

KP integrability of triple Hodge integrals

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KP HIERARCHY

The **Kadomtsev–Petviashvili (KP) hierarchy** was introduced by Sato about 40 years ago. It can be represented in terms of **tau-function** $\tau(\mathbf{t}) \in \mathbb{C}[[t_1, t_2, t_3, \dots]]$ by the **Hirota bilinear identity**

$$\oint_{\infty} e^{\sum_{k>0} (t_k - t'_k) z^k} \tau(\mathbf{t} - [z^{-1}]) \tau(\mathbf{t}' + [z^{-1}]) dz = 0,$$

which encodes all nonlinear equations of the KP hierarchy. Here we use the standard short-hand notation

$$\mathbf{t} \pm [z^{-1}] := \left\{ t_1 \pm z^{-1}, t_2 \pm \frac{1}{2} z^{-2}, t_3 \pm \frac{1}{3} z^{-3}, \dots \right\}.$$

Infinite hierarchy of PDEs.

KP EQUATION

The first nontrivial term in the expansion of the l.h.s. of the Hirota bilinear identity gives the **KP equation**

$$\tau\tau_{11111} - 4\tau_1\tau_{111} + 3(\tau_{11})^2 + 3\tau\tau_{22} - 3(\tau_2)^2 - 4\tau\tau_{13} + 4\tau_1\tau_3 = 0,$$

where $\tau_{i_1 i_2 \dots} = \frac{\partial}{\partial t_{i_1}} \frac{\partial}{\partial t_{i_2}} \dots \tau$. The second derivative of this equation with respect to t_1 gives the KP equation in its standard form

$$3u_{22} = (4u_3 - 12uu_1 - u_{111})_1,$$

where $u = \frac{\partial^2}{\partial t_1^2} \log(\tau)$.

A KP tau-function, independent on even times $t_{2k}, k = 1, 2, \dots$,

$$\frac{\partial}{\partial t_{2k}} \tau = 0,$$

is a tau-function of the **KdV hierarchy**.

SATO GRASSMANNIAN

Let us consider the description of the space of solutions for the KP hierarchy, introduced by [Sato '81]. Let us consider the space $H = H_+ \oplus H_-$ with

$$H_- = z^{-1}\mathbb{C}[[z^{-1}]], \quad H_+ = \mathbb{C}[z]$$

are generated by negative and nonnegative powers of z respectively. Then the Sato Grassmannian Gr consists of all closed linear spaces $\mathcal{W} \in H$, which are compatible with H_+ . Namely, an orthogonal projection $\pi_+ : \mathcal{W} \rightarrow H_+$ should be a Fredholm operator, i.e. both the kernel $\ker \pi_+ \in \mathcal{W}$ and the cokernel $\text{coker } \pi_+ \in H_+$ should be finite-dimensional vector spaces. The Grassmannian Gr consists of components $\text{Gr}^{(k)}$, parametrized by an index of the operator π_+ . We consider only the big cell $\text{Gr}_+^{(0)}$ of the component $\text{Gr}^{(0)}$, defined by the constraint $\ker \pi_+ = \text{coker } \pi_+ = 0$. We call $\text{Gr}_+^{(0)}$ the **Sato Grassmannian** for simplicity. There exists a bijection between the points of the Sato Grassmannian $\mathcal{W} \in \text{Gr}_+^{(0)}$ and the tau-functions with $\tau(0) = 1$.

A point of the Sato Grassmannian $\mathcal{W} \in \text{Gr}_+^{(0)}$ can be described by an **admissible basis**

$$\mathcal{W} = \text{span}\{\Phi_1^{\mathcal{W}}, \Phi_2^{\mathcal{W}}, \Phi_3^{\mathcal{W}}, \dots\}.$$

Convenient choice of the basis

$$\Phi_k(z) = z^{k-1}(1 + O(z^{-1})).$$

KAC-SCHWARZ OPERATORS

How to describe a given point of the Sato Grassmannian (=corresponding tau-function)?

Consider the ring of differential operators with coefficients formal Laurent series in the variable z^{-1}

$$\mathcal{D} := \mathbb{C}((z^{-1}))\left[\left[\frac{\partial}{\partial z}\right]\right]$$

and its subrings $\mathcal{D}_{\pm} := H_{\pm}\left[\left[\frac{\partial}{\partial z}\right]\right]$. A natural direct sum decomposition holds

$$\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-.$$

DEFINITION

For any point of the Sato Grassmannian \mathcal{W} the **Kac-Schwarz algebra**

$$\mathcal{A}_{\mathcal{W}} := \{A \in \mathcal{D} \mid A \cdot \mathcal{W} \subset \mathcal{W}\}$$

is the algebra of the differential operators which stabilize this point.

LINEAR CONSTRAINTS AND MIWA PARAMETRIZATION

To any Kac–Schwarz operator $A \in \mathcal{A}_{\mathcal{W}}$ one can associate an operator \widehat{A} acting of the space of functions of \mathbf{t} variables such that the tau-function is its eigenfunction

$$\widehat{A} \cdot \tau_{\mathcal{W}} = c \tau_{\mathcal{W}}.$$

Let $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ be a diagonal matrix. For any function f , dependent on the infinite set of variables $\mathbf{t} = (t_1, t_2, t_3, \dots)$, let

$$f([\Lambda^{-1}]) := f(\mathbf{t}) \Big|_{t_k = \frac{1}{k} \text{Tr} \Lambda^{-k}}$$

be the **Miwa parametrization**. For any admissible basis the tau-function of the KP hierarchy in the Miwa parametrization is equal to the ratio of the determinants

$$\tau_{\mathcal{W}}([\Lambda^{-1}]) = \frac{\det_{i,j=1}^N \Phi_i^{\mathcal{W}}(\lambda_j)}{\Delta(\lambda)},$$

where $\Delta(\lambda) := \prod_{i < j} (\lambda_j - \lambda_i)$ is the **Vandermonde determinant**. Moreover, if for some function $\tau_{\mathcal{W}}$ this equation holds for all $N \in \mathbb{Z}_{\geq 0}$, then $\tau_{\mathcal{W}}$ is a tau-function of the KP hierarchy.

Relation to matrix models.

CANONICAL KAC–SCHWARZ OPERATORS AND QUANTUM SPECTRAL CURVE

Consider the space of the pairs of operators ($\mathcal{D}_- = z^{-1}\mathbb{C}[[z^{-1}]]\left[\left[\frac{\partial}{\partial z}\right]\right]$)

$$\text{Gr}_{\mathcal{D}} := \left\{ (P, Q) \in \mathcal{D}^2 \mid [P, Q] = 1, P - \frac{\partial}{\partial z} \in z^{-1}\mathcal{D}_-, Q - z \in \mathcal{D}_- \right\}.$$

THEOREM (A.A. '21)

There is a bijection between $\text{Gr}_+^{(0)}$ and $\text{Gr}_{\mathcal{D}}$

$$\text{Gr}_+^{(0)} \ni \mathcal{W} \mapsto \rho(\mathcal{W}) = (P_{\mathcal{W}}, Q_{\mathcal{W}}) \in \text{Gr}_{\mathcal{D}}.$$

Operator $P_{\mathcal{W}}$, by construction, always annihilates the wave function

$$P_{\mathcal{W}} \cdot \Phi_1 = 0,$$

and it can be considered as the **quantum spectral curve** operator. Operator $Q_{\mathcal{W}}$ is the raising operator, which generates a basis for a point of the Sato Grassmannian

$$\mathcal{W} = \text{span}\{\Psi, Q_{\mathcal{W}} \cdot \Psi, Q_{\mathcal{W}}^2 \cdot \Psi, \dots\}.$$

The Kac–Schwarz algebra is generated by $P_{\mathcal{W}}$ and $Q_{\mathcal{W}}$.

SYMMETRIES OF KP HIERARCHY AND HEISENBERG-VIRASORO ALGEBRA

It is well known that a certain extension of the algebra $\mathfrak{gl}(\infty)$ and corresponding group $GL(\infty)$ act on the space of KP tau-functions (Sato's Grassmannian). Let us consider the important Heisenberg–Virasoro subalgebra of $\mathfrak{gl}(\infty)$. The **Heisenberg–Virasoro subalgebra** of (the central extension of) $\mathfrak{gl}(\infty)$ is generated by the operators

$$\widehat{J}_k = \begin{cases} \frac{\partial}{\partial t_k} & \text{for } k > 0, \\ 0 & \text{for } k = 0, \\ |k|t_{|k|} & \text{for } k < 0, \end{cases}$$

unit, and the Virasoro operators

$$\widehat{L}_m = \frac{1}{2} \sum_{a+b=-m} a b t_a t_b + \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2} \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b}.$$

These operators satisfy the commutation relations

$$\begin{aligned} [\widehat{J}_k, \widehat{J}_m] &= k \delta_{k+m,0}, \\ [\widehat{L}_k, \widehat{J}_m] &= -m \widehat{J}_{k+m}, \\ [\widehat{L}_k, \widehat{L}_m] &= (k-m) \widehat{L}_{k+m} + \frac{1}{12} \delta_{k+m,0} (k^3 - k). \end{aligned}$$

HEISENBERG-VIRASORO GROUP

Operators \widehat{J}_m and \widehat{L}_m correspond to the operators

$$j_m = z^m, \quad \mathbf{1}_m = -z^m \left(z \frac{\partial}{\partial z} + \frac{m+1}{2} \right)$$

acting on the Sato Grassmannian. The Heisenberg–Virasoro group \mathcal{V} is generated by the operators $\widehat{J}_k, \widehat{L}_k$ and a unit,

$$\mathcal{V} := \left\{ C e^{\sum a_k \widehat{J}_k + b_k \widehat{L}_k} \mid a_k, b_k, C \in \mathbb{C} \right\}.$$

LEMMA

For a KP tau-function $\tau(\mathbf{t})$,

$$\tilde{\tau}(\mathbf{t}) = C e^{\sum a_k \widehat{J}_k + b_k \widehat{L}_k} \cdot \tau(\mathbf{t})$$

is also a tau-function of KP (if well-defined).

LEMMA

If $\sum a_k j_k + b_k \mathbf{1}_k \in \mathcal{A}_{\mathcal{W}}$, then $\left(\sum a_k \widehat{J}_k + b_k \widehat{L}_k \right) \tau_{\mathcal{W}} = c \tau_{\mathcal{W}}$.

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INTERSECTION NUMBERS

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne–Mumford compactification of the moduli space of stable complex curves of genus g with n distinct marked points. The moduli space $\overline{\mathcal{M}}_{g,n}$ is defined to be empty unless the **stability condition**

$$2g - 2 + n > 0$$

is satisfied. We consider **Hodge integrals**

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_k} \psi_1^{m_1} \psi_2^{m_2} \cdots \psi_n^{m_n} \in \mathbb{Q},$$

where $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$ is the first Chern class of the line bundle corresponding to the cotangent space of the curve at the i -th marked point, and $\lambda_j \in H^{2j}(\overline{\mathcal{M}}_{g,n})$ is the j -th Chern class of the Hodge bundle \mathbb{E} . These integrals are trivial, unless the corresponding complex dimensions coincide

$$\sum_{l=1}^k j_l + \sum_{i=1}^n m_i = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n},$$

where $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$.

KONTSEVICH–WITTEN TAU-FUNCTION

In the early 90's Witten initiated new directions in the study of $\overline{\mathcal{M}}_{g,n}$. Consider the intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n}.$$

Let $T_j, j \geq 0$, be formal variables and let

$$\tau_1 := \exp \left(\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} h^{2g-2+n} F_{g,n}^1 \right),$$

where

$$F_{g,n}^1 := \sum_{a_1, \dots, a_n \geq 0} \frac{\prod T_{a_i}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n} \in \mathbb{Q}[T_1, T_3, T_5, \dots].$$

By Witten's conjecture the generating function τ_1 becomes a tau-function of the KdV hierarchy after the change of variables $T_n = (2n + 1)!! t_{2n+1}$, known as **times**.

THEOREM (KONTSEVICH, WITTEN)

The generating function τ_1 is a tau-function of the KdV hierarchy.

TOPOLOGICAL EXPANSION

Instead of the **genus expansion** of the generating function

$$\exp \left(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \hbar^{2g-2} \mathcal{F}_{g,n} \right),$$

where g is the genus, n is the number of the marked points, we consider the **topological expansion**

$$\exp \left(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \hbar^{2g-2+n} \mathcal{F}_{g,n} \right).$$

The latter is more convenient for the KP hierarchy description. Two descriptions are related by a change of the normalization of the variables t_k by an additional factor of \hbar .

$$\begin{aligned} \log(\tau_1) = & \hbar \left(\frac{1}{8} t_3 + \frac{1}{6} t_1^3 \right) + \hbar^2 \left(\frac{3}{16} t_3^2 + \frac{1}{2} t_3 t_1^3 + \frac{5}{8} t_1 t_5 \right) \\ & + \hbar^3 \left(\frac{15}{4} t_1 t_3 t_5 + \frac{105}{128} t_9 + \frac{35}{16} t_7 t_1^2 + \frac{3}{2} t_1^3 t_3^2 + \frac{5}{8} t_1^4 t_5 + \frac{3}{8} t_3^3 \right) + \dots \end{aligned}$$

NORBURY'S CLASSES AND BGW TAU-FUNCTION

The most natural alternative to the Kontsevich–Witten tau-function is the **Brézin–Gross–Witten (BGW) tau-function**. This tau-function governs the intersection theory with the insertions of the fascinating **Norbury's Θ -classes**,

$$\Theta_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n}).$$

Consider the generating function of the intersection numbers of Θ -classes and ψ -classes

$$F_{g,n}^0 = \sum_{a_1, \dots, a_n \geq 0} \frac{\prod T_{a_i}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{a_1} \psi_2^{a_2} \dots \psi_n^{a_n} \in \mathbb{Q}[T_1, T_3, T_5, \dots],$$
$$\tau_0 = \exp \left(\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} F_{g,n}^0 \right).$$

For this generating function **[Norbury '17]** has suggested a direct analog of Witten's conjecture.

THEOREM (N. K. CHIDAMBARAM, E. GARCIA-FAILDE, AND A. GIACCHETTO '22)

The generating function τ_{Θ} is the BGW tau-function of the KdV hierarchy in the variables $t_{2k+1} = \frac{T_k}{(2k+1)!!}$,

KONTSEVICH-TYPE MATRIX INTEGRAL

Generalized Kontsevich model

$$Z_V(\Lambda) = \mathcal{C}^{-1} \int [d\Phi] \exp \left(-\frac{1}{\hbar} \text{Tr} (V(\Phi) - \Phi V'(\Lambda)) \right).$$

For $V = z^3/3!$ the asymptotic expansion of the $N \times N$ **Kontsevich matrix model** describes the Kontsevich–Witten tau-function in the **Miwa parametrization**

$$\begin{aligned} \tau_1 \Big|_{t_k = \frac{1}{k} \text{Tr} \Lambda^{-k}} &= Z_{z^3/3!}(\Lambda) \\ &= \mathcal{C}^{-1} \int [d\Phi] \exp \left(-\frac{1}{\hbar} \text{Tr} \left(\frac{\Phi^3}{3!} + \frac{\Lambda \Phi^2}{2} \right) \right). \end{aligned}$$

τ_0 is nothing but a tau-function of the BGW model. KdV integrability of the BGW model follows from the relation to the generalized Kontsevich model **[Mironov, Morozov, and Semenov '96]**:

$$\tau_0 \Big|_{t_k = \frac{1}{k} \text{Tr} \Lambda^{-k}} = \tilde{\mathcal{C}}^{-1} \int [d\Phi] \exp \left(\frac{1}{2\hbar} \text{Tr} (\Lambda^2 \Phi + \Phi^{-1} - 2\hbar N \log \Phi) \right).$$

Admissible basis is given by the asymptotic expansion

$$\Phi_k^{KW}(z) = \sqrt{\frac{z}{2\pi\hbar}} \int_{\gamma} d\varphi (\varphi + z)^{k-1} e^{-\frac{1}{\hbar} \left(\frac{\varphi^3}{3!} + \frac{z\varphi^2}{2} \right)}.$$

Let us introduce

$$W := \frac{1}{z} \frac{\partial}{\partial z} - \frac{1}{2z^2}.$$

Using integration by parts one can get the canonical pair of the KS operators

$$Q_{KW} = z + \hbar W, \quad P_{KW} = \frac{\partial}{\partial z} + \hbar \frac{W^2}{2}.$$

The modified quantum spectral curve is given by the Airy equation

$$\left(\frac{\hat{y}^2}{2} - \hat{x} \right) \cdot \tilde{\Psi}_{KW} = 0,$$

where $\hat{x} = z^2/2$, $\hat{y} = \hbar \frac{\partial}{\partial x}$ and $\tilde{\Psi}_{KW} = \frac{e^{z^3/(3\hbar)}}{\sqrt{z}} \Phi_1^{KW}(z)$. The semi-classical limit yields the **Airy curve** $\frac{y^2}{2} = x$.

HEISENBERG–VIRASORO CONSTRAINTS

The Kontsevich–Witten ($\alpha = 1$) and Brézin–Gross–Witten ($\alpha = 0$) tau-functions τ_α satisfy the **Heisenberg–Virasoro constraints** [Dijkgraaf, Verlinde, Verlinde '91; Gross, Newman '92]

$$\begin{aligned}\frac{\partial}{\partial t_{2k}} \cdot \tau_\alpha &= 0, \quad k > 0, \\ \widehat{L}_k^\alpha \cdot \tau_\alpha &= 0, \quad k \geq -\alpha,\end{aligned}$$

where the Virasoro operators are given by

$$\widehat{L}_k^\alpha = \frac{1}{2} \widehat{L}_{2k} - \frac{1}{2\hbar} \frac{\partial}{\partial t_{2k+1+2\alpha}} + \frac{\delta_{k,0}}{16}.$$

These operators satisfy the commutation relations

$$\left[\widehat{L}_k^\alpha, \widehat{L}_m^\alpha \right] = (k - m) \widehat{L}_{k+m}^\alpha, \quad k, m \geq -\alpha.$$

The Virasoro constraints can be obtained from the Kac–Schwarz operators [Kac, Schwarz '91]. The Virasoro constraints completely specify the tau-functions τ_α , there are several ways to describe the solution.

ALGEBRAIC TOPOLOGICAL RECURSION

Consider the **cut-and-join operators**

$$\widehat{W}_0 = \sum_{k,m \in \mathbb{Z}_{\text{odd}}^+} \left(k m t_k t_m \frac{\partial}{\partial t_{k+m-1}} + \frac{1}{2} (k+m+1) t_{k+m+1} \frac{\partial^2}{\partial t_k \partial t_m} \right) + \frac{t_1}{8},$$

$$\widehat{W}_1 = \frac{1}{3} \sum_{k,m \in \mathbb{Z}_{\text{odd}}^+} \left(k m t_k t_m \frac{\partial}{\partial t_{k+m-3}} + \frac{1}{2} (k+m+3) t_{k+m+3} \frac{\partial^2}{\partial t_k \partial t_m} \right) + \frac{t_1^3}{3!} + \frac{t_3}{8}.$$

THEOREM

$$\tau_\alpha = \exp \left(\hbar \widehat{W}_\alpha \right) \cdot 1.$$

Topological expansion with $\tau_\alpha^{(0)} = 1$

$$\tau_\alpha = \sum_{k \geq 0} \hbar^k \tau_\alpha^{(k)}.$$

Algebraic topological recursion

$$\tau_\alpha^{(k)} = \frac{1}{k} \widehat{W}_\alpha \cdot \tau_\alpha^{(k-1)}.$$

GOAL

Goal: generalize the Kontsevich–Witten/Brézin–Gross–Witten theory to other interesting enumerative geometry problems

- **Tau-function of the solitonic integrable hierarchy**
- **Virasoro constraints**
- **Cut-and-join description**
- **Matrix model**

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GIVENTAL GROUP

Cohomological field theories capture the formal universal properties of the Gromov-Witten theory [Kontsevich, Manin '94]. Givental's group acts on semi-simple cohomological field theories [Givental '01, Teleman '12].

We consider the Givental group for the simplest, **rank one case**. It is parametrized by the **R-matrix**, $R(z) \in 1 + z\mathbb{C}[[z]]$, which satisfies the symplectic condition

$$R(z)R(-z) = 1$$

or, equivalently, $\log R(z)$ is an odd function of z .

QUANTIZATION OF THE R -MATRIX

Let

$$\widehat{z^{2k-1}} = \widehat{W}_k, \quad k \in \mathbb{Z},$$

where for positive k

$$\widehat{W}_k = - \sum_m \tilde{T}_m \frac{\partial}{\partial T_{m+2k-1}} + \frac{1}{2} \sum_{m=0}^{2k-2} (-1)^m \frac{\partial^2}{\partial T_m \partial T_{2k-m-2}}.$$

Here the so-called **dilaton shift** is given by $\tilde{T}_k = T_k - \hbar^{-1} \delta_{k,1}$. These operators commute for $k > 0$. Quantization

$$\widehat{R} := \exp \left(\log \widehat{R}(z) \right).$$

Below we call the group of such operators the **Givental group**.

Observation: operators \widehat{W}_k are similar to the Virasoro operators
($T_n = (2n + 1)!! t_{2n+1}$)

$$\widehat{L}_m = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2} \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b}.$$

FACTORIZATION

Operators \widehat{R} can be factorized. Let us introduce a formal series in two variables

$$V^R(z, w) := \frac{1 - R(-w)R(-z)}{w + z} \in \mathbb{C}[[z, w]].$$

The matrix of its coefficients $V^R(z, w) = \sum_{k, l=0}^{\infty} V_{kl}^R w^k z^l$ is symmetric, $V_{kl}^R = V_{lk}^R$. Let the linear change of the shifted variables from \mathbf{T} to \mathbf{T}^R and associated with it change of the dilaton shift are given by

$$\sum_{k=0}^{\infty} \tilde{T}_k^R z^k := R(-z) \sum_{k=0}^{\infty} \tilde{T}_k z^k, \quad \sum_{k=2}^{\infty} \delta_k z^k := z(1 - R(-z)).$$

Then, for any element of the Givental group and any series $Z(\mathbf{T})$ we have

LEMMA (GIVENTAL)

$$\widehat{R} \cdot Z(\mathbf{T}) = e^{\frac{1}{2} \sum_{i, j=0}^{\infty} V_{ij}^R \frac{\partial^2}{\partial T_i \partial T_j}} e^{\hbar^{-1} \sum_{k=2}^{\infty} \delta_k \frac{\partial}{\partial T_k}} Z(\mathbf{T}) \Big|_{T_k \mapsto T_k^R}.$$

Follows from the Baker–Campbell–Hausdorff formula.

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IDEA: COMPARE MATRICES v AND V^R .

Compare the action of the Givental operator

$$\widehat{R} \cdot Z(\mathbf{T}) = e^{\frac{1}{2} \sum_{i,j=0}^{\infty} V_{ij}^R \frac{\partial^2}{\partial T_i \partial T_j}} e^{\hbar^{-1} \sum_{k=2}^{\infty} \delta_k \frac{\partial}{\partial T_k}} Z(\mathbf{T}) \Big|_{T_k \mapsto T_k^R}$$

with the action of the Virasoro group element

$$\exp \left(\sum_{k=1}^{\infty} a_k \widehat{L}_k \right) = \exp \left(\sum_{k=1}^{\infty} a_k \sum_{m=1}^{\infty} m t_m \frac{\partial}{\partial t_{k+m}} \right) \exp \left(\frac{1}{2} \sum_{k,m=1}^{\infty} v_{km} \frac{\partial^2}{\partial t_k \partial t_m} \right).$$

The coefficients v_{nm} are the Grunsky coefficients of $h(z) := e^{\sum_{k>0} a_k z^{k+1} \partial_z} z$,

$$\sum_{k,m=1}^{\infty} v_{km} \eta_1^k \eta_2^m = \log \left(\frac{h(\eta_1) - h(\eta_2)}{\eta_1 - \eta_2} \right).$$

We identify $T_k \equiv (2k+1)!! t_{2k+1}$ and need to find $R(z)$ and $h(z)$ satisfying

$$V_{km}^R = (2k+1)!!(2m+1)!! v_{2k+1, 2m+1},$$

$$\frac{1 - R(-w)R(-z)}{w+z} = \frac{1}{2\pi(zw)^{3/2}} \int_{\gamma^2} d\eta_1 d\eta_2 \eta_1 \eta_2 e^{-\frac{\eta_1^2}{2z} - \frac{\eta_2^2}{2w}} \log \left(\frac{h(\eta_1) - h(\eta_2)}{\eta_1 - \eta_2} \right).$$

TWO-PARAMETRIC FAMILY OF GIVENTAL OPERATORS

Let us introduce two parameters, p and q , and

$$dx = f(z)df(z) = \frac{zdz}{(1 + \sqrt{p+qz})(1 + qz/\sqrt{p+q})},$$

$$y_0 = \frac{1}{z},$$

$$y_1 = \int^z y_0(\eta)dx(\eta) = \frac{\sqrt{p+q}}{p} \left(\log(1 + \sqrt{p+qz}) - \log\left(1 + \frac{qz}{\sqrt{p+q}}\right) \right),$$

where we assume that $p + q \neq 0$. The functions

$$\phi_k(z) := \left(-\frac{\partial}{\partial x}\right)^k \cdot \frac{1}{z} \in \mathbb{C}[z^{-1}]$$

define a change of variables if we associate kt_k with z^{-k} and $T_k^{q,p}$ with $\phi_k(z)$. This change of variables can be described by the recursion

$$T_0^{q,p}(\mathbf{t}) = t_1,$$

$$T_k^{q,p}(\mathbf{t}) = \left(q \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_k} + \frac{2q+p}{\sqrt{p+q}} \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k-1}} + \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k-2}} \right) T_{k-1}^{q,p}(\mathbf{t}).$$

IDENTIFICATION OF GIVENTAL AND HEISENBERG–VIRASORO OPERATORS

Let us consider a two-parametric family of functions satisfying symplectic condition

$$R_{q,p}(z) = \exp \left(- \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \left(p^{2k-1} + q^{2k-1} - \left(\frac{pq}{p+q} \right)^{2k-1} \right) z^{2k-1} \right),$$

Here B_{2k} are the Bernoulli numbers. We introduce the coefficients $v_k^{(\alpha)}$ and a_k

$$v_k^{(\alpha)} := [z^k] \int^z (f(\eta)^{2\alpha-1} - y_\alpha(\eta)) dx(\eta),$$

$$e^{-\sum_{k>0} a_k z^{k+1}} \frac{\partial}{\partial z} z = f(z) = \sqrt{2 \frac{p+q}{pq} \log \left(1 + \frac{qz}{\sqrt{p+q}} \right) - \frac{2}{p} \log(1 + \sqrt{p+q}z)}.$$

THEOREM

Rank one Givental operator coincides up to a linear change of variables with the element of the Heisenberg–Virasoro group of symmetries of the KP hierarchy iff $R(z)$ belongs to a family $R_{q,p}$ and then both act on any function of odd times $Z(\mathbf{T}) \in \mathbb{C}[[\mathbf{T}]]$

$$\widehat{R}_{q,p} \cdot Z(\mathbf{T}) \Big|_{\mathbf{T}=\mathbf{T}^{q,p}(\mathbf{t})} = e^{\hbar^{-1} \sum_{k=4} v_k^{(1)} \frac{\partial}{\partial t_k}} e^{\sum_{k \in \mathbb{Z}_{>0}} a_k \widehat{L}_k} \cdot Z(\mathbf{T}).$$

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GIVENTAL OPERATORS AND HODGE INTEGRALS

Let $\Lambda_g(u) = \sum_{i=0}^g u^i \lambda_i$ be the Chern polynomial of the Hodge bundle \mathbb{E} . Consider the m -parametric deformation of the Kontsevich–Witten tau-function

$$Z(\mathbf{T}; \mathbf{u}) = \exp \left(\sum_{g,n=0}^{\infty} \hbar^{2g-2+n} \sum_{a_1, \dots, a_n} \frac{\prod T_{a_i}}{n!} \int_{\mathcal{M}_{g,n}} \Lambda_g(u_1) \dots \Lambda_g(u_m) \psi_1^{a_1} \dots \psi_n^{a_n} \right).$$

In general this is not a KdV tau-function!

Insertion of the Hodge classes can be described by

$$Z(\mathbf{T}; \mathbf{u}) = \widehat{R}(\mathbf{u}) \cdot \tau_1(\mathbf{T}),$$

where

$$\widehat{R}(\mathbf{u}) := \exp \left(\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} \sum_{j=1}^m u_j^{2k-1} \widehat{W}_k \right)$$

is an element of the Givental group **[Mumford '83; Faber, Pandharipande '00]**.

TRIPLE HODGE INTEGRALS WITH CALABI–YAU CONDITION

Let us consider triple Hodge integrals with an additional **Calabi–Yau condition**

$$\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} = 0.$$

It is convenient to use the parametrization

$$u_1 = -p, u_2 = -q, u_3 = \frac{pq}{p+q}.$$

For $\alpha \in \{0, 1\}$ consider the generating function

$$\mathcal{F}_{g,n}^\alpha = \sum_{a_1, \dots, a_n \geq 0} \frac{\prod T_{a_i}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}^{1-\alpha} \Lambda_g(-q) \Lambda_g(-p) \Lambda_g\left(\frac{pq}{p+q}\right) \psi_1^{a_1} \psi_2^{a_2} \dots \psi_n^{a_n},$$

$$Z_{q,p}^\alpha(\mathbf{T}) = \exp\left(\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} \mathcal{F}_{g,n}^\alpha\right).$$

MAIN THEOREM

THEOREM

The generating functions of triple Hodge integrals, satisfying the Calabi–Yau condition, in the variables $\mathbf{T}^{q,p}$ are tau-functions of the KP hierarchy,

$$\tau_{q,p}^{(\alpha)}(\mathbf{t}) = Z_{q,p}^{\alpha}(\mathbf{T}^{q,p}(\mathbf{t})).$$

They are related to the Kontsevich–Witten ($\alpha = 1$) and Brézin–Gross–Witten ($\alpha = 0$) tau-functions by the elements of the Heisenberg–Virasoro group

$$\tau_{q,p}^{(\alpha)}(\mathbf{t}) = \exp\left(\hbar^{-1} \sum_{j=2+2\alpha}^{\infty} v_j^{(\alpha)} \frac{\partial}{\partial t_j}\right) \exp\left(\sum_{k=1}^{\infty} a_k \widehat{L}_k\right) \cdot \tau_{\alpha}(\mathbf{t}).$$

Linear Hodge integrals appear as a specification at $q = 0$ (or $p = 0$), because $\Lambda_g(0) = 1$. For this case with $\alpha = 1$ the KP integrability was proved by **[Kazarian '09]**.

Alternative proof of the integrability part of this theorem for $\alpha = 1$, based on the Mariño–Vafa formula, is given by **[Kramer '21]**.

We expect, that this is the most general family of the KP-integrable Hodge integrals.

HEISENBERG–VIRASORO CONSTRAINTS

To describe the Heisenberg–Virasoro constraints for the tau-functions $\tau_{q,p}^{(\alpha)}$ let us consider

$$\begin{aligned}\widehat{J}_k^{(q,p)} &= \frac{1}{2} \sum_{m=2k}^{\infty} \rho[2k, m] \widehat{J}_m, \\ \widehat{L}_k^{\alpha, (q,p)} &= \frac{1}{2} \sum_{m=2k}^{\infty} \sigma[2k, m] \widehat{L}_m - \frac{1}{2\hbar} \sum_{m=2k+1+2\alpha}^{\infty} \chi_{\alpha}[2k, m] \widehat{J}_m \\ &\quad + \frac{\delta_{k,0}}{16} - \frac{\delta_{k,-1}}{48} \frac{p^2 + pq + q^2}{p+q}.\end{aligned}$$

Here the coefficients $\sigma[k, m]$, $\rho[k, m]$, and $\chi_{\alpha}[k, m]$ are given by

$$\rho[k, m] = [z^m] f(z)^k, \quad \sigma[k, m] = [z^{m+1}] \frac{f(z)^{k+1}}{f'(z)}, \quad \chi_{\alpha}[k, m] = [z^m] f(z)^{k+2} y_{\alpha}(z),$$

where

$$f(z) = \sqrt{2 \frac{p+q}{pq} \log \left(1 + \frac{qz}{\sqrt{p+q}} \right) - \frac{2}{p} \log(1 + \sqrt{p+q}z)}.$$

HEISENBERG–VIRASORO CONSTRAINTS

THEOREM

The tau-functions of triple Hodge integrals satisfy the Heisenberg–Virasoro constraints

$$\begin{aligned}\widehat{J}_k^{(q,p)} \cdot \tau_{q,p}^{(\alpha)} &= 0, & k > 0, \\ \widehat{L}_k^{\alpha,(q,p)} \cdot \tau_{q,p}^{(\alpha)} &= 0, & k \geq -\alpha.\end{aligned}$$

These operators satisfy the commutation relations of the Heisenberg–Virasoro algebra

$$\begin{aligned}\left[\widehat{J}_k^{(q,p)}, \widehat{J}_m^{(q,p)} \right] &= 0, & k, m > 0, \\ \left[\widehat{L}_k^{\alpha,(q,p)}, \widehat{J}_m^{(q,p)} \right] &= k \widehat{J}_{k+m}^{(q,p)}, & k \geq -\alpha, m > 0, \\ \left[\widehat{L}_k^{\alpha,(q,p)}, \widehat{L}_m^{\alpha,(q,p)} \right] &= (k - m) \widehat{L}_{k+m}^{\alpha,(q,p)}, & k, m \geq -\alpha.\end{aligned}$$

String equation $\widehat{L}_{-1}^{1,(q,p)} \cdot \tau_{q,p}^{(1)} = 0$,

$$\widehat{L}_{-1}^{1,(q,p)} = \frac{1}{2} \widehat{L}_{-2} + \frac{2q+p}{2\sqrt{p+q}} \widehat{L}_{-1} + \frac{q}{2} \widehat{L}_0 - \frac{1}{2\hbar} \sum_{m=1}^{\infty} \chi_1[-2, m] \widehat{J}_m - \frac{1}{48} \frac{p^2 + pq + q^2}{p+q}.$$

COHOMOLOGICAL FIELD THEORIES

Cohomological field theories were introduced by [Kontsevich and Manin '94] to describe universal properties of Gromov–Witten theory. A **cohomological field theory (CohFT)** is given by a family of tensors $\Omega_{g,n}$. These tensors satisfy the compatibility conditions and are labeled by a number of the marked points n and genus g for all stable combinations $2g - 2 + n > 0$. For any CohFT Ω let us consider its partition function, dependent on the formal variables $\mathbf{T} = \{T_k^a; 1 \leq a \leq N, k \geq 0\}$,

$$Z_\Omega = \exp \left(\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \sum_{k_j} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n}) \prod_{j=1}^n \psi_j^{k_j} T_{k_j}^{a_j} \right) \in \mathbb{C}[\mathbf{T}][[\hbar]].$$

We also call it the (total) **ancestor potential**, because for CohFTs associated with the Gromov–Witten theory on some variety, function Z_Ω is the generating function of the so-called ancestor Gromov–Witten invariants.

Semi-simple CohFTs are classified by [Givental '01; Teleman '12].

CUT-AND-JOIN OPERATOR

Let

$$\widehat{J}_k^a := \begin{cases} (2k-1) \frac{\partial}{\partial T_{k-1}^a} & \text{for } k > 0, \\ T_{|k|}^a & \text{for } k \leq 0 \end{cases}$$

generate N copies of the Heisenberg algebra. For a semi-simple CohFT we construct a cubic operator

$$\widehat{W} = \sum_{\substack{i \leq j \leq k, \\ i+j+k \geq 0}} A_{a,b,c}^{i,j,k} \widehat{J}_i^a \widehat{J}_j^b \widehat{J}_k^c + \sum_{j \geq -1} B_a^j \widehat{J}_j^a,$$

where the coefficients $A_{a,b,c}^{i,j,k}$ and B_a^j are independent of \mathbf{T} and \hbar . We call it the **cut-and-join operator**.

THEOREM

A partition function of a semi-simple CohFT satisfies

$$Z_\Omega = \exp(\hbar \widehat{W}) \cdot 1.$$

ALGEBRAIC VERSION OF TOPOLOGICAL RECURSION

The **Chekhov–Eynard–Orantin topological recursion** is a universal procedure, which allows us to construct a family of symmetric differentials on a spectral curve with additional structures on it. For many known examples these differentials encode interesting invariants of enumerative geometry and mathematical physics.

The cubic cut-and-join formula is valid for the generating function of any Chekhov–Eynard–Orantin topological recursion with simple (possibly irregular) ramification points.

Consider the **topological expansion** (non-connected!)

$$Z = \sum_{k=0}^{\infty} Z^{(k)} \hbar^k.$$

COROLLARY

A partition function of the Chekhov–Eynard–Orantin topological recursion with simple (possibly irregular) ramification points (including all semi-simple CohFTs) satisfies the algebraic version of topological recursion with the initial condition $Z^{(0)} = 1$,

$$Z^{(k)} = \frac{1}{k} \widehat{W} \cdot Z^{(k-1)}.$$

CUT-AND-JOIN OPERATORS

Let

$$\tilde{h}(z) = \frac{p}{q\sqrt{p+q}} \frac{\exp\left(2\frac{q}{\sqrt{p+q}}z\right) - 1}{1 - \exp\left(-2\frac{p}{\sqrt{p+q}}z\right)} - \frac{1}{\sqrt{p+q}}.$$

and

$$f_0(z) = \tilde{f}(z), \quad \frac{f_1(z)^3}{3} = \int_0^z \tilde{f}(\eta) dx(\eta).$$

where $\tilde{f}(z)$ is inverse to $\tilde{h}(z)$, $\tilde{h}(\tilde{f}(z)) = z$ so that $f_\alpha(z) \in z + z\mathbb{C}[[z]]$. With coefficients

$$\rho_\alpha[k, m] = [z^m] f_\alpha(z)^k$$

we construct the operators

$$\widehat{J}_k^\alpha = \sum_{m=k}^{\infty} \rho_\alpha[k, m] \widehat{J}_m, \quad k \in \mathbb{Z}.$$

$$\widehat{J}_k = \begin{cases} \frac{\partial}{\partial t_k} & \text{for } k > 0, \\ |k|t_{|k|} & \text{for } k < 0, \end{cases}$$

CUT-AND-JOIN OPERATORS

$$\widehat{W}_0^{q,p} = \sum_{k,m \in \mathbb{Z}_{\text{odd}}^+}^{\infty} \left(\widehat{J}_{-k}^0 \widehat{J}_{-m}^0 \widehat{J}_{k+m-1}^0 + \frac{1}{2} \widehat{J}_{-k-m-1}^0 \widehat{J}_k^0 \widehat{J}_m^0 \right) + \frac{\widehat{J}_{-1}^0}{8},$$

$$\widehat{W}_1^{q,p} = \frac{1}{3} \sum_{k,m \in \mathbb{Z}_{\text{odd}}^+}^{\infty} \left(\widehat{J}_{-k}^1 \widehat{J}_{-m}^1 \widehat{J}_{k+m-3}^1 + \frac{1}{2} \widehat{J}_{-k-m-3}^1 \widehat{J}_k^1 \widehat{J}_m^1 \right) + \frac{(\widehat{J}_{-1}^1)^3}{3!} + \frac{\widehat{J}_{-3}^1}{24}.$$

THEOREM

$$\tau_{q,p}^{(\alpha)} = \exp\left(\hbar \widehat{W}_\alpha^{q,p}\right) \cdot 1$$

Algebraic topological recursion for topological expansion $\tau_{q,p}^{(\alpha)} = \sum_{k=0}^{\infty} \hbar^k \tau_{q,p}^{(\alpha,k)}$,

$$\tau_{q,p}^{(\alpha,k)} = \frac{1}{k} \widehat{W}_\alpha^{q,p} \cdot \tau_{q,p}^{(\alpha,k-1)}.$$

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It is also interesting to consider the specific values of the parameters q and p , for which the tau-function $\tau_{q,p}^{(\alpha)}$ describes k -reductions of the KP hierarchy.

COROLLARY

For $p = -2q$ the tau-functions $\tau_{q,p}^{(\alpha)}$ do not depend on even times,

$$\frac{\partial}{\partial t_{2k}} \tau_{q,-2q}^{(\alpha)}(\mathbf{t}) = 0, \quad k \in \mathbb{Z}_{>0},$$

that is $\tau_{q,-2q}^{(\alpha)}(\mathbf{t})$ (with $\Lambda_g(-q)\Lambda_g(2q)\Lambda_g(2q)$) are tau-functions of the KdV hierarchy.

It follows from the form of the linear change of variables

$$T_k^{q,p}(\mathbf{t}) = \left(q \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_k} + \frac{2q+p}{\sqrt{p+q}} \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k-1}} + \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k-2}} \right) T_{k-1}^{q,p}(\mathbf{t}).$$

In **[Dubrovin, Liu, Yang, Zhang, '20]** the relation between the generating function of the triple Hodge integrals for $\alpha = 1$ with $2q + p = 0$ and the **discrete** KdV was established. Our results show that there is a simpler relation between $Z_{q,-2q}^{(\alpha)}$ and the ordinary KdV hierarchy given by a linear change of variables.

MILLER–MORITA–MUMFORD CLASSES AND TRANSLATIONS

With the forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ we define the Miller–Morita–Mumford classes, $\kappa_k := \pi_* \psi_{n+1}^{k+1} \in H^{2k}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Let us consider the generating functions of the intersection numbers of ψ , κ , and Θ classes

$$F_{g,n}^\alpha(\mathbf{T}, \mathbf{s}) = \sum_{a_1, \dots, a_n \geq 0} \frac{\prod T_{a_i}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}^{1-\alpha} e^{\sum_{j=1}^\infty s_j \kappa_j} \psi_1^{a_1} \psi_2^{a_2} \dots \psi_n^{a_n},$$

$$\tau_\alpha(\mathbf{t}, \mathbf{s}) = \exp \left(\sum_{g=0}^\infty \sum_{n=0}^\infty \hbar^{2g-2+n} F_{g,n}^\alpha(\mathbf{T}, \mathbf{s}) \right) \Big|_{T_k = (2k+1)!! t_{2k+1}}.$$

We introduce the polynomials $q_j(\mathbf{s})$, defined by the generating function

$$1 - \exp \left(- \sum_{j=1}^\infty s_j z^j \right) = \sum_{j=1}^\infty q_j(\mathbf{s}) z^j,$$

THEOREM (MANIN, ZOGRAF; NORBURY)

$$\tau_\alpha(\mathbf{t}, \mathbf{s}) = \tau_\alpha \left(\left\{ t_{2k+1} + \frac{1}{\hbar} \frac{q_{k-\alpha}(\mathbf{s})}{(2k+1)!!} \right\} \right).$$

IDENTIFICATION OF TAU-FUNCTIONS

Let us put $q = -u^2$ with new parameter u , so that $p = 2u^2$. This parametrization corresponds to the insertion of the triple Hodge class of the form $\Lambda(-2u^2)\Lambda(-2u^2)\Lambda(u^2)$. Consider the parameters

$$\sum_{k=1}^{\infty} s_k^0 z^k = -\log \left(1 + \sum_{k=1}^{\infty} (2k-1)!! (zu^2)^k \right),$$

$$\sum_{k=1}^{\infty} s_k^1 z^k = -\log \left(1 + \sum_{k=1}^{\infty} (2k+1)!! \left(\sum_{j=0}^k \frac{1}{2j+1} \right) (zu^2)^k \right).$$

$$s_1^0 = -u^2, s_2^0 = -\frac{5}{2}u^4, s_3^0 = -\frac{37}{3}u^6; s_1^1 = -4u^2, s_2^1 = -15u^4, s_3^1 = -\frac{316}{3}u^6.$$

THEOREM

The generating functions of the triple Hodge integrals satisfying Calabi–Yau condition for $p = -2q = 2u^2$ coincide with the generation functions of the intersection numbers containing κ classes,

$$\tau_{-u^2, 2u^2}^{(\alpha)}(\mathbf{t}) = \tau_{\alpha}(\mathbf{t}, \mathbf{s}^{\alpha}).$$

REDUCTION TO KdV: CORRELATION FUNCTIONS

For $p = -2q = 2u^2$

$$\frac{\partial}{\partial x} = \frac{1 - u^2 z^2}{z} \frac{\partial}{\partial z}.$$

and

$$\Phi_k(z) := \left(-\frac{\partial}{\partial x} \right)^k \cdot \frac{1}{z}.$$

For $g, n \geq 0$ let us introduce the *correlation functions*

$$W_{g,n}^\alpha(z_1, \dots, z_n) = \sum_{a_1, \dots, a_n \geq 0} \int_{\mathcal{M}_{g,n}} \Theta_{g,n}^{1-\alpha} \Lambda(-2u^2) \Lambda(-2u^2) \Lambda(u^2) \prod_{j=1}^n \psi_j^{a_j} \Phi_{a_j}(z_j),$$

$$\tilde{W}_{g,n}^\alpha(z_1, \dots, z_n) = \sum_{a_1, \dots, a_n \geq 0} \int_{\mathcal{M}_{g,n}} \Theta_{g,n}^{1-\alpha} e^{\sum_{j=1}^{\infty} s_j^\alpha \kappa_j} \prod_{j=1}^n \psi_j^{a_j} \left(-\frac{1}{z_j} \frac{\partial}{\partial z_j} \right)^{a_j} z_j^{-1}.$$

COROLLARY

For the stable (g, n)

$$W_{g,n}^\alpha(z_1, \dots, z_n) = \tilde{W}_{g,n}^\alpha(z_1, \dots, z_n).$$

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How to generalize to the r-spin case?**Generalized Kontsevich model deformation!**

Tau-function for linear Hodge case ($q = 0, p = w^2, \alpha = 1$) is given by the Kac–Schwarz operators

$$\begin{aligned} Q_H &= z + \hbar \left(\frac{1}{z} - w \right) \frac{\partial}{\partial z} - \hbar \frac{1}{2z^2}, \\ X_H &= -\frac{z}{w} - \frac{1}{u^2} \log(1 - wz). \end{aligned}$$

THEOREM

Tau-function for this case is given by the the generalized Kontsevich integral with the potential

$$V = w^{-3}(1 - wz) \log(1 - wz) + \frac{2z - wz^2}{2w^2}$$

(or $V'' = \frac{z}{1-wz}$):

$$\tau_H(\mathbf{t}; w) = \tau_V(\mathbf{t}).$$

QUANTUM SPECTRAL CURVE

The modified wave function satisfies the **quantum spectral curve** equation

$$e^{w^2 \hat{x} + w \hat{y}} \left(1 - w \hat{y} - \hbar \frac{w^3}{2} \right) \cdot \tilde{\Psi} = \tilde{\Psi}.$$

The classical spectral curve

$$x = -w^{-2} (\log(1 - wy) + wy)$$

reduces to Airy curve $x = y^2/2$ at $w = 0$. It can be rewritten as

$$e^{-w^2 x} = (1 - wy)e^{wy},$$

a version of the Lambert curve.

KONTSEVICH INTEGRAL DEFORMATION: TRIPLE HODGE INTEGRALS

Tau-function for triple Hodge integral with Calabi–Yau constraint is described by the generalized Kontsevich model with the potential

$$V''(z) = \frac{z}{(1 + \sqrt{p + qz})(1 + qz/\sqrt{p + q})}$$

and extra translation of times

$$\tilde{v}_k = [z^k] \int_0^z (\eta - y(\eta)) dx(\eta).$$

THEOREM

Deformed generalized Kontsevich model is related to the generating function of cubic Hodge integrals satisfying the Calabi-Yau constraint by the translation of times

$$\tau_{q,p}^{(1)}(\mathbf{t}) = \tilde{C} e^{\sum \tilde{v}_k \frac{\partial}{\partial t_k}} \tau_V(\mathbf{t}).$$

Here \tilde{C} does not depend on \mathbf{t} .

Translation of times – insertion of κ -classes.

DEFORMED KONTSEVICH MATRIX MODEL

Cubic Hodge integrals with an additional Calabi-Yau condition are described by the potential

$$V''(z) = \frac{z}{(1 + \sqrt{p + qz})(1 + qz/\sqrt{p + q})}$$

up to some explicit translation of times.

Shifted r -spin case

$$V''(z) = z^{r-1} + b_{r-2}z^{r-2} + \dots + b_1z$$

Deformation of the shifted r -spin case

$$V''(z) = \frac{z^{r-1} + b_{r-2}z^{r-2} + \dots + b_1z}{(1 - w_1z)(1 - w_2z) \dots (1 - w_rz)}$$

Kac-Schwarz operators

$$Q = z + \frac{\hbar}{V''} \frac{\partial}{\partial z} - \frac{\hbar V'''}{2(V'')^2}, \quad X = V'(z).$$

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INTERSECTIONS OF THE SYMMETRY GROUPS

- The most general case for multicomponent KP, BKP, CKP, and 2D Toda hierarchies. Drinfeld–Sokolov, Kac–Wakimoto, . . .
- Integrability of the topological vertex (3-component KP hierarchy, [Aganagic, Dijkgraaf, Klemm, Mariño, Vafa '06])
- r -spin intersection numbers. From k copies of KdV to $k + 1$ reduction of the KP (For the Kac–Wakimoto hierarchies considered in [Givental '03; Frenkel, Givental, Milanov '10]), Insertion of Hodge, Chiodo, Norbury, κ classes, open r -spin, . . .
- Relation between KP integrability and other integrable structures for Hodge integrals: Dubrovin–Zhang, Volterra, Hodge, . . .
- Geometrical interpretation of the matrix integral and KdV reduction
- Applications

CohFT \cap Solitonic integrable hierarchies \in Interesting geometry

$$\tau_{q,p}^{(0)}(\mathbf{t}) = \exp\left(\sum_{k=1}^{\infty} h^k F_k^0(\mathbf{t})\right)$$

$$F_1^0 = \frac{1}{8} t_1,$$

$$F_2^0 = \frac{1}{16} t_1^2 - \frac{1}{128} \frac{p^2 + pq + q^2}{p + q},$$

$$F_3^0 = \frac{9}{128} t_3 + \frac{1}{24} t_1^3 + \frac{3}{64} \frac{(p + 2q) t_2}{\sqrt{p + q}} - \frac{1}{128} \frac{(2p^2 - pq - q^2) t_1}{p + q}$$

$$F_4^0 = \frac{27}{128} t_3 t_1 + \frac{1}{32} t_1^4 + \frac{9}{64} \frac{(p + 2q) t_1 t_2}{\sqrt{p + q}} - \frac{3}{128} \frac{(p^2 - 2pq - 2q^2) t_1^2}{p + q}$$

$$+ \frac{1}{512} \frac{p^4 + 2p^3q + 3p^2q^2 + 2pq^3 + q^4}{(p + q)^2}$$

$$F_5^0 = \frac{225}{1024} t_5 + \frac{27}{64} t_3 t_1^2 + \frac{1}{40} t_1^5 + \frac{75}{256} \frac{(p + 2q) t_4}{\sqrt{p + q}} + \frac{9}{32} \frac{(p + 2q) t_2 t_1^2}{\sqrt{p + q}}$$

$$+ \frac{3}{512} \frac{t_3 (4p^2 + 79pq + 79q^2)}{p + q} - \frac{1}{64} \frac{(2p^2 - 7pq - 7q^2) t_1^3}{p + q}$$

$$- \frac{1}{512} \frac{(22p^3 + 21p^2q - 69pq^2 - 46q^3) t_2}{(p + q)^{3/2}}$$

$$+ \frac{1}{1024} \frac{(8p^4 - 6p^3q - 5p^2q^2 + 2pq^3 + q^4) t_1}{(p + q)^2}$$

$$\tau_{q,p}^{(1)}(\mathbf{t}) = \exp\left(\sum_{k=1}^{\infty} h^k F_k^1(\mathbf{t})\right)$$

$$F_1^1 = \frac{1}{8}t_3 + \frac{1}{6}t_1^3 + \frac{1}{12} \frac{(p+2q)t_2}{\sqrt{p+q}} - \frac{1}{24} \frac{p^2 t_1}{p+q},$$

$$\begin{aligned} F_2^1 &= \frac{1}{2} t_3 t_1^3 + \frac{5}{8} t_5 t_1 + \frac{3}{16} t_3^2 \\ &+ \frac{5}{6} \frac{(p+2q)t_4 t_1}{\sqrt{p+q}} + \frac{1}{4} \frac{(p+2q)t_3 t_2}{\sqrt{p+q}} + \frac{1}{3} \frac{(p+2q)t_2 t_1^3}{\sqrt{p+q}} \\ &+ \frac{1}{8} \frac{(p^2 + 12pq + 12q^2)t_3 t_1}{p+q} + \frac{1}{12} \frac{(p^2 + 4pq + 4q^2)t_2^2}{p+q} + \frac{1}{6} q t_1^4 \\ &- \frac{1}{12} \frac{(p^3 - p^2 q - 9pq^2 - 6q^3)t_1 t_2}{(p+q)^{3/2}} \\ &- \frac{1}{48} \frac{q(2p^2 - pq - q^2)t_1^2}{p+q} \\ &- \frac{1}{5760} \frac{p^2 q^2}{p+q}, \end{aligned}$$

$$\begin{aligned} F_3^1 &= \frac{5}{8} t_5 t_1^4 + \frac{3}{2} t_3^2 t_1^3 + \frac{35}{16} t_7 t_1^2 + \frac{15}{4} t_5 t_3 t_1 + \frac{3}{8} t_3^3 + \frac{105}{128} t_9 \\ &+ \frac{35}{8} \frac{(p+2q)t_6 t_1^2}{\sqrt{p+q}} + \frac{3}{4} \frac{(p+2q)t_3^2 t_2}{\sqrt{p+q}} + \frac{35}{16} \frac{(p+2q)t_8}{\sqrt{p+q}} + \frac{5}{6} \frac{(p+2q)t_4 t_1^4}{\sqrt{p+q}} \\ &+ 2 \frac{(p+2q)t_3 t_2 t_1^3}{\sqrt{p+q}} + \frac{5}{2} \frac{(p+2q)t_5 t_2 t_1}{\sqrt{p+q}} + 5 \frac{(p+2q)t_4 t_3 t_1}{\sqrt{p+q}} + \dots \end{aligned}$$

Thank you!