiical 4-manfiolds and smooth 4-manifolds differ. In fact, they differ drastically.

Theorem 4.4. *(R. Friedman and J. Morgan) There are topological manifolds with infinitely many non-diffeomorphic smooth structures. One example is $\mathbb{C}P^2$ blown up 9 times.*

By contrast, in every other dimension any compact topological manifold has at most finitely many non-diffeomorphic smooth structures.

5 Lefschetz Duality

Let M be a compact, oriented n-manifold. Lefschetz duality is equivalent to the statement that the induced pairings

$$H_k(M, \partial M)/\mathrm{Tor} \otimes H_{n-k}(M)/\mathrm{Tor} \to \mathbb{Z}$$

and

$$\mathrm{Tor}(H_{k-1}(M,\partial M)) \otimes \mathrm{Tor}(H_{n-k}(M)) \to \mathbb{Q}/\mathbb{Z}$$

are perfect pairings. Of course they are still perfect pairings if we reverse the roles of relative and absolute homology. This means that

$$\cap[M,\partial M] \colon H^k(M) \to H_{n-k}(M,\partial M)$$

is also an isomorphism.

Thus, Lefschetz duality tells us that the long exact sequences of homology and cohomology are dual:

$$\begin{array}{ccccccccc} H^k(M,\partial M) & \longrightarrow & H^k(M) & \longrightarrow & H^k(\partial M) & \longrightarrow & H^{k+1}(M,\partial M) & \longrightarrow \\ {\scriptstyle \cap[M,\partial M]}\downarrow & & {\scriptstyle \cap[M,\partial M]}\downarrow & & {\scriptstyle \cap[\partial M]}\downarrow & & {\scriptstyle \cap[M,\partial M]}\downarrow & \\ H_{n-k}(M) & \longrightarrow & H_{n-k}(M,\partial M) & \longrightarrow & H_{n-k-1}(\partial M) & \longrightarrow & H_{n-k-1}(M) & \longrightarrow \end{array}$$

Proposition 5.1. *Now suppose that M is a $4k+1$ compact, oriented manifold with boundary. Then the signature of the intersection on $H_{2k}(\partial M)$ is zero.*

Proof. Consider the exact sequence

$$H_{2k+1}(M,\partial M) \xrightarrow{\partial} H_{2k}\partial M) \xrightarrow{i_*} H_{2k}(M).$$

Modulo torsion, the first term is dual to the last and i_* is the adjoint of ∂ with respect to the intersection pairing on $H_{2k}(M)$. That is to say $\partial(a)\cdot b = \langle a, i_*(b)\rangle$. In particular the rank of the image of ∂ is equal to the rank of the image of i_*. But the rank of the image of i_* is the rank of $H_{2k}(\partial M)$ minus the rank of the $\mathrm{Ker}(i_*)$, which by exactness is the rank of $H_{2k}(\partial M)$ minus the rank of the image of ∂. We conclude that the rank of the image of ∂ is equal to one-half the rank of $H_{2k}(\partial M)$.

Since $i_* \circ \partial = 0$, we see that the image of ∂ is a self-annihilating subspace of $H_{2k}(\partial M)$ (meaning that any two elements in this subspace has intersection product 0). Let $L \subset H_{2k}(\partial M)$ be the subgroup of all elements with the property that some positive multiple of the element is in the image of ∂. The image of L/Tor in $H_{2k}(\partial M)/\mathrm{Tor}$ is a direct summand which is self-annihilating under the intersection pairing. Furthermore, the rank of L/Tor is one-half the rank of $H_{2k}(\partial M)/\mathrm{Tor}$. It follows that the pairing is isomorphic to an orthogonal direct sum of pairings of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus, the signature of the pairing is trivial.

□

Corollary 5.2. *For any $k \geq 1$ the manifold $\mathbb{C}P^{2k}$ is not the boundary of a compact, oriented $(4k+1)$-manifold.*

Proof. The signature of the pairing on $H_{2k}(\mathbb{C}P^{2k})$ is $+1$. □

Consider the closed, oriented 5-manifold obtained by taking $\mathbb{C}P^2 \times I$ and gluing the ends together by the map induced by complex conjugation. The result is a closed oriented 5-manifold. Using the linking pairing on H_2 one can show that this manifold is not the boundary of a compact, oriented 6-manifold.