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Title: First Steps in Global Lie Theory: wild Riemann surfaces, their character varieties and topological symplectic structures

Abstract: Stokes data, properly understood (e.g. as wild monodromy representations), are nonabelian analogues of periods, generalising the better known monodromy representations of the topological fundamental group into nonabelian Lie groups. This has many applications due to the prevalence of irregular connections on curves in geometry and physics. I'll review some aspects of this theory I have found amusing, as well as some work extending many familiar construction to the wild world:

a) To start I'll describe some of the story leading up to the construction of the topological symplectic structures (P.B. Oxford thesis 1999, Adv. Math. 2001) and subsequent evolution leading to the general, purely algebraic approach (B. 2002, 2009, 2014, B.-Yamakawa 2015). They generalise the holomorphic version of the symplectic structures of Narasimhan, Atiyah-Bott, Goldman involving the topological fundamental group. Our approach gives a TQFT approach to moduli of meromorphic connections on curves, involving Lie group valued moment maps. This gives many new algebraic Poisson varieties: first of all the wild character varieties (which were the original aim), but more generally these techniques immediately yield many more examples: the fission varieties (e.g. allowing the structure group to change on different pieces of surface). Classification of these varieties, as "global analogues of Lie groups", is still at a quite elementary stage, but a rich theory of Dynkin diagrams exists for many examples.

b) I'll also discuss the intrinsic language to describe (wild) boundary conditions in 2d gauge theory, and how this leads to the notion of Stokes local system, yielding the explicit matrix presentations of the wild character varieties (and is really just the topological description of what Stokes-Borel-Ecalle multisummation does, complementary to the asymptotic/filtered viewpoint of Deligne-Malgrange---this story is reviewed in more detail in the paper "Topology of the Stokes phenomenon").

c) Concerning the modular deformation parameters (isomonodromy times), the right point of view seems to be to generalise the notion of Riemann surface to the notion of wild Riemann surface, in the spirit of Weil's 1958 Bourbaki talk, and view these symplectic varieties as their character varieties (in the spirit of Weil's 1948 text "Sur les courbes algébriques et les variétés qui s'en déduisent"), leading to the wild mapping class groups, generalising the usual (tame) setting. The simplest irregular example (involving the wild fundamental group) underlies the Drinfeld-Jimbo quantum group (and deformations of the underlying wild Riemann surface explain the natural G-braid group action of Lusztig, Kirillov-Reshetikhin and Soibelman). I'll briefly review how the study and generalisation of the nonlinear local systems formed by the wild character varieties can by viewed as "pure wall-crossing", motivated more by a desire to understand the new natural flat nonlinear connections than specific examples.

d) If time permits I'll recall how these topological two-forms fit together with the Bottacin-

Markman Poisson structure on the meromorphic Higgs bundle moduli spaces to give the wild nonabelian Hodge hyperkahler manifolds (Biquard-B. 2004). This gives in particular a rich bestiary of new special Lagrangian fibrations and a way to classify wild harmonic bundles in terms of holomorphic or topological data. Surprisingly these hyperkahler metrics are often complete even though the corresponding harmonic maps have infinite energy. The simplest examples, certain hyperkahler manifolds of real dimension four, are the "spaces of initial conditions" of the Painlevé equations. Painlevé knew his equations were deformations of equations for elliptic functions, and so we can now see this "Painlevé simplification" as a hyperkahler rotation, from meromorphic connections to meromorphic Higgs bundles. Not only does this story encompass many famous classical integrable systems like the Lagrange top (2 poles of order 2), and those studied by Mumford (in Tata lectures on Theta II), but several of these Painlevé integrable systems were used in Seiberg-Witten's 1994 solution of 4d N=2 super Yang-Mills theory for SU(2), and one of the higher rank generalizations, introduced by Garnier in 1919 (the simplified Schlesinger system), underlies the famous Gaudin model. It was solved by Garnier in terms of abelian functions by defining spectral curves, a method rediscovered in the soliton literature in the 1970s (see e.g. Adler-Van Moerbeke 1980, Linearization of Hamiltonian systems, Jacobi varieties and representation theory, p.337, or Verdier's 1980 Séminaire Bourbaki), before being generalised by Hitchin to the case where the base curve has genus >1 and then connected to the harmonic theory, which ultimately led to the understanding of Painleve's simplification as a hyperkahler rotation.