Hamiltonian Dynamics of Monodromy of the maximal degenerate family of CY manifolds 3

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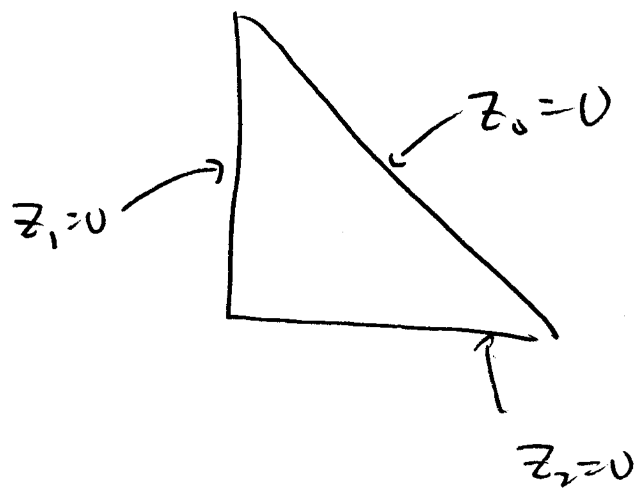
SCGP

$\mathbb{P}^2$

$\mu : \mathbb{P}^2 \longrightarrow \mathbb{R}$ moment map



$$\mu([z_0 : z_1 : z_2]) = \left(\frac{|z_1|^2}{|z_0^2 + z_1^2 + z_2^2|} , \frac{|z_2|^2}{|z_0^2 + z_1^2 + z_2^2|} \right)$$

$$f([z_0 : z_1 : z_2]) = \frac{z_0 z_1 z_2}{z_0^3 + z_1^3 + z_2^3}$$

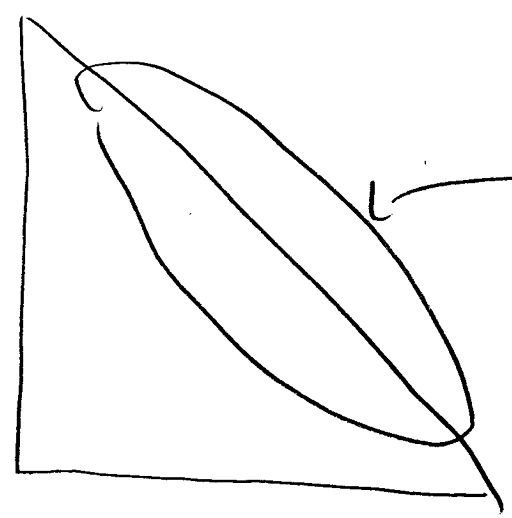
$\check{\pi} = \check{f} : \check{\mathbb{P}}^2 \longrightarrow \mathbb{C} \cup \{\infty\}$ $\quad \mathcal{B}$

9 pt blow up.

$H = |f| = |\pi| : \mathbb{P}^2 \longrightarrow \mathbb{R} \cup \infty$

$M_\varepsilon = \pi^{-1}(\varepsilon)$ cubic curve $\cong T^2$

$\varphi_\varepsilon : M_\varepsilon \to M_\varepsilon$ Poincaré return map of X_H



want to study X_H there.

$[z_0 : z_1 : z_2)$ $|z_0|$ small

$$H = |z_0| \frac{|z_1 z_2|}{|z_0^3 + z_1^3 + z_2^3|}$$

$$w = \frac{z_0}{z_1}$$

$$\approx |z_0| \frac{|z_1 \cdot z_2|}{|z_1^3 + z_2^3|}$$

$$z = \frac{z_2}{z_1}$$

z_0 small

coordinate of $\mathbb{P}^1$

$\mathbb{P}^1$

$$= |w| \frac{|z|}{|1+z^3|}$$

w : sym form = blow up of Fubini Study form

at this divisor it is $\frac{1}{1+|z|^2}\delta_{00}$ in the normal direction

So

$$H_0 = \underbrace{|z_0|}_{H_{00}} \; \underbrace{\frac{|z|\sqrt{1+|z|^2}}{|1+z^3|}}_{H_{01}}$$

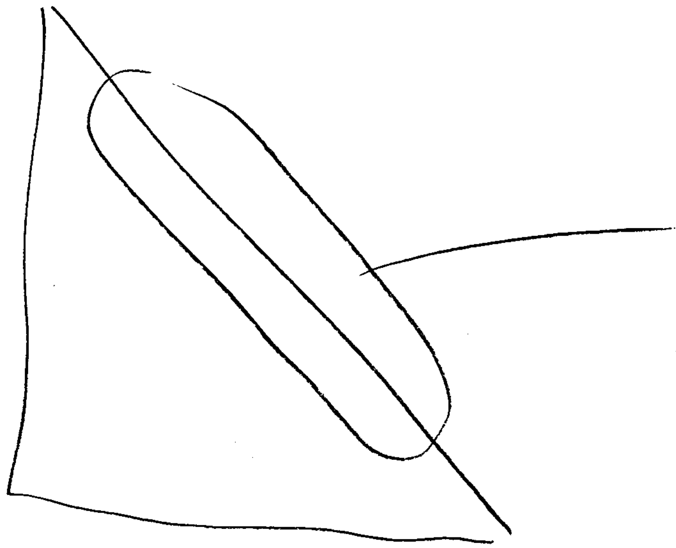
is the leading order term in Darboux coordinate

$$\{H_{00}, H_{01}\} = 0$$

(H_{00} does not contain z
H_{01} does not contain z_0)

H_{00}, H_{01} is preserved by X_{H_0}

$$\gamma \in M_\varepsilon \Rightarrow |H_0(\gamma)| \sim |H(\gamma)| \sim \varepsilon$$



We are here

$z \in \mathbb{P}^1 \setminus$ nbd of 0

$\setminus$ nbd of ∞

$z^3+1=0 \iff$ linear locus $\quad z_0 z_1 z_2 = 0$

$z_0^3 + z_1^3 + z_2^3 = 0$

(at $z_0=0$)

We blow up $\mathbb{P}^2$ here (Discuss it later)

So $0 < C' < H_{01} < C'' < \infty$ uniform in ε

$$X_{H_0} = X_{H_{00}} \cdot H_{01} + H_{00} X_{H_{01}}$$

bounded and bounded away from 0

$\sim \varepsilon$

$z_0 = re^{2\pi i \theta}$

$X_{H_{01}}$

$$= C \cdot \frac{1}{r}\frac{\partial}{\partial \theta} + \varepsilon X_{H_{01}}$$

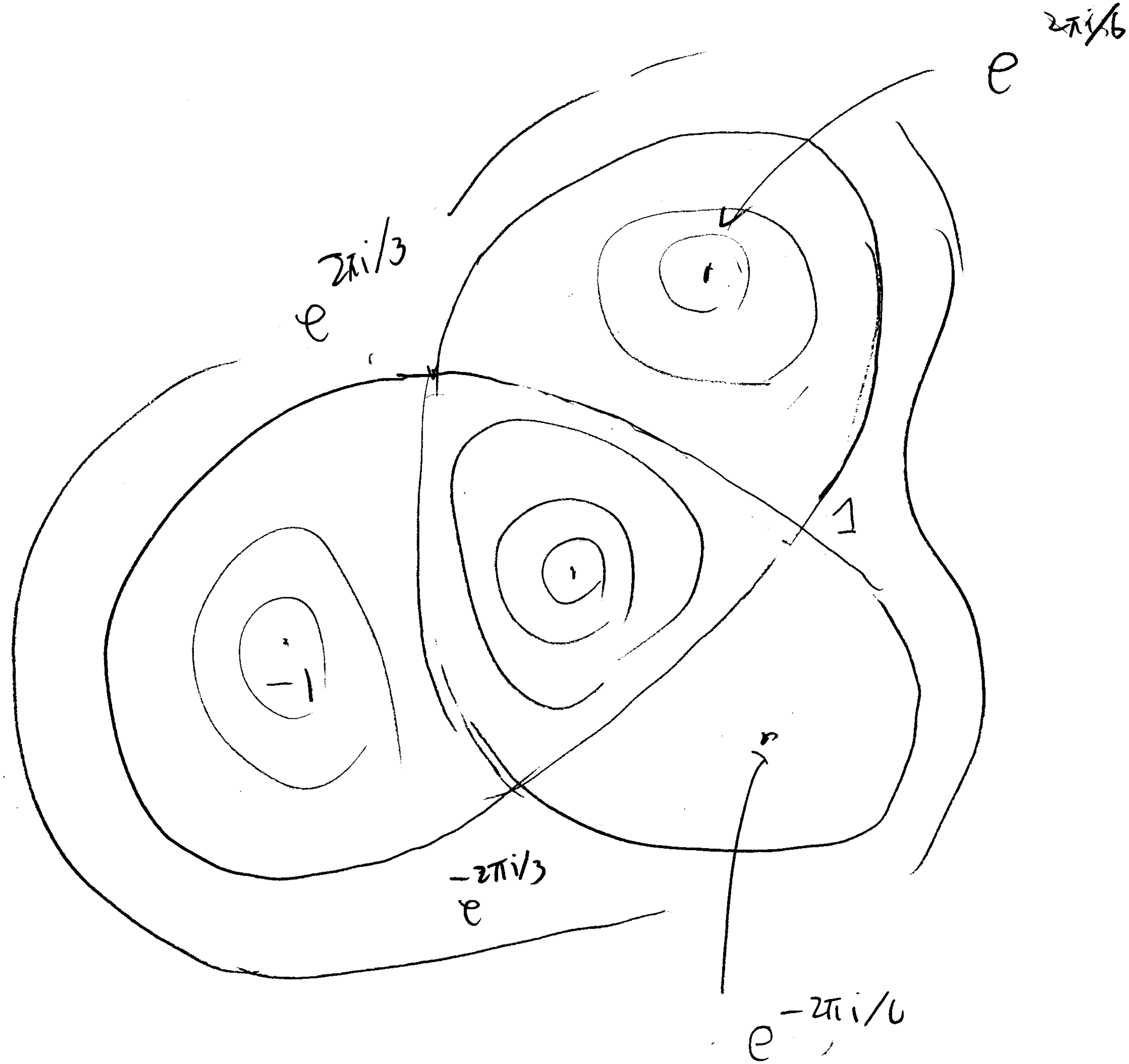
$X_{H_{01}}$ is defined on $\mathbb{P}^1 \setminus$ nbd of 0

$\setminus$ nbd of ∞

$\setminus$ three horns

$$H_{01} = \frac{|z| \sqrt{1+|z|^3}}{|1+z^3|}$$

level set of H_{01} is drawn as follows


$e^{2\pi i/6}$
$e^{2\pi i/3}$
1
-1
$e^{-2\pi i/3}$
$e^{-2\pi i/6}$

X_{H_0} in our domain is an integrable system

two period

$C \frac{1}{r} \frac{\partial}{\partial \theta} \longleftrightarrow \sim \varepsilon$ short fast mode

$\varepsilon X_{H_{01}} \longleftrightarrow \sim \frac{1}{\varepsilon}$ period of $X_{H_{01}}$

long slow mode

This is the situation of Arnold's theorem

(Yesterday)

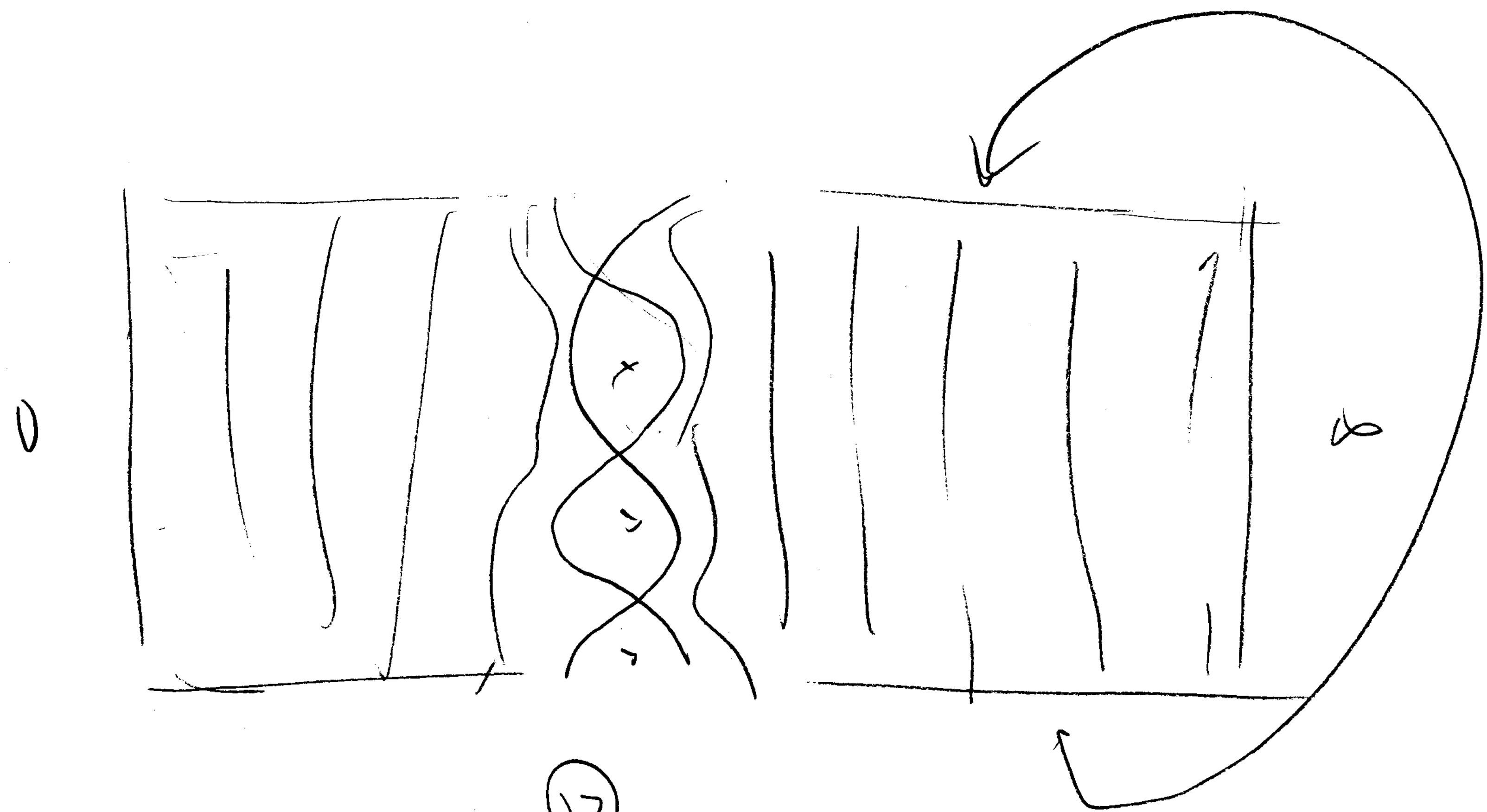
We may directly apply it by considering

$\log H$ in place of H

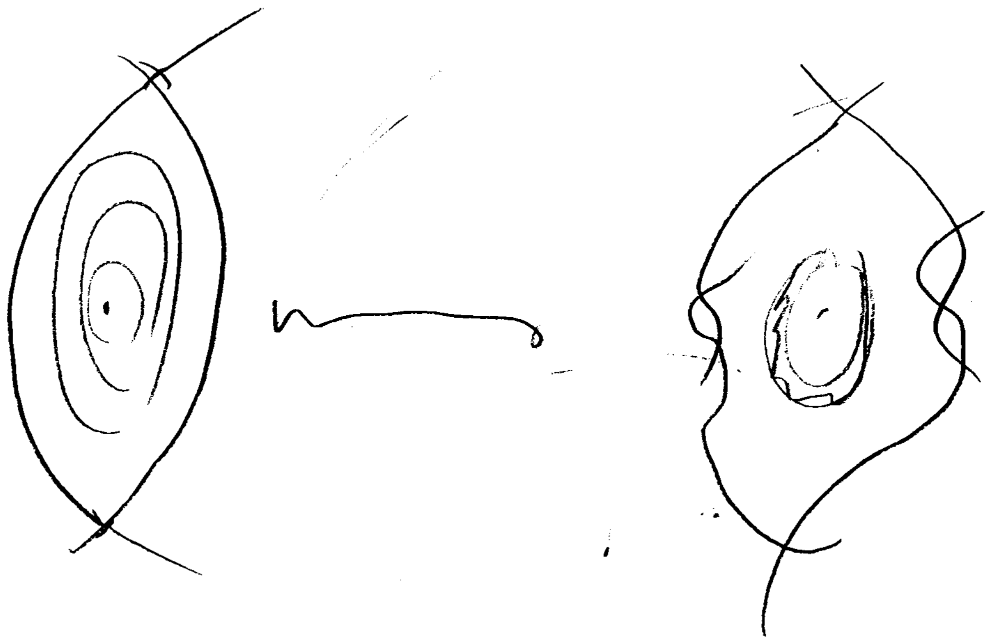
$$\left(X_{\log H} = \frac{1}{H} X_H . \right)$$

The first return map is a small perturbation

of the Hamiltonian system $X_{H_{01}}$ on $\mathbb{P}^1 \setminus \#$



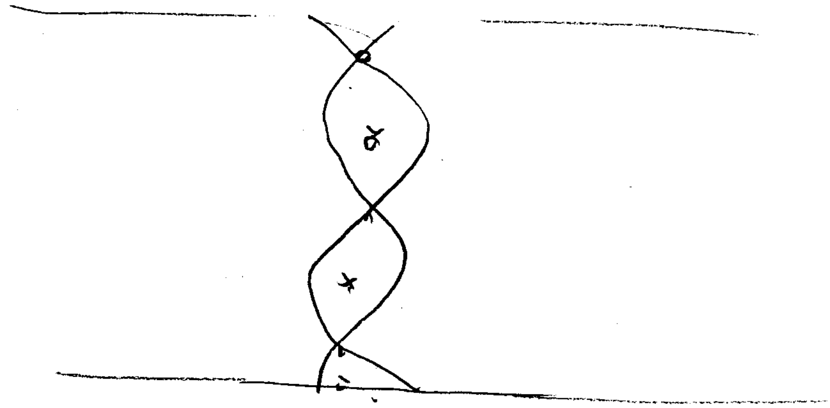
H is expected to be KAM system



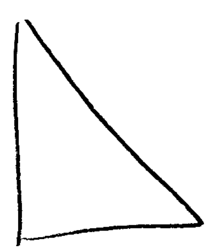
<u>Problem</u> Show X_H is not completely integrable (Certain invariant tori actually destroyed.)

There are 6 fixed points of $\varphi_\varepsilon : M_\varepsilon \to M_\varepsilon$

on our domain



3 edges



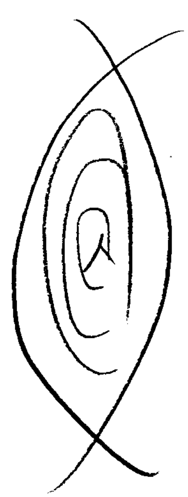
of moment polytope.

$\Rightarrow$ $3 \times 6 = 18$ fixed points.

Remarks

$z^3 + 1 = 0$ base locus

we blow up Kähler form the



However this does not change the behavior of X_H there

$\exists$ still elliptic fixed point there

We thus proved Thm 2.

We next study the case of a bit higher dimension.

The case of quartic surface

$\mathbb{P}^3$

$$[z_0 : z_1 : z_2 : z_3] \xrightarrow{\mu} \mathbb{R}^3$$

$\mu(\mathbb{P}^3) =$ 3 dimensional simplex

$\{ (x_1, x_2, x_3)$

$\mid 0 \leq x_i, \ x_1 + x_2 + x_3 \leq 1 \}$

$f : \mathbb{P}^3 \longrightarrow \mathbb{C}$ meromorphic

$$f([z_0 : z_1 : z_2 : z_3]) = \frac{z_0 z_1 z_2 z_3}{z_0^4 + z_1^4 + z_2^4 + z_3^4}$$

need to blow up

Baer locus

$$z_0 z_1 z_2 z_3 = 0 = z_0^4 + z_1^4 + z_2^4 + z_3^4$$

B_0 : $z_0 = 0$ $\quad z_1^4 + z_2^4 + z_3^4 = 0$

(curve in $\mathbb{P}^2 = \{z_0 = 0\}$)

B_1 $\quad z_1 = 0$ $\quad z_0^4 + z_2^4 + z_3^4 = 0$

B_2 $\quad z_2 = 0$ $\quad z_0^4 + z_1^4 + z_3^4 = 0$

B_3 $\quad z_3 = 0$ $\quad z_0^4 + z_1^4 + z_2^4 = 0$

B_i are non singular.

$B_i \cap B_j$ 4 points $i \neq j$ ($B_i \pitchfork B_j$)

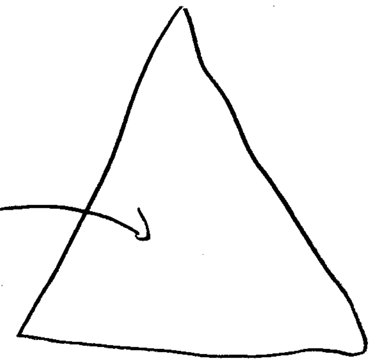
ex

$B_1 \cap B_2$ $\quad \underbrace{z_0 = z_1 = 0}_{\Downarrow \; \mathbb{P}^1 \subset \mathbb{P}^3}$ $\quad z_2^4 + z_3^4 = 0$

$$\frac{z_3}{z_2} \in \{ \pm x, \pm i x \}$$

$$x = e^{2\pi i/8}$$

$Z_0 = 0$ $\quad \mathbb{P}^2$



corresponds to this face of $\mu(\mathbb{P}^3)$

$$\bigcup_{i \neq j} B_i \cap B_j = 4 \times 12 = 48 \text{ points}$$

We blow these 48 points up

$\check{f} : \check{\mathbb{P}}^3 \longrightarrow \mathbb{C}$ rational function

Basic locus of $\check{f} = \check{B}_i$ $\quad \pi : \check{\mathbb{P}}^3 \to \mathbb{P}^3$

$$\left(\check{B}_i = \overline{\pi^{-1}\Big(B_i \setminus \bigcup_{j \neq i} B_i\Big)} \right.$$

$\check{B}_i \cap \check{B}_j = \emptyset \quad i \neq j$

$\check{B}_i$ are smooth divisors

We blow up $\check{B}_0, \check{B}_1, \check{B}_2, \check{B}_3$ on $\mathbb{P}^3$

$$\check{f} : \check{\mathbb{P}}^3 \longrightarrow \mathbb{C} \cup \{\infty\}$$

now is well defined holomorphic map

Note the fiber $\check{f}^{-1}(\varepsilon)$ $(\varepsilon \neq 0)$ is not

quartic $z_0 z_1 z_2 z_3 = \varepsilon (z_0^4 + z_1^4 + z_2^4 + z_3^4)$ but its

blow up at 16 pts $(\cup B_i \cap B_j)$

We blow down $\check{\check{\check{\mathbb{P}}}}^3$ as follows.

$\mathbb{P}^2_{ij} \subset \check{\mathbb{P}}^3$ 16 copies (at 16 pts $P_{ij} = B_i \cap B_j$)

$\check{\mathbb{P}}^2_{ij} \subset \check{\check{\mathbb{P}}}^3$ $\Rightarrow$ 2 pts blow up of $\mathbb{P}^2_{ij}$

$$\| \atop (\hat{B}_i \cap \mathbb{P}^2_{ij} , \hat{B}_j \cap \mathbb{P}^2_{ij})$$

$\check{\mathbb{P}}^2_{ij} \longrightarrow \mathbb{P}^1$ restriction of $\check{f}$

We blow down $\check{\mathbb{P}}^3_{ij}$ to $\mathbb{P}^1$. Get

$$\check{\check{\check{\mathbb{P}}}}^3 \xrightarrow{\check{\check{\check{f}}}} \mathbb{P}^1$$

$\varepsilon \in \mathbb{P}^1 \quad \varepsilon \notin U \Rightarrow \quad \check{\check{\check{f}}}^{-1}(\varepsilon)$ is quartic

From now on we write

$\check{\mathbb{P}}^3$, $\check{F}$ in place of $\check{\check{\check{\mathbb{P}}}}^3$, $\check{\check{\check{F}}}$

$\check{f}: \check{\mathbb{P}}^3 \longrightarrow \mathbb{C} \cup \{\infty\}$ holomorphic map

Take appropriate blow up Kähler metric on $\check{\mathbb{P}}^3$ (obtained from the standard Kähler metric on $\mathbb{P}^3$)

$X = \check{f}(D^2)$ $\qquad \pi = \check{f} : X \longrightarrow D^2$

$H = |\check{f}|$

$$M_\varepsilon = f^{-1}(\varepsilon) \subset X \qquad z_0 z_1 z_2 z_3 = \varepsilon(z_0^4 + z_1^4 + z_2^4 + z_3^4)$$

quartic surface

$(\simeq K3)$

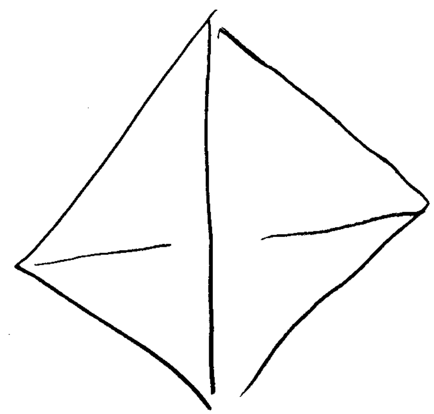
$\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon$ Poincaré map of X_H

<u>Thm</u> φ_ε has 408 fixed points

The proof mostly shows that the conclusion of KAM does not hold for φ_ε

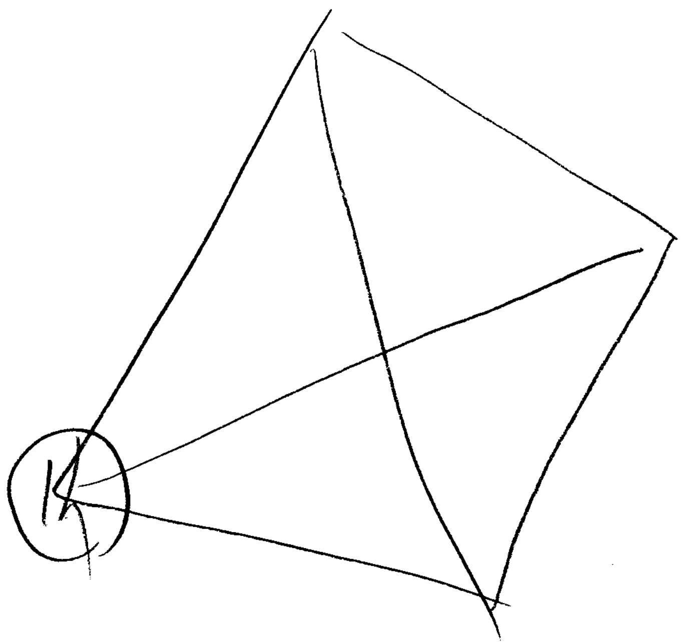
Let us study φ_ε by perturbation theory of Hamiltonian dynamics.

$\mu(X) = P \hookleftarrow$ Simplex



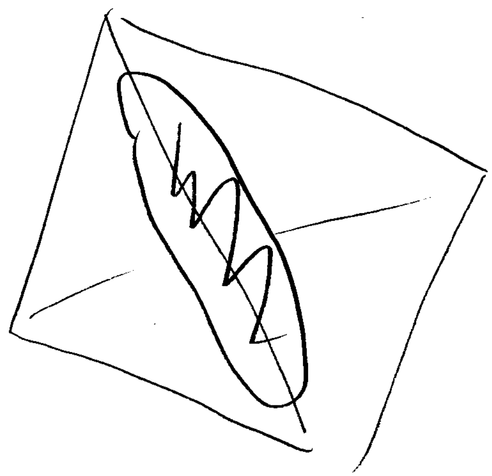
$\mu(M_c) \subset$ in a ε nbd of ∂P

$\partial P =$ vertex $\cup$ edge $\cup$ faces



In a nbd of vertex, say $z_0 = z_1 = z_2 = 0$

We can apply KAM. (Proposition 1, first day)



A neighborhood of edges.

The situation is mostly the same as the case of cubic curve.

But let me repeat it.

Consider the edge corresponding to $\underline{z_0 - z_1 = 0}$

$$f = |z_0 z_1| \frac{|z_2 z_3|}{|z_0^4 + z_1^4 + z_2^4 + z_3^4|}$$

$$f \sim |z_0 z_1| \frac{|z_2 z_3|}{|z_2^4 + z_3^4|}$$

approximate

H_{00}

H_{01}

$H_{00}(z_0, z_1) = |z_0 z_1|$

completely integrable

$\{H_{00}, H_a\} = 0$

$z_0 = z_1 = 0$

$\Leftrightarrow \mathbb{P}^1$

construct $\frac{z_3}{z_2} = z$

To use Darboux chart

we need to take into account FS metric

in the mor

$$H_{uv} = w_1 w_2$$

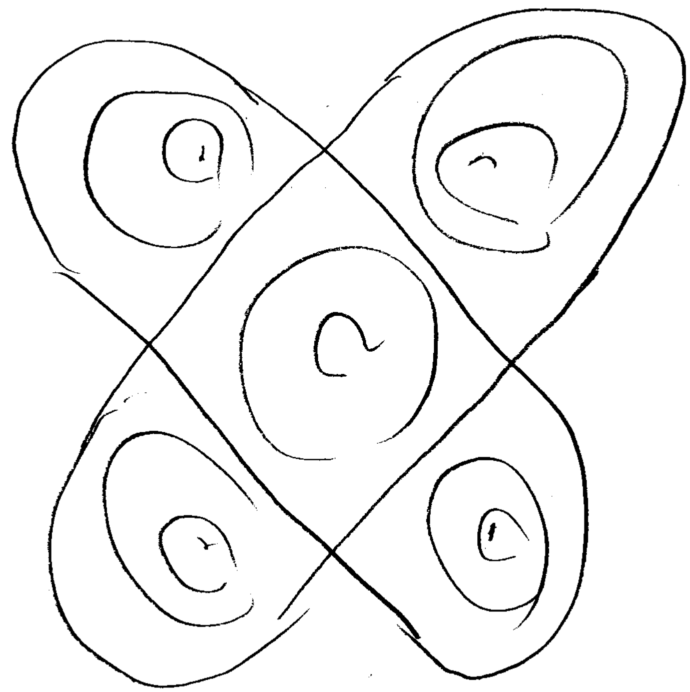
$$w_1 = z_0/z_3$$

$$w_2 = z_1/z_3$$

$$H_{u1} = \frac{|z|(1+|z|^2)}{|1+z^4|}$$

$$\left(H_{u1}\left(\frac{1}{z}\right) = H_{u1}(z) \right)$$

Level set of $H_{0,1}$



$$X_{H_{\omega}} = -\frac{r_2}{r_1}\frac{\partial}{\partial\theta_1} + \frac{r_1}{r_2}\frac{\partial}{\partial\theta_2}$$

$X_{H_{\omega 2}}$ = as the above diagram

$$X_{H_{\omega\times\omega_1}} = c_1 X_{H_{\omega 0}} + \varepsilon X_{H_{\omega 1}}$$

Integrable system

w_1, w_2

fast mode

slow mode

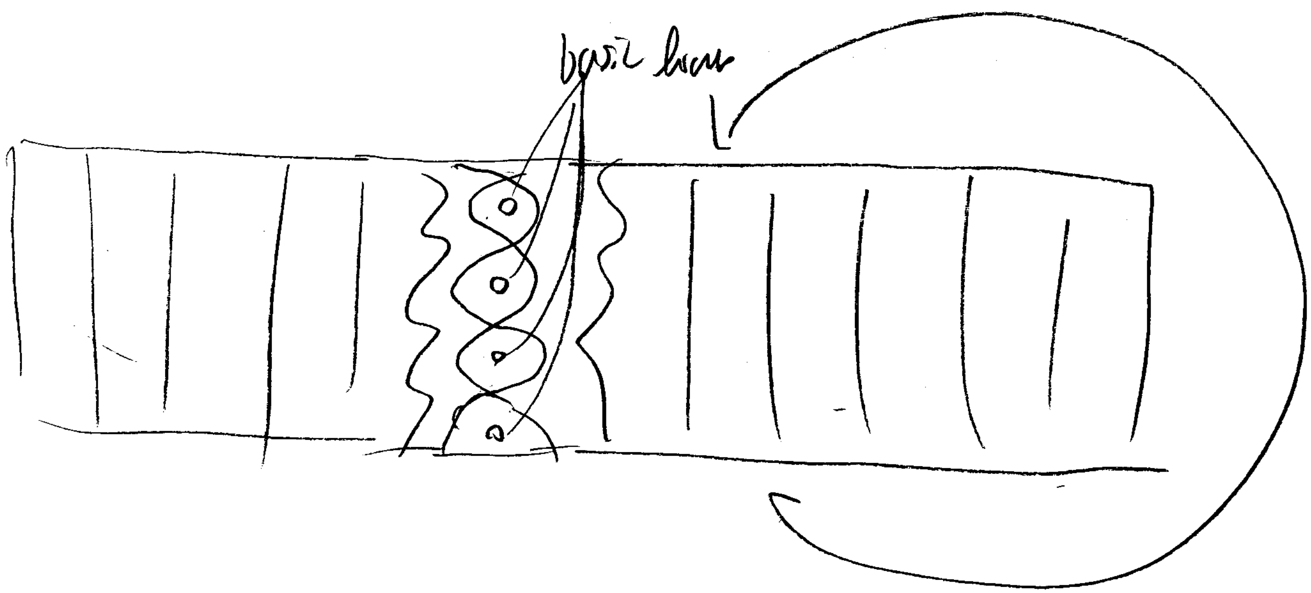
We can still apply Arnold's theorem.
KAM holds in this domain.

—

first move here

Poincaré map of $X_{H_0 \cup X_{01}}$. domain S^1 bundle $\times (-\delta, \delta)$

over $S^2 \setminus \{0,\infty\} \setminus 4$ pts (the basic locus)



$\times S^1$

$\times (-\delta, \delta)$

What happens at the faces?

We consider the face $z_0 = 0$

$$H = |z_0| \frac{|z_1 z_2 z_3|}{|z_0^4 + z_1^4 + z_2^4 + z_3^4|}$$

small

$$= |v| \frac{|z w|}{|1 + z^4 + w^4|}$$

$$v = \frac{z_0}{z_1}$$

$$z = \frac{z_2}{z_1}$$

$$w = \frac{z_3}{z_1}$$

z, w inhomogenous coordinate of

$\mathbb{P}^2$ $\quad [Z_1 : Z_2 : Z_3]$ $\quad Z_0 = 0$

FS metric of normal direction

$$H_0 = H_{00} H_{01}$$

in Darboux chart

$$H_{00} = |v|$$

$$H_{01} = \frac{|zw| \sqrt{1+|z|^2+|w|^2}}{|1+z^4+w^4|}$$

$$\{H_{00}, H_{01}\} = 0$$

$$X_{H_{00}} = \frac{1}{r}\frac{\partial}{\partial \theta} \qquad v = r e^{2\pi i \theta}$$

$$X_{H_{01}} = ?$$

<u>Problem</u> Show that $X_{H_{01}}$ is <u>not</u> completely integrable (on $\mathbb{P}^2 \setminus 3\ \mathbb{P}^1$'s $\setminus B_0 \leftarrow$ Base locus)

Problem is OK $\Longrightarrow$ $\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon$ does <u>**not**</u> satisfy the conclusion of KAM.

I do not know how to solve this problem.

However I think

$$H_{\omega_1} = \frac{|zw|\sqrt{1+|z|^2+|w|^2}}{|1+z^4+w^4|} = h(z,w)$$

Let us study fixed point of

$X_{H_{0,1}}$.

<u>Lemma</u>

h has exactly 16 critical points

$\{(z,w) \mid \{z,w\} \subset \{\pm 1, \pm i\}\}$