Hamiltonian Dynamics of monodromy of the maximal degenerate family of CY manifolds 1

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SCGP

$\pi : X^{n+1} \longrightarrow D^2$ holomorphic & proper.

$c \in D^2 \Rightarrow M_c^n := \pi^{-1}(c)$.

Assumption

① M_c is smooth for $c \neq 0$

② M_0 is normal crossing divisor.

($M_0 = \cup D_i$ $D_i \subset X$ codim 1 cpx submanifold

intersection is transversal)

③ $\exists p \in M_0$ st $p \in \cap$ $n+1$ of D_i's.

We call such $\pi : X^{n+1} \longrightarrow D^2$

the maximal degenerate family.

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Let ω be a Kähler form of X.

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Define $H : X \longrightarrow \mathbb{R}$

$H(x) = |\pi(x)|$. ← We consider H as a Hamiltonian

Let X_H be the Hamiltonian vector field generated by H.

i.e. $\omega(V, X_H) = V(H) \quad V \in TX.$

H is well known $X_H(H) = 0$

Let $\varepsilon \in \mathbb{R}_+$

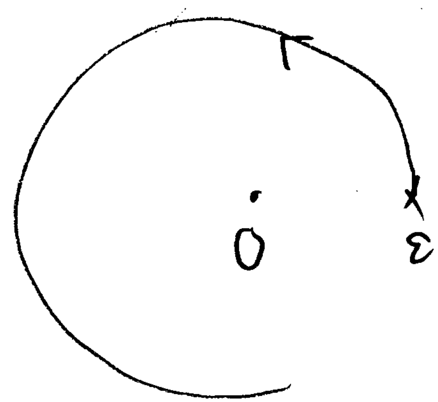
We will define Poincaré map $\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon$
which is a symple diffeomorphism below

For $p \in M_\varepsilon$ (or $p \in X$), Let

$\gamma_p : \mathbb{R} \to X$ the curve $\begin{cases} \dfrac{d\gamma_p}{dt} = X_H(\gamma_p) \\ \gamma_p(0) = p \end{cases}$

Since $X_H(H) = 0$, $H(\gamma_p) = |\pi(\gamma_p)| \equiv \varepsilon$ (constant)

$t \mapsto \pi(\gamma_p(t))$ looks like



Therefore $\exists\ t_p$ s.t

$$t_p = \inf \{ t > 0 \mid \gamma_p(t) \in M_\varepsilon \}$$

We put

$$\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon \quad \text{by} \quad \varphi_\varepsilon(p) = \gamma_p(t_p).$$

This is the Poincaré map

Lemma $\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon$ is a symplectic diffeomorphism

i.e put $w = \omega|_{M_\varepsilon}$ then $\varphi_\varepsilon^* w = w$.

Proof $p \in M_\varepsilon$ $t_p > 0$ $q = \varphi_\varepsilon(p) = \gamma_p(t_p)$

U_ε nbd of p in M_ε.

Let $V_q = \{x \in X \mid d(q,x) < \delta,\ H(\xi) = \varepsilon\}$

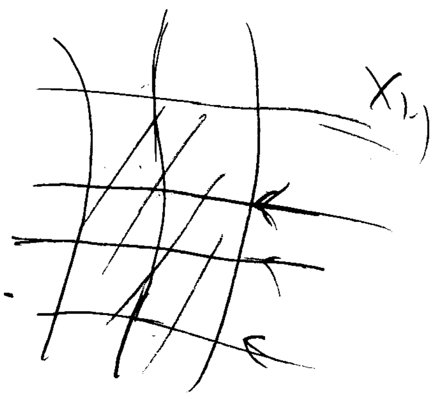
$n+1$ dimensional

$P_r : V_\varepsilon \longrightarrow U_\varepsilon =$ a nbd of q in M_ε

$x \longmapsto \gamma_x(p)$

Here $t \longmapsto \gamma_x(t)$ satisfies $\begin{cases} \dfrac{d\gamma_x}{dt} = X_H \\ \gamma_x(0) = x. \end{cases}$

M_ε



V_ξ

$$V_\varepsilon \longrightarrow V_\xi \xrightarrow{\Pr} U_\xi$$

$$x \longmapsto \varphi_x(t_p)$$

The composition is φ_ε ⊗

Note $x \longmapsto \varphi_x(t_p)$ is a (local) sym

differmorphism (from a nbd of p in X to a

neighborhood of Σ)

It is also easy to see $\omega|_{V_\Sigma} = P_r^*(\omega|_{U_c})$

They imply $\varphi_\varepsilon^* \omega = \omega$.

QED

We want to study:

<u>Problem</u>

Study the symplectic diffeomorphism

$\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon$

as a Hamiltonian Dynamics.

Turkish J. Math 2003

<u>Conj</u> (Fukaya ; Galois Symmetry on Floer Cohomology / Conjecture 4.3)

As $\varepsilon \to 0$ φ_ε "converges" to a complete integrable system. In particular (using KAM)

$\exists$ $M_\varepsilon^0 \subset M_\varepsilon$ st,

1) M_ε^0 is φ_ε invariant and foliated by invariant tori. (Lag $T^n \subset M_\varepsilon$)

2) $\mathrm{Vol}(M_\varepsilon \setminus M_\varepsilon^0) \to 0$ as $\varepsilon \to 0$

I explained it as a conjecture again last year in the Colloquium Kyoto.

Actually as I will explain this week, the conjecture seems to be completely wrong.

In place of ω a Kähler form on X,

we might use certain Kähler form ω' on $X \setminus M_0$

which diverge at M_0

the use Hamiltonian vector field X'_H

by this symplectic structure.

The I believe the conjecture is correct

I will explain certain points related to it

in the second half of this week.

In the first half I use ω a Kähler form ω on X and study $\varphi_\Sigma : M_\Sigma \to M_0$ via examples etc.

Examples

$\mathbb{P}^{n+1}$ $n+1$ dimensional complex projective space

$\cup$

$[z_0 : \cdots : z_{n+1}]$ $z_i \in \mathbb{C}$ $(z_0, \cdots, z_{n+1}) \neq (0, \cdots, 0)$

$[z_0 : \cdots : z_{n+1}] = [z_0' : \cdots : z_{n+1}']$

$\Leftrightarrow \exists c \in \mathbb{C} \setminus \{0\}$ $z_i' = c z_i$

Define $f : \mathbb{P}^{n+1} \longrightarrow \mathbb{C}$

$$f([z_0 : \cdots : z_{n+1}]) = \frac{z_0 z_1 \cdots z_{n+1}}{z_0^{n+2} + \cdots + z_{n+1}^{n+2}}$$

f is a meromorphic function

$f([z_0 \cdots z_{n+1}])$ is well define $\in \mathbb{C} \cup \{\infty\}$

unless

$$z_0 \cdots z_{n+1} = z_0^{n+2} + \cdots + z_{n+1}^{n+2} = 0$$

$\exists$ appropriate blow up $\check{\mathbb{P}}^{n+1}$ of $\mathbb{P}^{n+1}$

st f is well defined holomorphic map

$f : \check{\mathbb{P}}^{n+1} \longrightarrow \mathbb{C} \cup \{\infty\}$ (This point will be explained more later)

$\varepsilon \in \mathbb{C} \setminus \{0\}$ $\quad |\varepsilon|$ small

$M_\varepsilon = f^{-1}(c)$ is a hypersurface $\subset \mathbb{P}^{n+1}$ defined by an equation

$$z_0 \cdots z_{n+1} = \varepsilon (z_0^{n+2} + \cdots + z_{n+1}^{n+2})$$

M_ε is a Calabi-Yau manifold and is its typical example

$n = 3 \Rightarrow$ quintic 3 fold

This is a famous example in Mirror symmetry

$D^2 \subset \mathbb{C} \cup \{\infty\}$ a nbd of 0

$X = f^{-1}(D^2) \subset \check{\mathbb{P}}^{(n+1)}$

$f = \pi : X \longrightarrow D^2$

We may take D^2 small st $\pi^{-1}(z) = M_z$

is smooth for $z \neq 0$

$M_0 = \pi^{-1}(\tau)$ is given by

$z_0 \cdots z_{n+1} = 0$ (its blow up actually)

$D_i \longleftrightarrow z_i = 0$

$\overline{M}_0 = D_0 \cup \cdots \cup D_{n+1}$ normal crossing divisor

$\exists\ p$ for example $p = [1 : 0 : \cdots : 0]$

$p \in D_1 \cap \cdots \cap D_{n+1}$

Thus $\pi : X \longrightarrow D^2$ is a maximal degenerate family.

Take an appropriate Kähler form on $X \subset \mathbb{P}^{n+1}$

We are in the situation discussed before

$\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon$ is the induced map.

We want to study $\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon$ as a symplectic diffeomorphism.

We state a theorem in case $n=2$

$$M^2_\varepsilon \subset \check{\mathbb{P}}^3 \qquad z_0 \cdots z_3 = \varepsilon(z_0^4 + z_1^4 + z_2^4 + z_3^4)$$

This is a quartic surface

(An example of K3 surface.)

Th'm

$\varphi_\varepsilon : M^2_\varepsilon \longrightarrow M^2_\varepsilon$ has 408 fixed points

for $\varepsilon \neq 0$ small.

Remark $\mathrm{rk}\, H(M^2_\varepsilon, \mathbb{Q}) = 24 = \chi(M^2_\varepsilon)$

408 is much bigger.

We start studying $\varphi_\varepsilon : M^n_\varepsilon \longrightarrow M^n_\varepsilon$ via perturbation theory in Hamiltonian Dynamics.

Reference

Encyclopedia of Math. Science
Dynamical Systems $\underline{\text{III}}$ (ed Arnold)
Chapter 5 (Perturbation theory of Integrable Systems)

We consider $\varphi_\varepsilon : M^n_\varepsilon \longrightarrow M^n_\varepsilon$ (for maximal degenerate family $X \xrightarrow{\pi} D^2$)

Proposition 1:

$\exists\ M^{00}_\varepsilon \subset M_\varepsilon$ st.

① $\varphi_\varepsilon(M^{00}_\varepsilon) = M^{00}_\varepsilon$

② M^{00}_ε is foliated by the φ_ε invariant Lagrangian tori.

③ $\mathrm{Vol}(M^{00}_\varepsilon) > 0$.

<u>Note</u> $\mathrm{Vol}(M_\varepsilon - M_0^{\omega}) \not\to 0$.

This proposition is actually an easy consequence of KAM theory

Review of KAM

We begin with a definition of complete integrable system.

$$\begin{array}{ccc} T^n & \hookrightarrow & X \\ & & \downarrow \\ & & B \end{array}$$

$$\begin{array}{ccc} T^n & \hookrightarrow & M^{2n} \\ & & \downarrow \\ & & B \end{array}$$

(M, ω) symplectic

fiber bundle

fiber = n dim compact Lag. sub manifold $(\omega|_{\text{fiber}} = 0)$

$\Downarrow$ ← Liouville-Arnold

fiber $\cong T^n = \mathbb{R}^n / \mathbb{Z}^n$

Action angle coordinate $q_1, \cdots, q_n, p_1, \cdots p_n$

$\pi(p, q) = q$

$q_1 \cdots q_n$ coordinate of B

$\frac{\partial}{\partial p_1}, \cdots \frac{\partial}{\partial p_n}$ tangent to the fiber

$\omega = \sum dp_i \wedge dq_i$

$T^n = \mathbb{R}^n / \mathbb{Z}^n$ fibers $p_1 \cdots p_n$ coordinate of $\mathbb{R}^n$

$H_0 : M^{2n} \longrightarrow \mathbb{R}$ Hamiltonian

$H_0(q)$ H_0 depends only on q

Namely $H_0 = \bar{H}_0 \circ \pi$

in other words $\{P_i, H_0\} = 0 \quad i = 1 \cdots m$

Note $\{\ \}$ is Poisson bracket $\{f, g\} = X_f(g)$.

$\overline{X}_{H_0}$: Hamiltonian vector field

on each fiber

$$X_{H_0} = \sum \frac{\partial H_0}{\partial q_i} \frac{\partial}{\partial p_i}$$

$\Uparrow$

const on a fiber

X_{H_0} is linear on the fibers

Thm (KAM)

Let H_0, M etc as above. $H_1 : M \to \mathbb{R}$

$H_\varepsilon = H_0 + \varepsilon H_1$ $\quad X_{H_\varepsilon}$ its Hamiltonian vector field

We assume certain non-degeneracy condition for H_0 (described later)

$\Rightarrow \exists\, M_\varepsilon^0 \subset M$ st

1) M_ε^0 is X_{H_ε} invariant

2) M_ε^0 is foliated by Lagrangian tori. (close to the fibers of $\pi : M \to B$)

3) $\mathrm{Vol}(M - M_\varepsilon^0) \to 0$ as $\varepsilon \to 0$

Non degeneracy condition

$\vec{q} \in B$ $\quad X_{H_J} = \sum w_i(\vec{q}) \frac{\partial}{\partial p_i}$ $\quad p_1 \cdots p_n$

coordinate of $T^n = \mathbb{R}^n / \mathbb{Z}^n$

$w_1(\vec{q}) \cdots w_n(\vec{q})$ frequences

Non-degeneracy $\det\left(\frac{\partial w_i}{\partial q_j}\right) \neq 0.$

We go back to the point of Proposition 1,

$$\begin{array}{c} X^{n+1} \\ \downarrow \pi \\ D^2 \end{array}$$

Take $p_0 \in \pi^{-1}(0) = M_0$ s.t. $p_0 \in D_1 \cap \cdots \cap D_{n+1}$

D_i irreducible component of M_0

A neighborhood U of p_0 $U \subset \mathbb{C}^{n+1}$

$$D_1 \cap U \cong (\mathbb{C} \times \cdots \times \mathbb{C} \times 0 \times \mathbb{C} \times \cdots \times \mathbb{C}) \cap U \subset \mathbb{C}^{n+1} \qquad p_0 = 0$$

Rescale by $R = \frac{1}{\varepsilon}$, replace ω by $\varepsilon^2\omega$

ω on 1 ball is $\qquad z_i = x_i + \sqrt{-1}\, y_i$

$$\sum_{i=1}^{n+1} dx_i \wedge dy_i + \text{term of order } \varepsilon$$

$$H : X^{n+1} \longrightarrow D \overset{\parallel}{\longrightarrow} \mathbb{R}$$

H on 1 ball ($\simeq \varepsilon$ ball before scaling) is

$H = |z_1, \ldots, z_{n+1}| +$ term of order ε

<u>Darboux's theorem</u>

$$(x_i, y_i) \xmapsto{\psi} (x_i', y_i')$$

$$\omega = \sum dx_i' \wedge dy_i' \qquad |\psi - id|_{C^1} < \varepsilon$$

In new coordinate

$$H = \underbrace{\prod \sqrt{(x_i'^2 + y_i'^2)}}_{= H_0} + \text{term of order } \varepsilon.$$

$$r_i = \sqrt{x_i'^2 + y_i'^2} \qquad \{r_i, r_j\} = 0$$

$$H_0 = r_1 \sim r_{n+1}$$

$$\therefore \{H_0, r_i\} = 0, \quad i = 1 \sim n+1$$

$$U \setminus M_0 \xrightarrow{r_1 \sim r_n} \mathbb{R}^{n+1}$$

$$\|$$

$$X$$

fiber = Lagrangian tori

So H_0 is a complete integrable system.

H satisfies somewhat weaker non-degeneracy condition.

Reference Dynamical System III page 183

$H_0 : M^n \longrightarrow \mathbb{R}$ completely integrable as before.

$$(M^n \longrightarrow B)$$
$$(\gamma, I) \qquad \xi$$

$\omega_1(\xi), \dots, \omega_n(\xi)$ frequencies.

Def isoenergetically nondegenerate

$\omega_1(\xi) \neq 0 \qquad \overline{\omega}_i = \omega_i / \omega_1 \qquad i = 2, \dots, n$

$$\operatorname{rk}\left(\frac{\partial \overline{\omega}_i}{\partial q_j}\right) = n-1.$$

KAM theorem still holds under this weaker condition.

(Theorem 13 of Dynamical System III page 183)

<u>Note</u> Why nondegeneracy condition?

If $w_1, \cdots, w_n$ has rationally related

that is $\exists\, k_i \in \mathbb{Z}$ st

$$\sum k_i w_i = 0 \qquad (k_1 \cdots k_n) \neq 0$$

Then the occur resonance which makes system unstable.

(that is small perturbation destroy invariant tori.)

Non-degeneracy $\Rightarrow$ resonance occurs in measure 0 value of $\vec{I}$.

For this "isoenergetically non-degenerate" is enough

Going back to our situation

$$H_0 = r_1 \cdots r_{n+1}$$

$$r_i = \sqrt{x_i'^2 + y_i'^2}$$

$$T^{n+1} \hookrightarrow X^{n+1} \to B$$

Fiber is given by

$$r_1 = c_1 \quad \cdots \quad r_{n+1} = c_{n+1}$$

$$X_{r_i} = \frac{1}{r_i} \frac{\partial}{\partial \theta_i} \qquad r_i e^{2\pi i \theta_i} = x_i' + \sqrt{-1} y_i$$

θ_i's are angle coordinate (coordinate of $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$)

$$\omega_i = \frac{r_1 \cdots \hat{r}_i \cdots r_{n+1}}{r_i}$$

r_i are action coordinate

__Lemma__

Our H_0 is isoenergetically non-degenerate

$\because$) $\dfrac{w_i}{w_1} = \dfrac{r_1^3}{r_i^2}$ //

This conclusion of KAM holds in s nbd of γ_ω

(This is about $X^{aH} \longrightarrow D^3$ and X_H)

The KAM tori for this system gives a KAM tori of $M_\varepsilon \xrightarrow{\varphi_\varepsilon} M_\varepsilon$.

QED