Hamiltonian Dynamics of monodromy of the maximal degenerate family of CY manifolds 2

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Kenji Fukaya

SCGP

The proposition (existence of M_ε° of positive measure which is φ_ε invariant and is foliated by Lagrangian tori) we discussed yesterday is a bit superficial since it studies only a neighborhood of points p_0 st $p_0 = D_1 \cap \cdots \cap D_{n+1}$ (intersection of $n+1$ $= \dim_{\mathbb{C}} X$ irreducible components).

We want to study φ_ε outside such (finitely many) points.

We will study by examples.

We first study a pencil of cubic curves.

$$f : \mathbb{P}^2 \longrightarrow \mathbb{C} \qquad [z_0, z_1, z_2] \longmapsto \frac{z_0 z_1 z_2}{z_0^3 + z_1^3 + z_2^3}$$

f is a meromorphic function, which is defined as a map to $\mathbb{C} \cup \{\infty\}$ outside the basic locus.

(B)

Basic locus

$\Longleftrightarrow \quad z_0 z_1 z_2 = 0, \quad z_0^3 + z_1^3 + z_2^3 = 0$

$\Longleftrightarrow$ 9 pts $\quad [0 : 1 : \exp((1+2k)\pi\sqrt{-1}/3)]$ $\quad k = 0, 1, 2$

permutation of $0, 1, 2$

$\checkmark$

$\mathbb{P}^2$ blow up of $\mathbb{P}^2$ at these 9 points

Lemma holomorphic map

$$\begin{tikzcd} \hat{\mathbb{P}}^2 \arrow[rr, "\check{f}"] \arrow[dr] & & \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \\ & \mathbb{P}^2 \arrow[ur] & \end{tikzcd}$$

$$f = \frac{z_0 z_1 z_2}{z_0^3 + z_1^3 + z_2^3}$$

∵) Well known. But let us recall blow up and prove this lemma.

Review of Blow up

M^n complex mfd, $Z^k \subset M^n$ complex submanifold

$N_Z M$ normal bundle (complex vector bundle)

$\mathbb{P} N_Z M = \{ v \in N_Z M \mid v \neq 0 \} / \sim$

$v \sim v' \iff \exists c \in \mathbb{C} \setminus 0 \quad v' = cv.$

$\check{M} := (M \setminus Z) \amalg \mathbb{P} N_Z M$ set theoretically

blow up of M at Z

$$\check{M} \longrightarrow M \quad \begin{cases} \text{identity on } M \setminus Z \\ \text{Obvious projection on } \mathbb{P}N_Z M \end{cases}$$

<u>Lemma</u> $\exists$ complex structure on $\check{M}$ s.t. $\check{M} \to M$ is holomorphic

$\because$) complex str. on $\check{M} \setminus \mathbb{P}N_Z M = M \setminus Z$ is induced by the right hand side.

We define a (complex) line bundle $\mathcal{L} \longrightarrow \mathbb{P}N_2M$ as follows

$$\begin{array}{ccl} \mathcal{L} = & \left\{ ([v], w) \;\middle|\; \begin{array}{l} [v] \in \mathbb{P}N_2M \\ w \in N_2M \\ \exists c \in \mathbb{C} \quad w = cv \end{array} \right\} \\ \Big\downarrow & ([v], w) & \\ & \Big\downarrow & \\ \mathbb{P}N_2M & [v] & \end{array}$$

$$B_\varepsilon(\mathcal{L}) = \{([v], w) \mid \|w\| < \varepsilon\}$$

$B_\varepsilon N_Z M$ ε ball bundle $\subset N_Z M$

$B_\varepsilon N_Z M \xhookrightarrow{i} M$ C^∞ embedding tubular neighborhood

$B_\varepsilon(\mathcal{L}) \xrightarrow{I} \check{M}$

$I([v], w) = i(w)$ if $w \neq 0$ $\in M \setminus Z$

$= [v]$ if $w = 0$ $\in \mathbb{P} N_Z W$

9

This is bijection onto an ε-neighborhood of $Z \subset \check{M}$

We use it to define a structure of C^∞ mfd on $\check{M}$. (Namely I becomes a diffeomorphism)

Looking locally we can show that there exists a complex structure on $B_\varepsilon(\mathcal{L})$ st.

$$I\big|_{B_\varepsilon(\mathcal{L}) \setminus 0 \text{ section}} \quad B_\varepsilon(\mathcal{L}) \setminus Z \longrightarrow M \setminus Z$$

is holomorphic embedding

We use it to define a complex structure on $\check{M}$

QED

Let ω be a Kähler form on M.

$pr : \check{M} \longrightarrow M$ as above.

$pr^*\omega$ a differential form on $\check{M}$

It is a Kähler form on $\check{M} \setminus \mathbb{P}N_Z M = M \setminus Z$.

but is degenerate on $\mathbb{P}N_2M$.

We take

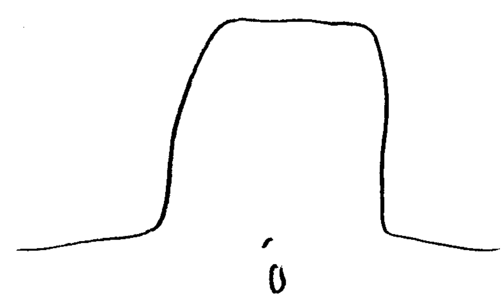
$$pr^*\omega + \varepsilon\chi\cdot\pi^*\omega_{\mathbb{P}N_2M} = \tilde{\omega}$$

as a Kähler form on $\tilde{W}$.

Here $\omega_{\mathbb{P}N_2W}$ is a Kähler form on $\mathbb{P}N_2M$

$\pi : \mathcal{L} \longrightarrow \mathbb{P}N_2M$

χ is a bump function



χ

It is well know that $\tilde{\omega}$ is a Kähler form.

<u>Note</u> The Kähler form $\tilde{\omega}$ or its homology class

$[\tilde{\omega}] \in H^2(\check{M})$ is not unique.

It depends on $\displaystyle\int_{\mathbb{P}N_Z M} \tilde{\omega} \in \mathbb{R}$

In case $Z = pt$ $\quad \check{M} = M \# \overline{\mathbb{C}P^n}$

$$H^2(\check{M}) = H^2(M) \oplus \mathbb{Z} \xleftarrow{PD} \mathbb{C}P^{n-1} \subset \overline{\mathbb{C}P^n}$$

$[\tilde{\omega}]$ on $H^2(M)$ is the same

but $\int_{\mathbb{C}P^{n-1}} \tilde{\omega}$ is a parameter.

Going back to our example,

$$f : \mathbb{P}^2 \longrightarrow \mathbb{C} \qquad [z_0 : z_1 : z_2] \longmapsto \frac{z_0 z_1 z_2}{z_0^3 + z_1^3 + z_2^3}$$

$\check{\mathbb{P}}^2$ blow up of $\mathbb{P}^2$ at Σ : $z_0 z_1 z_2 = z_0^3 + z_1^3 + z_3^3 = 0$

9 points

$$\begin{tikzcd}
\check{\mathbb{P}}^2 \arrow[r, "\hat{f}"] \arrow[dr] & \mathbb{C} \cup \{\infty\} \quad \text{holo} \\
& \mathbb{P}^2 \arrow[u, "f"'] \arrow[u, phantom, shift right=6, "\circlearrowleft"]
\end{tikzcd}$$

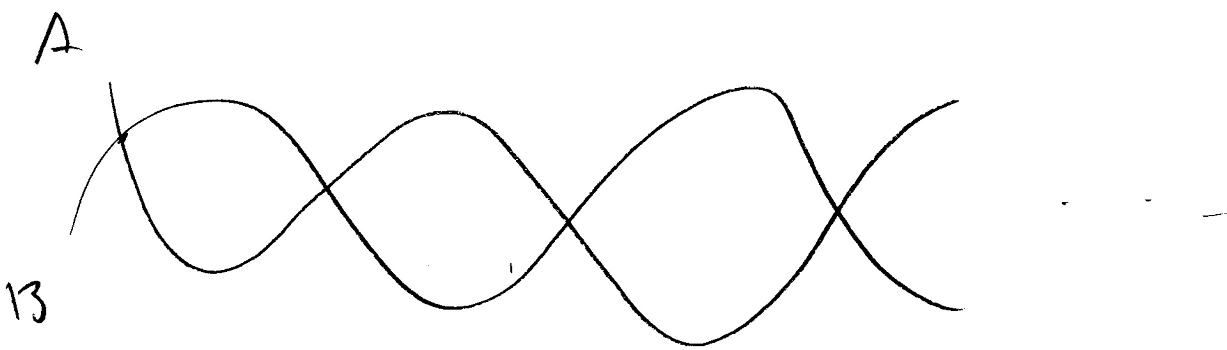
$\exists \hat{f}$

f meromorphic

ii)

A : $z_0 z_1 z_2 \subset \mathbb{P}^2$

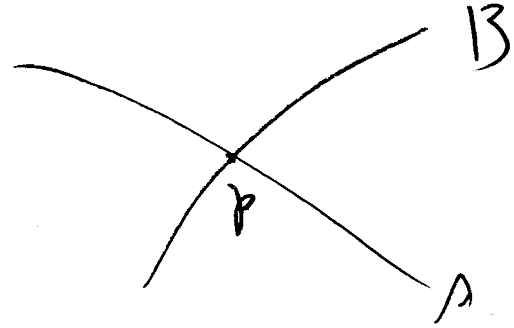
B $z_0^3 + z_1^3 + z_2^3 \subset \mathbb{P}^2$



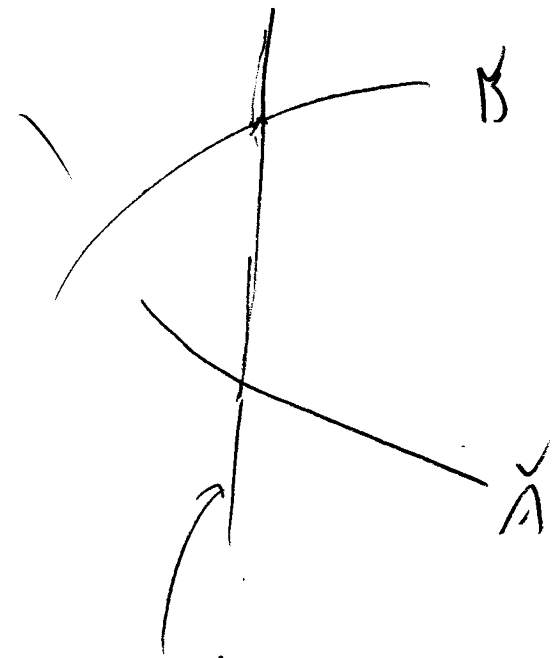
$\check{A} = \overline{(A \setminus Z)}$ in $\check{\mathbb{P}}^2$

$\check{B} = \overline{(B \setminus Z)}$ in $\check{\mathbb{P}}^2$

$\check{A} \cap \check{B} = \emptyset$



$p \in A \cap B$



$p_r^{-1}(p) \cong \mathbb{P}^1$

So $\dfrac{z_0 z_1 z_2}{z_0^3 + z_1^3 + z_2^3}$ is extended to a map to

$(\mathbb{C} \cup \{\infty\})$ on $\check{\mathbb{P}}^2$

//

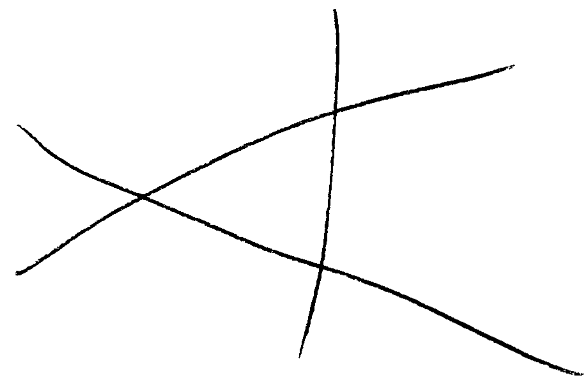
We thus have $\check{f} : \check{\mathbb{P}}^2 \longrightarrow \mathbb{C} \cup \{\infty\}$

and a Kähler form $\tilde{\omega}$ on $\check{\mathbb{P}}^2$

Let $D^2 \subset \mathbb{C} \cup \{\infty\}$ a small nbd of $0 \in \mathbb{C}$.

$$X = \check{f}^{-1}(D^2)$$

$$(X, \tilde{\omega}) \xrightarrow{\check{f}} D^2$$



$\check{f}^{-1}(0) \cong Z_0 Z_1 Z_2 = 0$ in $\mathbb{P}^2$ union of 3 $\mathbb{P}^1$'s

<u>Note</u> $\varepsilon \neq 0$

M_ε $\check{f}^{-1}(\varepsilon) \Longleftrightarrow z_0 z_1 z_2 = \varepsilon(z_0^3 + z_1^3 + z_2^3)$

cubic curve

$(:= \text{torus } T^2)$

$\varphi_\varepsilon : M_\varepsilon \longrightarrow M_0 \quad : \quad T^2 \longrightarrow T^2.$

<u>Thm 1</u>

$$M_\varepsilon$$
$$\parallel$$
∘ $\varphi_\varepsilon : T^2 \longrightarrow T^2$ has 18 fixed points

⊙ φ_ε satisfies the conclusion of KAM i.e.

1) $\exists\ M_\varepsilon^0 \subset M_\varepsilon$ φ_ε invariant.

2) M_ε^0 is foliated by φ_ε invariant S^1's.

3) $\lim_{\varepsilon \to 0} Vol(M_\varepsilon \setminus M_\varepsilon^0) = 0$

i.e. Conjecture 1 mentioned holds in this 2 dimensional case.

For the proof of this theorem (and some more results)

we review toric geometry.

Def (X, ω) sym mfld $T^n \curvearrowright X$ act preserving ω.

$\mu : X \to \mathbb{R}^n$ is said to be a moment map

$\Leftrightarrow$ $\mu = (\mu_1 \cdots \mu_n)$ $\mu_i : X \to \mathbb{R}$

X_{μ_i} Hamiltonian vector field

$=$ vector field generated by the action of i-th factor $S^1 \subset T^n$

(2)

$T^n \subset (\mathbb{C}_*)^n$ $\quad (S^1 \subset \mathbb{C})$

(X, ω, g, J) Kähler $\quad T^n \curvearrowright (X, \omega, g)$

is toric manifold

$\Leftrightarrow$ $\exists$ $(\mathbb{C}_*)^n$ action

① extending T^n action

② $(\mathbb{C}_*)^n$ action preserves complex structure

③ $\exists\, p_0 \in X$ st $(\mathbb{C}_*)^n \cdot p_0$ is dense in X

<u>Fact</u> (proof omitted)

1) $\exists$ moment map $\mu : X \longrightarrow \mathbb{R}^n$

2) $\operatorname{Im} \mu = P \subset \mathbb{R}^n$ is a convex polygon (moment polytope)

3) $\mu^{-1}(\operatorname{Int} P)$ is the $(\mathbb{C}_*)^n$ orbit $(\mathbb{C}_*)^n p_0$

4) $X \setminus \mu^{-1}(\operatorname{Int} P) = \mu^{-1}(\partial P)$ is a normal crossing divisor D (toric divisor)

5) $\partial P = \bigcup_i \partial_i P$ (faces).

$\Rightarrow \mu^{-1}(\partial_i P)$ is an irreducible component of D

Example

$$X = \mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}_*$$

$$T^n \curvearrowright X \qquad (\rho_1 \cdots \rho_n)[z_0 : \cdots : z_n]$$

$$= [z_0 : \rho_1 z_1 : \cdots : \rho_n z_n]$$

$$\rho_i \in S^1 = \{\rho \in \mathbb{C} \mid |\rho| = 1\}$$

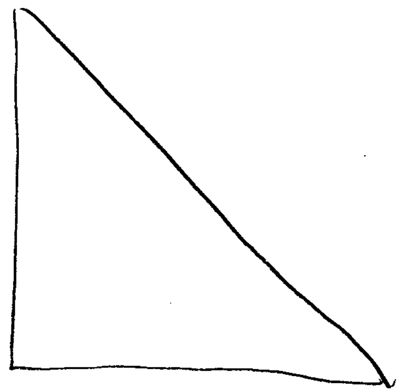
$$\mu : X \to \mathbb{R}^n \quad \text{is}$$

$$\mu = (\mu_1 \cdots \mu_n)$$

$$\mu_i([z_0 : \cdots : z_n]) \qquad i = 1, \cdots, n$$

$$= \frac{|z_i|^2}{|z_0|^2 + \cdots + |z_n|^2}$$

$P = \operatorname{Im}\mu = \{(r_1 \cdots r_n) \mid 0 \leq r_i,\ \Sigma r_i \leq 1\}$



$n = 2$

∂P consists of $n+1$ component $\partial_i P = (r_i = 0)$ $i = 1 \cdots n$

$\partial_0 P = (r_1 + \cdots + r_n = 1)$

$$D_i = \mu^{-1}(\partial_i P) = \{[z_0 : \cdots : z_n] \mid z_i = 0\} \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$$

$(X,\omega) \circlearrowleft T^n \quad X \xrightarrow{\mu} P \subset \mathbb{R}^n$

Toric manifold

$D = \mu^{-1}(\partial P)$

$\exists\, \mathcal{L} \to X$ a holomorphic line bundle associated with D.

$p \in X$ write D $\quad s_p = 0$ in a nbd U_p of p

$\bigcup_i U_{p_i} = X \qquad \mathcal{L}|_{U_{p_i}} \cong \mathbb{C} \times U_{p_i}$

on $U_{p_i} \cap U_{p_j} \longrightarrow \mathbb{C}_*$ $\qquad z \longmapsto \dfrac{s_{p_j}(z)}{s_{p_i}(z)} = s_{ji}(z) \in \mathbb{C}_*$

Assume D is effective

$\Leftrightarrow \exists\ s_0$ holomorph section of $\mathcal{L}$

s.t. $s_0^{-1}(0) = D$.

Exmpl. $X = \mathbb{P}^n$ as above $\mathcal{L} = \mathcal{O}(n+1)$

$s_0 \longleftrightarrow z_0 \cdots z_n$ deg $n+1$ polynomial.

S another generic section of $\mathcal{L}$.

$$f : X \longrightarrow \mathbb{C} \qquad z \longmapsto \frac{S_0(z)}{S(z)}$$

meromorphic function

$\exists\, \check{X}$ blowup of X f induces

Lefschetz pencil

$$\check{f} : \check{X} \longrightarrow \mathbb{C} \cup \{\infty\} \quad \text{holomorphic map}$$

$(X \setminus Z = \check{X} \setminus Z \quad Z = \{z \mid S_0(z) = S(z) = 0\}$ basic locus.$)$

D^2 a nbd of 0 in $\mathbb{C}$.

Replace $\check{X}$ by $\mu^{-1}(D^2) \subset \check{X}$

We have $\pi : X \longrightarrow D^2$ maximal degenerate

family.

The case $X = \mathbb{P}^n$

$$s_0 = z_0 \cdots z_n$$

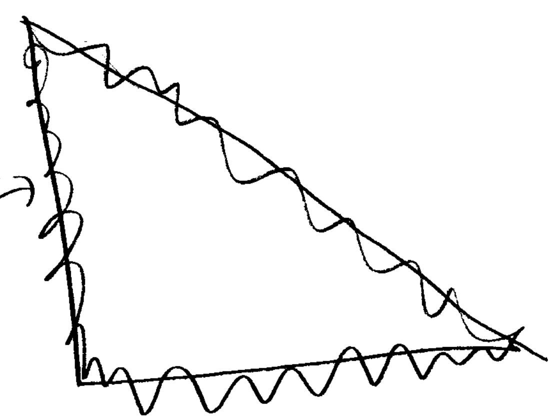
$$s = z_0^{n+1} + \cdots + z_n^{n+1}$$

is the case we have been studying

Note $M_\varepsilon = f^{-1}(\varepsilon)$.

$\mu(M_\varepsilon)$ is in a nbd of ∂P

Ex $\mathbb{P}^2$

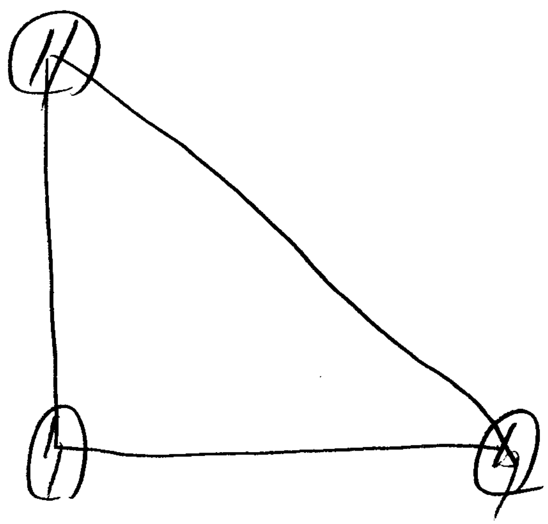


P

$\mu(M_\varepsilon)$ is here

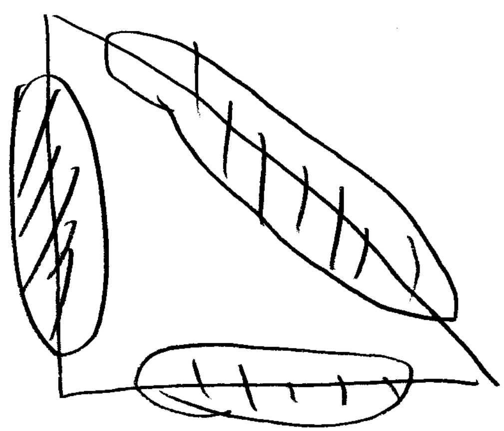
We we proved proposition we studied X_H

$(H = -|f|)$ in a nbd of vertices



KAM was easily applied there.

To prove Theorem 1, we study X_H in a neighborhood of edges.



We use again perturbation theory of Hamiltonian Dynamics.

Arnold's papers

- Proof of a theorem of A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbation of the Hamiltonian

  Russ. Math Surv. 18 (1963) 13–40

  KAM

- Small denominators and problems of stability of motion in classical and celestial mechanics

  Russian Math Surv. 18 (1963) 91–192

  I want to explain this a bit

Dynamical System III page 185 Thm 14.

KAM $\quad H_\varepsilon = \underbrace{H_0}_{\text{completely integrable}} + \varepsilon H_1 \quad H_0$ non degenerate

Here we generalize

$$H_\varepsilon = \underbrace{H_{00} + \varepsilon H_{01}}_{\text{completely integrable}} + \varepsilon^2 H_{11}$$

H_{00} alone is degenerate,
H_{00} together with H_{01} is non degenerate

$$\begin{array}{ccc} T^n & \hookrightarrow & M \\ & & \downarrow \\ & & B \end{array}$$

completely integrable system

(Lagrangian fibration)

$q_1, \cdots, q_n$ coordinate of B

$\overline{H}_0(q_1 \cdots q_k)$

$k < n$ variables

$\overline{H}_{01}(q_1 - q_k \, q_{k+1} \cdots q_n)$ n variables

$$H_{0,\varepsilon}(q_1 - q_n \, p_1 - p_n) = \overline{H}_0(q_1 \cdots q_k) + \varepsilon H_{01}(q_1 \cdots - q_n)$$

Thm (Arnold)

If $\left(\frac{\partial^2 H_0}{\partial q_i \partial q_j}\right)_{i,j=1}^{n}$ and $\left(\frac{\partial^2 H_{0,2}}{\partial q_i \partial q_j}\right)_{i,j=k+1}^{n}$

are non degenerate then

$H_\varepsilon = H_{0,2} + \varepsilon^2 H_1$ satisfies the

same conclusion as KAM.

This actually occurs in celestial mechanics.

I will explain how to apply this theorem to study

$$\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon$$

in the case of pencil of cubic curv.

$$\check{\mathbb{P}}^2 \xrightarrow{f} \mathbb{C} \qquad f([z_0:z_1:z_2])$$

tomorrow

$$= \frac{z_0 z_1 z_2}{z_0^3 + z_1^3 + z_2^3}$$