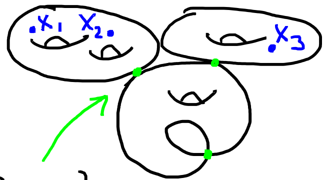
Tautological relations and integrable systems

Goal: getting an ultimate generalisation of Witten's conjecture

Witten's conjecture (1991)

$$\overline{\mathcal{M}}_{g,n} := \left\{ (C;x_1,\dots,x_n) = \right.$$



$\{xy=0\} \subset \mathbb{C}^2$

$\left. \begin{array}{l} \bullet\ g(C)=g \\ \bullet\ x_i \text{ lie on the smooth part of } C \\ \bullet\ |\mathrm{Aut}(C;x_1,\dots,x_n)|<\infty \end{array} \right\} \Big/ \text{isom.}$

moduli space of stable curves

Cotangent line bundles $\mathcal{L}_i \longrightarrow \overline{\mathcal{M}}_{g,n}$

$$\psi_i := c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n},\mathbb{Q}),\quad i=1,\dots,n$$

$$\mathcal{F}(t_0, t_1, t_2, \dots, \varepsilon) := \sum_{g,n\geq 0} \frac{\varepsilon^{2g}}{n!} \sum_{d_1,\dots,d_n \geq 0} \overbrace{\underbrace{\left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right)}_{\text{zero unless } \sum d_i = \dim \overline{\mathcal{M}}_{g,n} = 3g-3+n}}^{\text{intersection number}} t_{d_1} \cdots t_{d_n}$$

formal variables

Witten's conjecture/Kontsevich's theorem (1992)

$w = \dfrac{\partial^2 \mathcal{F}}{\partial t_0^2}$ satisfies the KdV hierarchy $(x = t_0)$

$$\frac{\partial w}{\partial t_1} = w w_x + \frac{\varepsilon^2}{12} w_{xxx}$$

$$\frac{\partial w}{\partial t_n} = \frac{w^n w_x}{n!} + \dots, \quad n \geq 2.$$

flows pairwise commute (integrability)

Possible development

$\overline{\mathcal{M}}_{g,n} \underset{\text{theory}}{\overset{\text{Gromov-Witten}}{\rightsquigarrow}} \overline{\mathcal{M}}_{g,n}(X,\beta) \rightsquigarrow$ generating series of intersection numbers $\mathcal{F}^X$

$\beta \in H_2(X,\mathbb{Z})$

$\boxed{\text{moduli space of stable maps}}$

X: smooth projective variety

Example: $\mathcal{F}^{\mathbb{CP}^1}$ is controlled by

- extended Toda hierarchy (Carlet-Dubrovin-Zhang, 2004)
- 2D Toda hierarchy (Okounkov-Pandharipande, equivariant extension, 2006)

General framework for curve counting theories:

the notion of a CohFT (Kontsevich-Manin, 1994)

unit

$$\left(V \ , \ e \in V \ , \ \eta : V^{\otimes 2} \to \mathbb{C} \ , \ c_{g,n} : V^{\otimes n} \to H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) \right)$$

finite dim. vector space

symmetric nondegenerate

linear maps

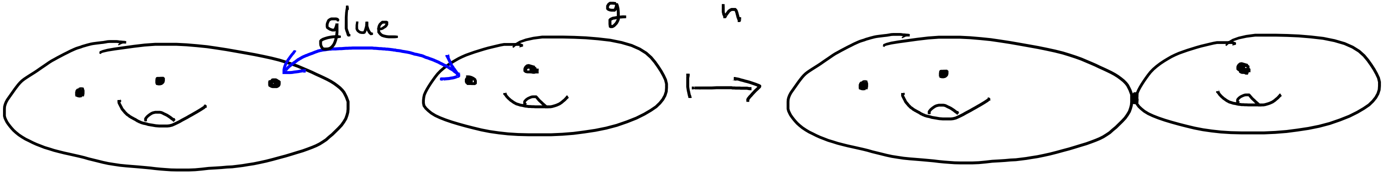
Should satisfy the following:

1. $c_{g,n}$ is S_n-equivariant
2. $\pi^* c_{g,n}(v_1 \otimes \ldots \otimes v_n) = c_{g,n}(v_1 \otimes \ldots \otimes v_n \otimes e)$, $v_i \in V$.

   $c_{0,3}(v_1 \otimes v_2 \otimes e) = \eta(v_1 \otimes v_2)$

$$\boxed{\begin{array}{l} \overline{\mathcal{M}}_{g,n+1} \\ \downarrow \pi \\ \overline{\mathcal{M}}_{g,n} \end{array} \quad \text{forgets the last marked point}}$$

③ $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{\underbrace{g_1+g_2}_{g},\underbrace{n_1+n_2}_{n}}$



$$gl^* c_{g,n}(v_1 \otimes \cdots \otimes v_n) = c_{g_1,n_1+1}(v_1 \otimes \cdots \otimes v_{n_1} \otimes e_\mu)\, \eta^{\mu\nu} \otimes$$
$$\otimes\, c_{g_2,n_2+1}(v_{n_1+1} \otimes \cdots \otimes v_n \otimes e_\nu),$$

where $e_1, \ldots, e_{\dim} \in V$ is a basis

④ ...

$\boxed{\text{① + ② + ③ : partial CohFT}}$

$1 \leq \alpha \leq \dim V \qquad d \geqslant 0$

Form a potential $\mathcal{F}(t^{\alpha}_{d}, \varepsilon) = \sum_{g \geqslant 0} \mathcal{F}_g(t^{\alpha}_{d})\varepsilon^{2g} :=$

$$:= \sum \frac{\varepsilon^{2g}}{n!} \left(\int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n}) \prod \psi_i^{d_i} \right) \prod t^{\alpha_i}_{d_i} \in \mathbb{C}[[t^{\alpha}_{d}, \varepsilon]].$$

Is $\mathcal{F}$ controlled by an integrable system?

Often, yes!

Can be different integrable systems!

the vector bundle formed by the spaces of holomorphic 1-forms

Basic example: $V = \mathbb{C} = \langle e_1 = e \rangle$, $\eta_{1,1} = 1$, $c_{g,n}(e_1^{\otimes n}) = 1 + c_1(E) + \dots + c_g(E)$

$\mathcal{F}$ is controlled by

$\xrightarrow{\text{Kazarian, 2009}}$ KP hierarchy

$\xrightarrow{\text{B., 2015}}$ ILW hierarchy

ILW hierarchy is canonically associated to $\mathcal{F}$ through a general construction (DZ theory)

<u>Dubrovin–Zhang theory</u>: canonical construction of an integrable system controlling $\mathcal{F}$

Fix a partial CohFT and consider its potential $\mathcal{F}(t^*_*, \varepsilon)$, where we fix a basis $e_i \in V$ with $e_1 = e$, $\dim V =: N$.

Define $w^1, \ldots, w^N \in \mathbb{C}[[t^*_*, \varepsilon]]$ by

$$w^\alpha := \eta^{\alpha\mu} \frac{\partial^2 \mathcal{F}}{\partial t^1_0 \partial t^\mu_0}, \qquad w^\alpha_d = \partial_x^d w^\alpha, \text{ where } \partial_x := \frac{\partial}{\partial t^1_0}.$$

<u>Generalized DZ conjecture</u>

w^α satisfy a system of pairwise compatible PDEs of the form

$$\frac{\partial w^\alpha}{\partial t^\beta_d} = \partial_x P^\alpha_{\beta,d},$$

elements of $\mathbb{C}[[w^*]][w^*_{\geq 1}][[\varepsilon]]$

$\deg w^\alpha_d = d$
$\deg \varepsilon = -1$

where $P^\alpha_{\beta,d}$ are differential polynomials of degree 0.

Discussion of the conjecture

Two basic equations for $\mathcal{F}$:

$$\frac{\partial \mathcal{F}}{\partial t^1_0} = \sum_{n\geq 0} t^\alpha_{n+1} \frac{\partial \mathcal{F}}{\partial t^\alpha_n} + \frac{1}{2}\eta_{\alpha\beta} t^\alpha_0 t^\beta_0 + \varepsilon^2 C \qquad \boxed{\text{string equation}}$$

$C \in \mathbb{C}$

$$\frac{\partial \mathcal{F}}{\partial t^1_1} = \sum_{n\geq 0} t^\alpha_n \frac{\partial \mathcal{F}}{\partial t^\alpha_n} + \varepsilon\frac{\partial \mathcal{F}}{\partial \varepsilon} - 2\mathcal{F} + \varepsilon^2 \frac{N}{24} \qquad \boxed{\text{dilaton equation}}$$

$$w^\alpha_d = t^\alpha_d + \delta^{\alpha,1}\delta_{d,1} + \begin{pmatrix} \text{linear combination of} \\ \text{monomials } t^{\alpha_1}_{d_1}\cdots t^{\alpha_n}_{d_n} \\ \text{with } \sum d_i \geq d+1 \end{pmatrix} + O(\varepsilon^2)$$

Therefore, any formal power series from $\mathbb{C}[[t^*_*,\varepsilon]]$ can be expressed as a formal power series in $w^\alpha_d - \delta^{\alpha,1}\delta_{d,1}$ and ε in a unique way.

$$\{\text{Differentials polynomials}\} \subsetneq \mathbb{C}[[w^\alpha_d - \delta^{\alpha,1}\delta_{d,1}, \varepsilon]]$$

$$\frac{1}{w^1_x} = \frac{1}{1+(w^1_x-1)} = \sum_{i\geq 0} (-1)^i (w^1_x - 1)^i \quad \text{not a differential polynomial!}$$

Thus, the generalised DZ conjecture $\iff$ $\dfrac{\partial w^\alpha}{\partial t^\beta_b} = \eta^{\alpha\mu} \dfrac{\partial^3 \mathcal{F}}{\partial t^1_0 \partial t^\mu_0 \partial t^\beta_b} = \partial_x \left(\begin{array}{c}\text{differential}\\ \text{polynomial}\end{array} \right)$

$$\overset{\text{easy}}{\iff} \quad \frac{\partial^2 \mathcal{F}}{\partial t^\alpha_a \partial t^\beta_b} = \text{differential polynomial}$$

Dilaton equation $\overset{\text{easy}}{\Longrightarrow}$

$$\Rightarrow \frac{\partial^2 \mathcal{F}}{\partial t^\alpha_a \partial t^\beta_b} = \overbrace{\text{differential polynomial}}^{\deg = 0} + \sum_{i\geq 1} \frac{\overbrace{Q_{\alpha,a;\beta,b;i}}}{(w^1_x)^i}$$

- dif. pol
- doesn't depend on w^1_x
- $\deg = i$

Thus, the generalised DZ conjecture $\Longleftrightarrow$ $Q_{\alpha,a;\beta,b;i}=0$, $i\geq 1$.

How to compute $Q_{\alpha,a;\beta,b;i}$?

Example in genus 0

$$\frac{\partial^2 \mathcal{F}}{\partial t^\alpha_a \partial t^\beta_b} - \sum \left(\frac{\partial^{n+2}\mathcal{F}_0}{\partial t^\alpha_a \partial t^\beta_b \partial t^{\alpha_1}_0 \cdots \partial t^{\alpha_n}_0} \Bigg|_{t^*_*=0} \right) \frac{\prod w^\alpha}{n!} =$$

$$= \varepsilon^2 (\text{dif. pol.}) + (\text{rational part}).$$

The generalised DZ conjecture in genus 0 $\iff$

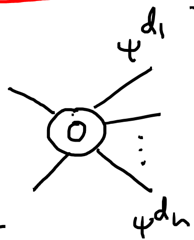
$$\iff \operatorname{Coef}_{\varepsilon^0}\left(\frac{\partial^2 \mathcal{F}}{\partial t^\alpha_a\,\partial t^\beta_b} - \sum\left(\frac{\partial^{n+2}\mathcal{F}_0}{\partial t^\alpha_a\,\partial t^\beta_b\,\partial t^{\alpha_1}_0\cdots\partial t^{\alpha_n}_0}\bigg|_{t^*_*=0}\right)\frac{\prod w^{\alpha}}{n!}\right)=0$$

$$= \sum_n \sum_{\substack{d_1,\dots,d_n\\ \sum d_i\geq 1}} \sum_{\alpha_1,\dots,\alpha_n} \frac{\prod t^{\alpha_i}_{d_i}}{n!}\times$$

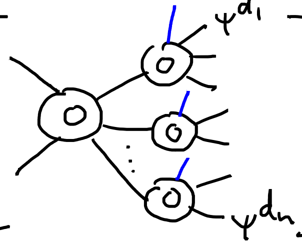
$$\boxed{\text{main observation}}$$

$$=: B_{0,(d_1,\dots,d_n)} \in H^{2\sum d_i}(\overline{\mathcal{M}}_{0,n+2})$$

$$\times \int_{\overline{\mathcal{M}}_{0,n+2}} \Bigg(\Big[$$



$$\Big] - \sum \pi_* \Big[$$



$$\Big]\Bigg)\, c_{0,n+2}\big(\otimes e_{\alpha_i} \otimes e_\alpha \otimes e_\beta\big)\cdot \psi^a_{n+1}\psi^b_{n+2}$$

forgets blue legs

Thus, $\boxed{B_{0;(d_1,\ldots,d_n)}=0 \text{ for all } d_1,\ldots,d_n\geq 0 \text{ with } \sum d_i \geq 1}$

$$\Downarrow$$

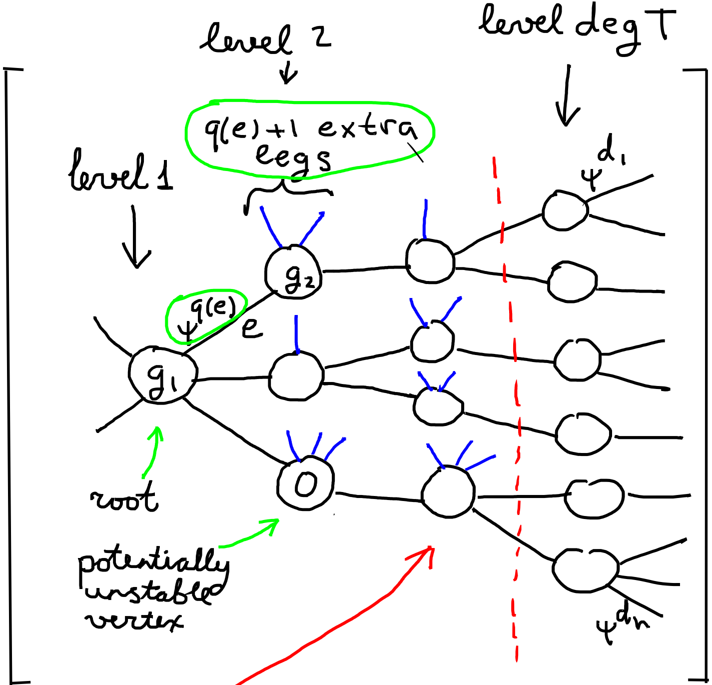
$$\boxed{\text{the generalised DZ conjecture in genus } 0}$$

In all genera, considerations are similar, but combinatorially more complicated.

$g, d_1, \dots, d_n \geq 0$

$$H^{2\sum d_i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ni B_{g;(d_1,\dots,d_n)} :=$$

$$:= \sum_{\substack{\text{rooted}\\ \text{trees } T\\ \text{of genus } g}} \; \sum_{q:\text{edges}\to\mathbb{Z}_{\geq 0}} (-1)^{\overbrace{\#\text{levels}-1}^{\deg T \,=}} \pi_*$$



not all potentially unstable vertices in each level

forgets blue legs

$$\sum_{\text{level(edge)}=i} q(\text{edge}) \leq 2 \sum_{\text{level}(v)\leq i} g(v), \quad i < \deg T$$

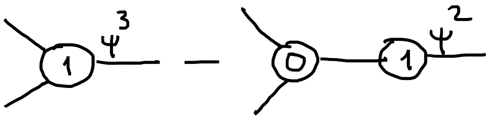
$\boxed{\text{Conjecture}}$ (B.–Shadrin, paper in progress)

$$B_{g,(d_1,\dots,d_n)} = 0 \in H^{2\sum d_i}(\overline{\mathcal{M}}_{g,n+2}, \mathbb{Q}) \text{ for all } g, d_i \geq 0$$

satisfying $\sum d_i > 2g$

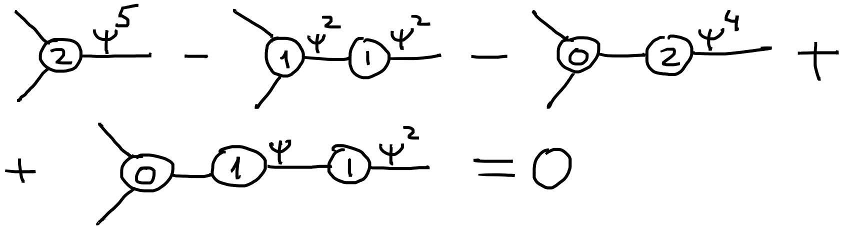
$\boxed{\text{Theorem}}$ (B.–Shadrin, paper in progress)

1. Conjecture $\Longrightarrow$ the generalised DZ conjecture
2. Conjecture is true for a) $g = 0, 1$
   b) $n = 1$, g arbitrary

Example $B_{1,3} =$  $= 0$

Example

$B_{2,5} =$



$= 0$