The DR hierarchy for the partial CohFT formed by the DR cycles

The DR cycles

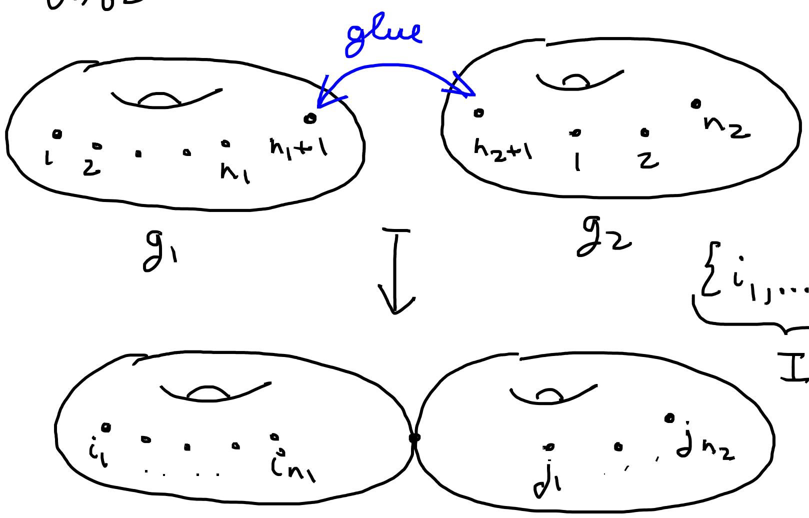
$$\mathrm{DR}_g(a_1,\dots,a_n)\in H^{2g}(\overline{\mathcal{M}}_{g,n}),\quad a_1,\dots,a_n\in\mathbb{Z},\quad \sum a_i=0.$$

Main properties

- $\mathrm{DR}_0(a_1,\dots,a_n)=1\in H^0(\overline{\mathcal{M}}_{0,n}) \Rightarrow \boxed{\mathrm{DR}_0(a,b,0)=\delta_{a+b,0}\in H^0(\overline{\mathcal{M}}_{0,3})}$
- $\mathrm{DR}_g(0,\dots,0)=(-1)^g\lambda_g\in H^{2g}(\overline{\mathcal{M}}_{g,n})$
- $\boxed{\mathrm{DR}_g(a_1,\dots,a_n,0)=\pi^*\mathrm{DR}_g(a_1,\dots,a_n)}$, where $\pi:\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ forgets the last marked point
- $gl:\overline{\mathcal{M}}_{g_1,n_1+1}\times\overline{\mathcal{M}}_{g_2,n_2+1}\to\overline{\mathcal{M}}_{\underbrace{g_1+g_2}_{g},\underbrace{n_1+n_2}_{n}}$

$\quad \| $

$gl^{I,J}_{g_1,g_2}$



$$\underbrace{\{i_1,\dots,i_{n_1}\}}_{I}\sqcup\underbrace{\{j_1,\dots,j_{n_2}\}}_{J}=\{1,\dots,n_1+n_2\}$$

$$\boxed{\begin{aligned}&gl^*\mathrm{DR}_g(a_1,\dots,a_n)=\\&=\mathrm{DR}_{g_1}(\overbrace{a_{i_1},\dots,a_{i_{n_1}}}^{A_I},-\sum_k a_{i_k})\otimes\mathrm{DR}_{g_2}(\overbrace{a_{j_1},\dots,a_{j_{n_2}}}^{A_J},-\sum_k a_{j_k})\end{aligned}}$$

Green properties imply the DR cycles form a partial CohFT:

- $V:=\langle e_a\rangle_{a\in\mathbb{Z}}$
- $e:=e_0$
- $\eta_{ab}:=\delta_{a+b,0}$
- $c_{g,n}(e_{a_1}\otimes\dots\otimes e_{a_n}):=\mathrm{DR}_g(a_1,\dots,a_n)$

Let us compute the DR hierarchy for this partial CohFT!

$$\frac{\partial u^\alpha}{\partial t^\beta_d} = \eta^{\alpha\mu}\partial_x \frac{\delta \bar g_{\beta,d}}{\delta u^\mu}, \quad \alpha,\beta\in\mathbb{Z},\quad d\geq 0.$$

We will determine only the flows $\boxed{\frac{\partial}{\partial t^0_d}}$, $d\geq 0$

Let us first compute the flow $\frac{\partial}{\partial t^0_1}$

$$u^\alpha=\sum_{a\in\mathbb{Z}} p^\alpha_a e^{iax}$$

$$\bar g_{0,1} = \sum_{g,n} \frac{(-\varepsilon^2)^g}{n!}\sum_{\substack{\alpha_i,\\ a_i}}\left[\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1\lambda_g c_{g,n+1}\Big(e_0\otimes \bigotimes_{i=1}^n e_{\alpha_i}\Big) DR_g(0,a_1,\ldots,a_n)\right]\times p^{\alpha_1}_{a_1}\cdots p^{\alpha_n}_{a_n}$$

$$= \sum_{g,n} \frac{(-\varepsilon^2)^g}{n!}(2g-2+n)\int_{\underbrace{\overline{\mathcal{M}}_{g,n}}_{\dim=3g-3+n}} \overbrace{\lambda_g}^{\deg=g} \underbrace{c_{g,n}\Big(\bigotimes_{i=1}^n e_{\alpha_i}\Big)}_{\deg=g} \underbrace{DR_g(a_1,\ldots,a_n)}_{\deg=g}\times p^{\alpha_1}_{a_1}\cdots p^{\alpha_n}_{a_n}$$

Integral $=0$ unless $\boxed{n=3}$

$$\int_{\overline{\mathcal{M}}_{g,3}} \lambda_g DR_g(a_1,a_2,a_3) DR_g(b_1,b_2,b_3) \overset{?}{=} \qquad \sum a_i=\sum b_i=0.$$

Theorem

$$\int_{\overline{\mathcal{M}}_{g,3}} \lambda_g DR_g(a_1,a_2,a_3) DR_g(b_1,b_2,b_3) = \frac{(a_1b_2-a_2b_1)^{2g}}{2^{3g}\, g!\,(2g+1)!!}, \quad g\geq 0.$$

Proof

Hain's formula $\quad DR_g(a_1,\ldots,a_n)\Big|_{\mathcal{M}^{ct}_{g,n}} = \frac{1}{g!}\,\Theta(a_1,\ldots,a_n)^g\Big|_{\mathcal{M}^{ct}_{g,n}}$

where $\quad \Theta(a_1,\ldots,a_n) = \sum_{j=1}^n \frac{a_j^2\psi_j}{2} - \frac{1}{4}\sum_{h=0}^g \sum_{J\subset[n]} a_J^2\,\delta_h^J$

$[n]:=\{1,\ldots,n\}, \qquad a_J:=\sum_{j\in J} a_j$

$\Bigg[$ 

$\Bigg] \in H^2(\overline{\mathcal{M}}_{g,n})$

Moreover, $\Theta(a_1,\ldots,a_n)^{g+1}\Big|_{\mathcal{M}^{ct}_{g,n}} = 0.$

$$\lambda_g\big|_{\overline{\mathcal{M}}_{g,n}\setminus\mathcal{M}^{ct}_{g,n}}=0$$

$$\Downarrow$$

Have to prove that

$$\underbrace{\int_{\overline{\mathcal{M}}_{g,3}}\lambda_g\,\Theta(\underset{(a_1,a_2,a_3)}{\underset{\parallel}{\bar a}})^g\,DR_g(\underset{(b_1,b_2,b_3)}{\underset{\parallel}{\bar b}})}_{f_g(\bar a,\bar b)}=\frac{(a_1b_2-a_2b_1)^{2g}}{2^{3g}(2g+1)!!}$$

Obviously, $f_0(\bar a,\bar b)=1$

Suppose $g\geq 1$.

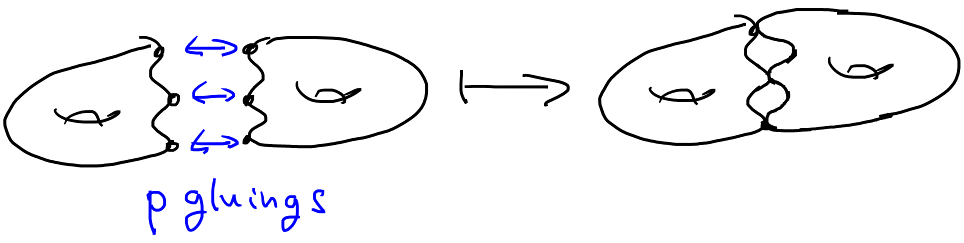
Main tool

$$\boxed{a_s\psi_s DR_g(a_1,\dots,a_n)=\sum_{\substack{I\sqcup J=[n]\\ a_I>0}}\sum_{p\geq 1}\sum_{g_1+g_2+p-1=g}\sum_{k_1,\dots,k_p\geq 1}\frac{\rho}{2g-2+n}\frac{\prod_{i=1}^p k_i}{p!}\times\underbrace{DR_{g_1}(A_I,-k_1,\dots,-k_p)\boxtimes DR_{g_2}(A_J,k_1,\dots,k_p)}_{\substack{\parallel\\ gl_*\left(DR_{g_1}(A_I,-k_1,\dots,-k_p)\otimes DR_{g_2}(A_J,k_1,\dots,k_p)\right)}}}$$

glue glue

$$\boxed{\rho=\begin{cases}2g_2-2+|J|+p, & \text{if } s\in I\\ -(2g_1-2+|I|+p), & \text{if } s\in J\end{cases}}$$

$$gl:\overline{\mathcal{M}}_{g_1,|I|+p}\times\overline{\mathcal{M}}_{g_2,|J|+p}\longrightarrow\overline{\mathcal{M}}_{g,n}$$



Corollary

$$\boxed{\lambda_g a_s\psi_s DR_g(a_1,\dots,a_n)=\sum_{\substack{I\sqcup J=[n]\\ s\in I}}\sum_{g_1+g_2=g}\frac{2g_2-1+|J|}{2g-2+n}\,a_I\times\left(\lambda_{g_1}DR_{g_1}(A_I,-a_I)\right)\boxtimes\left(\lambda_{g_2}DR_{g_2}(A_J,a_I)\right)}$$

In order to compute

$$\int_{\overline{\mathcal{M}}_{g,3}}\lambda_g\,\Theta(\bar a)^g\,DR_g(\bar b),$$

we express one $\Theta(\bar a)$ using Hain's formula.

We first have to compute $\{i,j,k\}=\{1,2,3\}$

$$\int_{\overline{\mathcal{M}}_{g,3}}\Theta(\bar a)^{g-1}\lambda_g\psi_i DR_g(\bar b)=$$

$$=\frac{2g-1}{2g+1}\boxed{\sum_{g_1+g_2=g}\int_{\overline{\mathcal{M}}_{g,3}}\lambda_g\Theta(\bar a)^{g-1}DR_{g_1}(b_i,-b_i)\boxtimes DR_{g_2}(b_j,b_k,b_i)}$$

1

$$-\frac{2}{2g+1}\sum_{g_1+g_2=g}\frac{b_j}{b_i}\int_{\overline{\mathcal{M}}_{g,3}}\lambda_g\Theta(\bar a)^{g-1}DR_{g_1}(b_i,b_k,b_j)\boxtimes DR_{g_2}(b_j,-b_j)$$

$$-\frac{2}{2g+1}\sum_{g_1+g_2=g}\frac{b_k}{b_i}\int_{\overline{\mathcal{M}}_{g,3}}\lambda_g\Theta(\bar a)^{g-1}DR_{g_1}(b_i,b_j,b_k)\boxtimes DR_{g_2}(b_k,-b_k)$$

$$\boxed{\text{note that } \left(gl^{I,J}_{g_1,g_2}\right)^*\Theta(a_1,\dots,a_n)=\Theta(A_I,-a_I)\otimes 1+1\otimes\Theta(A_J,-a_J)}$$

$$① = \sum_{\substack{h_1+h_2=g-1\\ g_1+g_2=g}} \binom{g-1}{h_1} \int_{\overline{\mathcal{M}}_{g,3}} \left(\Theta(a_i,-a_i)^{h_1} \lambda_{g_1} DR_{g_1}(b_i,-b_i)\right) \boxtimes$$

$$\boxtimes \left(\Theta(a_j,a_k,a_i)^{h_2} \lambda_{g_2} DR_{g_2}(b_j,b_k,b_i)\right)$$

$$\Bigg[\underbrace{\int_{\overline{\mathcal{M}}_{g_1,2}}}_{\dim = 3g_1-1} \underbrace{\Theta(a_i,-a_i)^{h_1} \lambda_{g_1} DR_{g_1}(b_i,-b_i)}_{\deg = 2g_1+h_1,\ =②} \Bigg] \times$$

$$\times \left[\int_{\overline{\mathcal{M}}_{g_2,3}} \Theta(a_j,a_k,a_i)^{h_2} \lambda_{g_2} DR_{g_2}(b_j,b_k,b_i) \right]$$

zero, unless $h_1 = g_1 - 1$ and $h_1 = 0$

$\Updownarrow$

$h_2 = g_2$

$$② = a_i^{2h_1} \Theta(1,-1)^{h_1} \lambda_{g_1} b_i^{2g_1} \Theta(1,-1)^{g_1} \frac{1}{g_1!}$$

product is 0 in compact type if $h_1 \geq 1$

So

$$① = \underbrace{\left[\int_{\overline{\mathcal{M}}_{1,2}} \lambda_1 DR_1(b_i,-b_i) \right]}_{= \frac{b_i^2}{24}} \times \underbrace{\left[\int_{\overline{\mathcal{M}}_{g-1,3}} \Theta(a_j,a_k,a_i)^{g-1} \lambda_{g-1} DR_{g-1}(b_j,b_k,b_i) \right]}_{= f_{g-1}(\bar{a},\bar{b})}$$

Using the same ideas for other terms, we obtain

$$\int_{\overline{\mathcal{M}}_{g,3}} \Theta(\bar{a})^{g-1} \lambda_g \psi_i DR_g(\bar{b}) = \frac{(2g+1)b_i^2 - 6 b_j b_k}{24(2g+1)} f_{g-1}(\bar{a},\bar{b})$$

We also have to compute

$$\int_{\overline{\mathcal{M}}_{g,3}} \lambda_g \Theta(\bar{a})^{g-1} \delta_h^J DR_g(\bar{b}),$$

which can be done using the same ideas.

After careful computations, we obtain

$$f_g(\bar{a},\bar{b}) = \frac{(a_1 b_2 - a_2 b_1)^2}{8(2g+1)} f_{g-1}(\bar{a},\bar{b}),$$

which completes the proof. $\square$

We obtain, $a \in \mathbb{Z}$

$$\frac{\partial u^a}{\partial t^0_1} = \partial_x \sum_{g\geq 0} \frac{\varepsilon^{2g}}{2} \sum_{a_1+a_2=a} \sum_{d_1+d_2=2g}$$

$$\operatorname{Coef}_{b_1^{d_1} b_2^{d_2}} \int_{\overline{\mathcal{M}}_{g,4}} \psi_1 \lambda_g DR_g(0,-a,a_1,a_2) DR_g(0,-b_1-b_2,b_1,b_2) \times u^{a_1}_{b_1} u^{a_2}_{b_2} =$$

$$= \partial_x \sum_{g\geq 0} \frac{\varepsilon^{2g}}{2} \sum_{a_1+a_2=a} \sum_{d_1+d_2=2g} \operatorname{Coef}_{b_1^{d_1} b_2^{d_2}} \Bigg[\underbrace{\frac{(a_1b_2-a_2b_1)^{2g}}{2^{3g} g!(2g-1)!!}}_{2^{2g}(2g)!} \Bigg] u^{a_1}_{d_1} u^{a_2}_{d_2} =$$

$$= \partial_x \sum_{g\geq 0} \frac{\varepsilon^{2g}}{2} \sum_{a_1+a_2=a} \sum_{d_1+d_2=2g} \operatorname{Coef}_{b_1^{d_1} b_2^{d_2}} \left[\frac{(a_1b_2-a_2b_1)^{2g}}{2^{2g}(2g)!} \right] u^{a_1}_{d_1} u^{a_2}_{d_2} =$$

$$= \partial_x \sum \frac{\varepsilon^{2g}}{2\cdot 2^{2g}} \operatorname{Coef}_{b_1^{d_1} b_2^{d_2}} \left[\sum_{h_1+h_2=2g} \frac{1}{h_1!h_2!} (a_1b_2)^{h_1} (-a_2b_1)^{h_2} \right] u^{a_1}_{d_1} u^{a_2}_{d_2} =$$

$$= \boxed{\partial_x \sum_{a_1+a_2=a} \frac{\varepsilon^{2g}}{2\cdot 2^{2g}} \sum_{h_1+h_2=2g} \frac{1}{h_1!h_2!} a_1^{h_1} (-a_2)^{h_2} (\partial_x^{h_2} u^{a_1})(\partial_x^{h_1} u^{a_2}) = \frac{\partial u^a}{\partial t^0_1}}$$

Main trick: multiply both sides by e^{iay} and denote $u = u(x,y) = \sum u^a(x) e^{iay} = \sum p^a_b e^{iay} e^{ibx}$.

We obtain

$$\frac{\partial u}{\partial t^0_1} = \frac{1}{2} \partial_x \sum_{\substack{g\geq 0 \\ a_1,a_2}} \frac{\varepsilon^{2g}}{2^{2g}} \sum_{h_1+h_2=2g} \frac{1}{h_1!h_2!} a_1^{h_1} (-a_2)^{h_2} (\partial_x^{h_2} u^{a_1}) e^{ia_1y} (\partial_x^{h_1} u^{a_2}) e^{ia_2y}$$

$$= \frac{1}{2} \partial_x \sum_{g\geq 0} \frac{1}{2^{2g}} \sum_{h_1+h_2=2g} \frac{(i\varepsilon)^{2g}}{h_1!h_2!} (-1)^{h_2} (\partial_x^{h_2} \partial_y^{h_1} u)(\partial_x^{h_1} \partial_y^{h_2} u)$$

One recognizes here the Moyal star-product in the space $C^\infty((S^1)^2)[[\hbar]]$:

$$f *_\hbar g := \sum_{n\geq 0} \sum_{n_1+n_2=n} \frac{(-1)^{n_2}(i\hbar)^n}{2^n n_1! n_2!} (\partial_x^{n_1}\partial_y^{n_2} f)(\partial_x^{n_2}\partial_y^{n_1} g) =$$

$$= f \exp\left(\frac{i\hbar}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x})\right) g$$

So we obtain

$$\boxed{\frac{\partial u}{\partial t^0_1} = \frac{1}{2}\partial_x(u *_\varepsilon u).}$$

More generally,

$$\boxed{\frac{\partial u}{\partial t^0_d} = \frac{1}{(d+1)!}\partial_x(\underbrace{u *_\varepsilon u *_\varepsilon \cdots *_\varepsilon u}_{d+1 \text{ times}}), \quad d\geq 1}$$

$$\boxed{\text{The DR hierarchy for } c_{g,n}(\otimes e_{a_i}) = DR_g(a_1,\ldots,a_n)}$$

$$\|$$

$$\boxed{\text{The DR hierarchy for } c_{g,n}(\otimes e_{a_i}) = \frac{\Theta(a_1,\ldots,a_n)^g}{g!}}$$

<u>Question</u>: what if we take

$$c_{g,n}(\otimes e_{a_i}) = \exp(\mu^2 \Theta(a_1,\ldots,a_n))$$

formal variable

<u>Answer</u>

The flows $\frac{\partial}{\partial t^0_d}$ of the DR hierarchy for are

$$\frac{\partial u}{\partial t^0_1} = \frac{1}{2}\partial_x(u *_{\varepsilon\mu} u) + \frac{\varepsilon^2}{12}u_{xxx} \quad \underbrace{\text{noncommutative KdV}}_{\text{ncKdV}}$$

$$\frac{\partial u}{\partial t^0_d} = \text{higher flows of the ncKdV hierarchy}$$

can be described using the Lax representation

Problem

How to describe the flows $\frac{\partial}{\partial t^\alpha_d}$ with $\alpha \neq 0$?

Relation with DZ hierarchy

$$\text{DR hierarchy} \underset{\text{equivalent}}{\overset{\text{conj.}}{\longleftrightarrow}} \text{DZ hierarchy}$$

$$\Downarrow \text{ B.-Rossi}$$

Conjecture (B.-Rossi) (noncommutative Witten's conjecture)
Intersection numbers with Pixton's class give a solution of the ncKdV hierarchy.