Hamiltonian Dynamics of Monodromy of the maximal degenerate family of CY manifolds 4

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$$\check{f} : \check{\mathbb{P}}^3 \longrightarrow \mathbb{C} \cup \{\infty\}$$

$$\check{f} = \frac{z_0 z_1 z_2 z_3}{z_0^4 + z_1^4 + z_2^4 + z_3^4}$$

$$H = |\check{f}|$$

$$\varphi_\varepsilon : \check{f}^{-1}(\varepsilon) = M_\varepsilon \longrightarrow M_\varepsilon$$

Poincaré map

$\check{D}_0$: nbd of $z_0 = 0 \subset \mathbb{P}^2$

$v = z_0/z_1, \quad z = z_2/z_1, \quad w = z_3/z_1$

$H = H_{00} H_{01}$ + small perturbation

$H_{00} = |v|$

$$H_{01} = \frac{|zw| \sqrt{1+|z|^2+|w|^2}}{|1+z^4+w^4|} = h(z,w)$$

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$\mathbb{P}^2 \setminus \begin{matrix} z=0 \\ w=0 \\ \infty\ \mathbb{P}^1 \end{matrix} \setminus$ basic locus

$(1+z^4+w^4=0)$

φ_ε is approximated by the

integration of

$H_{01} = h(z,w)$

Lemma

(z,w) is a critical point of h

$\Leftrightarrow$ $\{z,w\} \in \{\pm 1, \pm i\}$

Sublemma (z,w) is a critical point of h

$\Rightarrow$ $z^4, w^4 \in \mathbb{R}$

It suffices to show

$$\frac{\partial g}{\partial \rho} = 0 \quad \frac{\partial g}{\partial \sigma} = 0 \implies z', w' \in \mathbb{R}$$

at $(\rho, \sigma) = 0$

$\implies$

$$\frac{\partial g}{\partial \rho}(0,0) = 0 \implies \frac{z'}{1+w} \in \mathbb{R}$$

$$\frac{\partial g}{\partial \sigma}(0,0) = 0 \implies \frac{w'}{1+z'} \in \mathbb{R}$$

$$z \mapsto z e^{i\theta} \qquad w \longmapsto z e^{i\sigma}$$

$|zw|\sqrt{1+|z|^2+|w|^2}$ does not change

$|1+z^4+w^4|$ change

$$z^4 = z' \qquad w^4 = w'$$

$$|1+e^{i\theta}z'+e^{i\sigma}w'| = g(\theta,\sigma)$$

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$$\frac{z'}{1+w'}, \ \frac{w'}{1+z'} \in \mathbb{R} \implies z', w' \in \mathbb{R}$$

Easy exercise //

$$h(z,w) = \frac{|zw|\sqrt{1+|z|^2+|w|^2}}{|1+z^4+w^4|}$$

$z \mapsto \pm z, \pm iz$
$w \mapsto \pm w, \pm iw$
} symmetry of h

Thus by lemma it suffices to study the case

$$x^4, y^4 > 0 \qquad (1)$$

$$x^4 > 0 > y^4 \qquad (2)$$

$$0 > x^4, y^4 \qquad (3)$$

(1) Lemma 2

$(x, y) \in \mathbb{R}_+^2$ is a critical pt of $\Rightarrow x = y = 1$

$$\frac{xy\sqrt{1+x^2+y^2}}{1+x^4+y^4} = h(x\, y) \qquad (4)$$

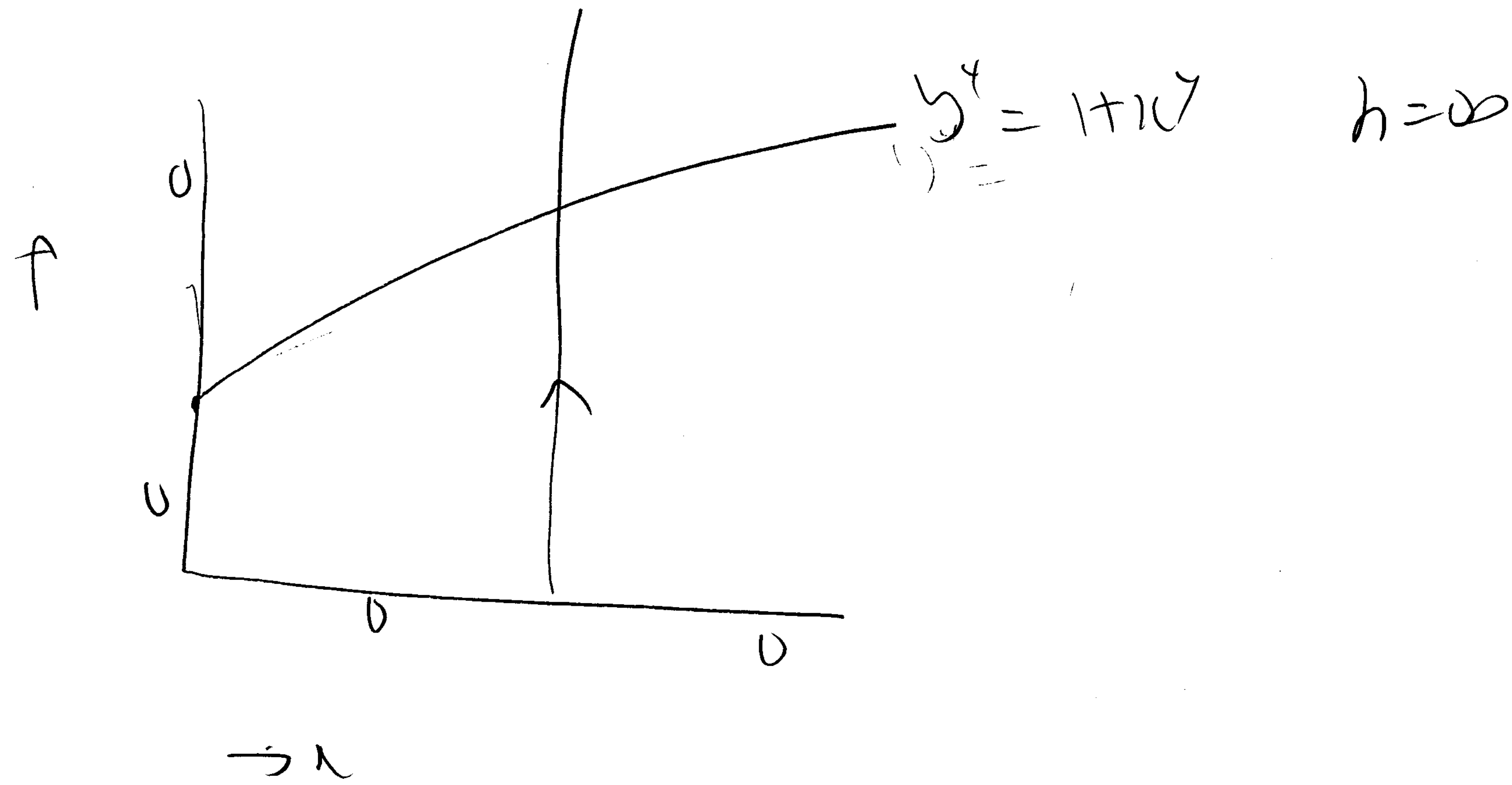
ii) High school Math

(· calculat $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial b}$)

Lemma 3

$$\frac{xy\sqrt{1+x^2+y^2}}{1+x^4-y^4}$$ has no critical

point for $(x,y) \in \mathbb{R}_+^2$.



$\frac{\partial h}{\partial y} > 0$ for $y^4 < 1 + x^4$

< 0 for $y^4 > 1 + x^4$

Direct calculation

$$\frac{xy\sqrt{1+x^2+y^2}}{1+x^4-y^4}$$

$$\Rightarrow \frac{\frac{y}{x}\frac{1}{x}\sqrt{1+\frac{y^2}{x^2}+\frac{1}{x^2}}}{1+\frac{y^4}{x^4}-\frac{1}{x^4}}$$

$x \mapsto \frac{y}{x}$ $y \mapsto \frac{1}{x}$

$$\parallel$$

$$-\frac{x\sqrt{1+x^2+y^2}}{1-x^4-y^4}$$

$\therefore$ <u>Can</u> $\dfrac{xy\sqrt{1+x^2+y^2}}{1-x^4-y^4}$ has no critical point for $(x,y) \in \mathbb{R}^2$?

Q!

Thus we proved

that the critical point of

$$\frac{|zw|\sqrt{1+|z|^2+|w|^2}}{|1+z^4+w^4|}$$ are 16 points

$\{z, w\} \in \{\pm 1, \pm i\}$.

There are 4 faces $z_0=0$, $z_1=0$ $z_2=0$, $z_3=0$

So there are $4\times 16=64$ fixed pts.

near the face

There are $6\times 8=48$ fixed pts

near the edge.

(No near the vertex)

It remains to study φ_2 near the basic locus.

$$B_0 \longleftrightarrow \begin{array}{l} z_0 = 0 \\ z_1^4 + z_2^4 + z_3^4 = 0 \end{array}$$

We take a short cut by using symmetry

$$R = \frac{|zw|\sqrt{1+|z|^2+|w|^2}}{|1+z^4+w^4|}$$

R is invariant by a group G

$$1 \to (\mathbb{Z}_4)^2 \to G \to S_3 \to 1$$

$z \mapsto \pm z, \pm iz$
$w \mapsto \pm w, \pm iw$

$$\Sigma = \{(z,w) \mid 1 + z^4 + w^4 = 0\}$$

genus 3 curve

$$\| $$

$$\{[z_1 : z_2 : z_3] \mid z_1^4 + z_2^4 + z_3^4 = 0\}$$

$$\Sigma / G = ?$$

Fixed point of G action

$x = e^{2\pi i/8}$

3 kinds

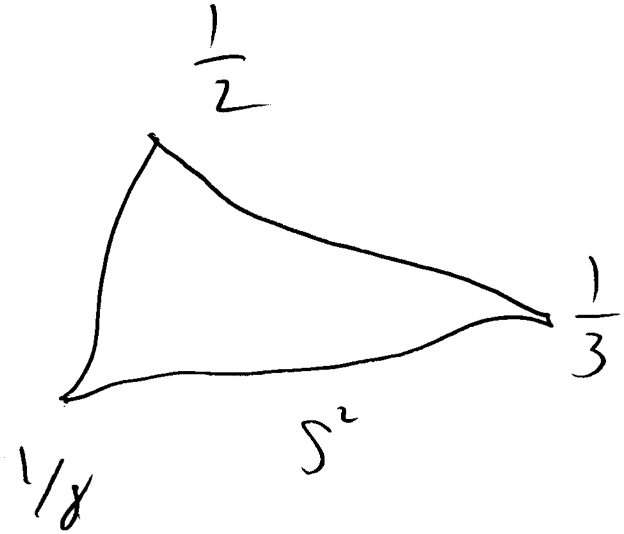
$a \qquad \# I_a = 8 \qquad [1 : 0 : 0]$

$b \qquad \# I_b = 2 \qquad [1, x 2^{-1/4}, x 2^{-1/4}]$

$c \qquad \# I_c = 3 \qquad [1, \zeta, \zeta^2]$

$\zeta = e^{2\pi i/3}$

$\overline{\Sigma}_3 / G \simeq$



check $\quad \chi(S^2 - 3\,pt) = -1$

$\# G = 96$ $\qquad\qquad 2 - 3 \cdot 2 = -4$

$$-96 + \frac{96}{8} + \frac{96}{2} + \frac{96}{3}$$

$$= -96 + 12 + 48 + 32 = -4 = \chi(\overline{\Sigma}_3)$$

ok

$\overline{Z}_3 = \quad 1 + z^4 + w^4 = 0$

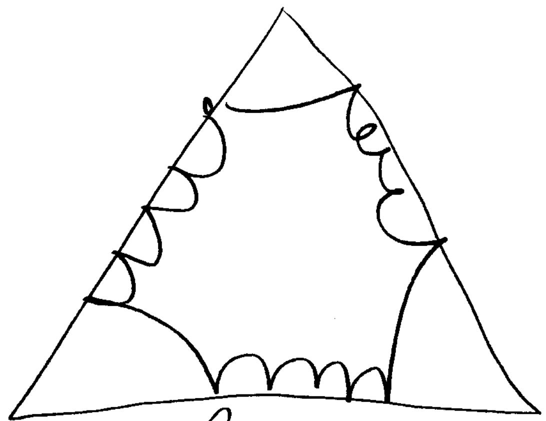
$\cup$

12 pts	a	$\# I_a = 8$
48 pts	b	$\# I_b = 2$
32 pts	c	$\# I_c = 3$

12 pts a corresponds $D_0 \cap D_i$ $\quad i = 1, 2, 3$

$\Uparrow$

4 pts

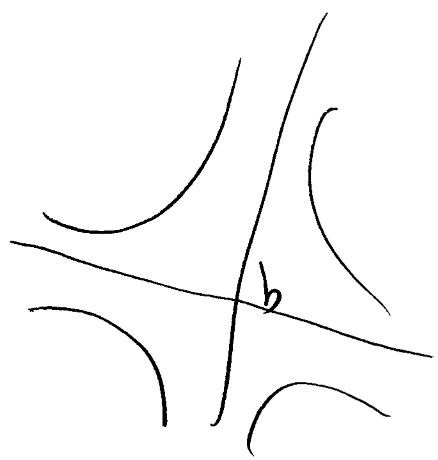


$z_0 = z_1 = 0$

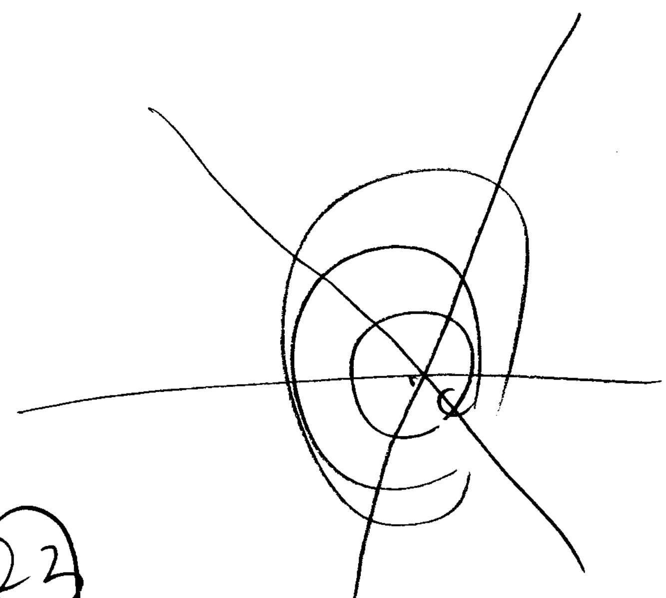
$z_2^4 + z_3^4 = 0$

4 pts

b $\qquad H\,I_b = 2$



H is symmetri



c $\qquad H\,I_c = 3$

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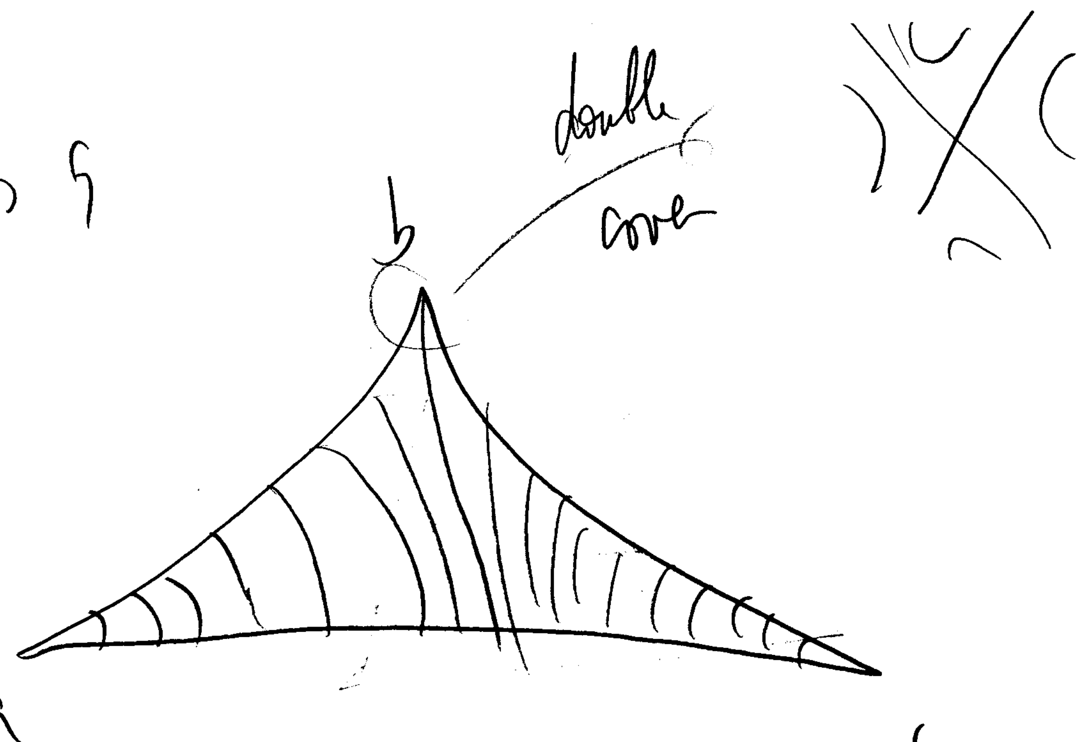
fixed pts $\Longleftrightarrow$ fixed pts of G
of X_{1-1}

often chords G



$h \iff$ hyperbolic $\qquad 96/2 = 48$

$c \iff$ elliptic $\qquad 96/3 = 32$

$16 \iff$ cnt pts of h

$16 + 48 + 32 = 96$ fixed points near each face

4 fixed points near each edge

$96 \times 4 + 4 \times 6 = 408$

<u>Thm</u> $\varphi_2 : M_2 \to M_2$ has 408 fixed points

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A bit more explanation on the dynamics near the basic locus.

$\mathbb{P}^3 \rightsquigarrow \check{\mathbb{P}}^3$ blow up 48 pts

$\cup$ $\check{B}_0$ $\check{B}_1$ $\check{B}_2$ $\check{B}_3$ disjoint basic locus

$\check{B}_1 \cong \Sigma_3$

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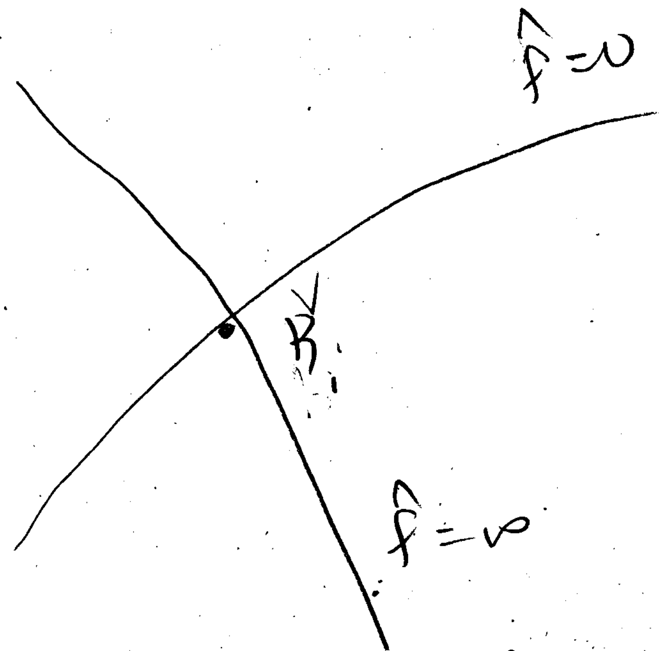
$$\check{B}_1 = \{z_0 z_1 z_2 z_3 = 0\} \cap \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

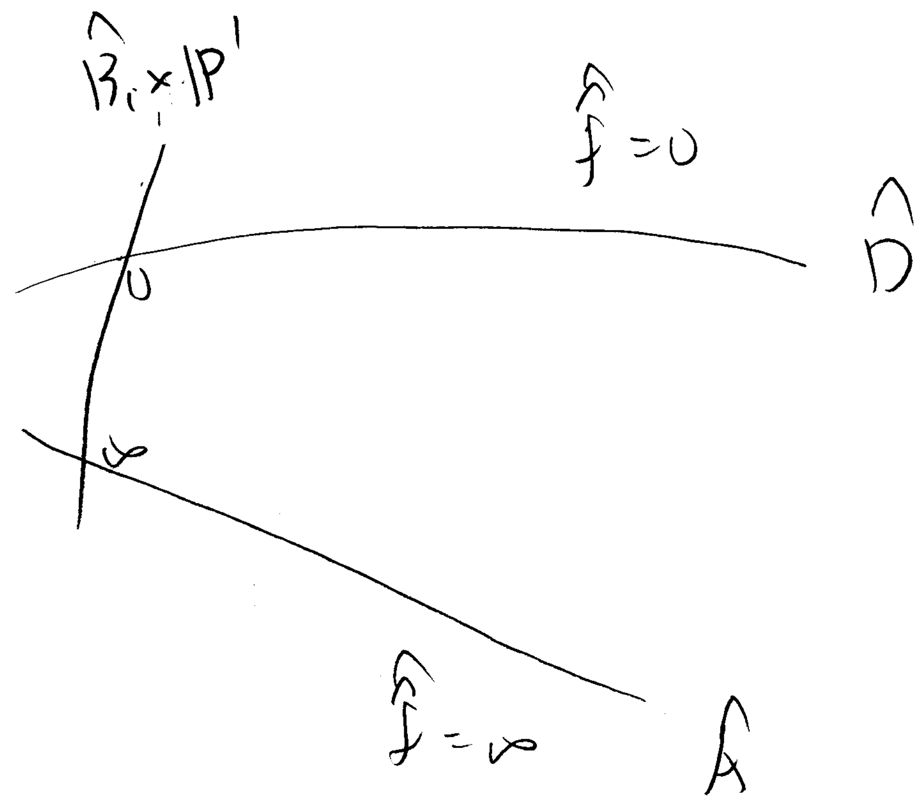


D $\quad z_0 z_1 z_2 z_3 = 0$

A $\quad z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$

$$\check{f} = \frac{z_0 z_1 z_2 z_3}{z_0^4 + z_1^4 + z_2^4 + z_3^4}$$

Blow up $\hat{B}_i$



$$\hat{f} : \hat{B}_i \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

projection to 2nd
factor

$$\hat{f}^{-1}(\varepsilon) \cap (\hat{B}_i \times \mathbb{P}^?) \cong \hat{B}_i$$

Σ_3

X_H is tangent to it

Poincaré map on X_H on $\hat{B}_i$ is

$\varphi_{\varepsilon,i} : \hat{B}_i \longrightarrow \hat{B}_i$ some slow dynamics

(I do not know the precise form.)

X_h : h is G inv.

This is enough to show

$\exists$ 12 type a fixed points

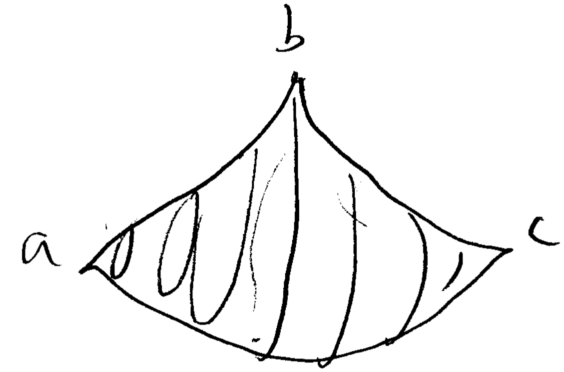
48 type b fixed points

32 type c fixed points on it.

It may be small but unlikely.

The dynamics on $\overline{\Sigma}_3$ is likely

96 fold cover of this picture.



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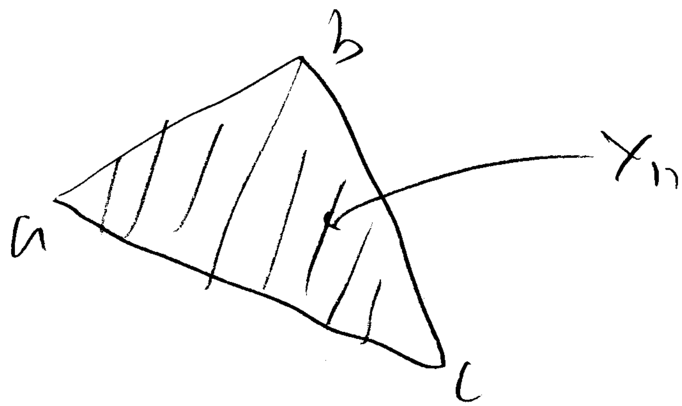
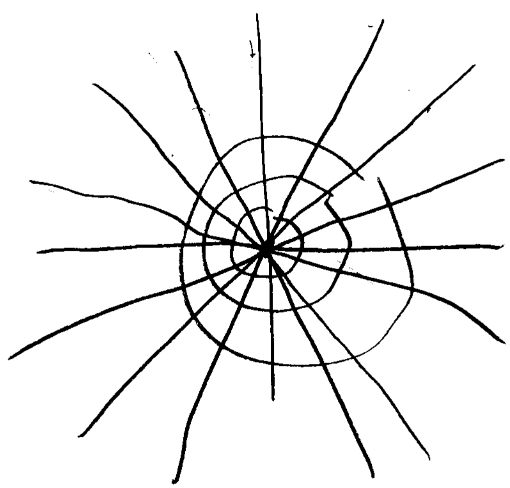
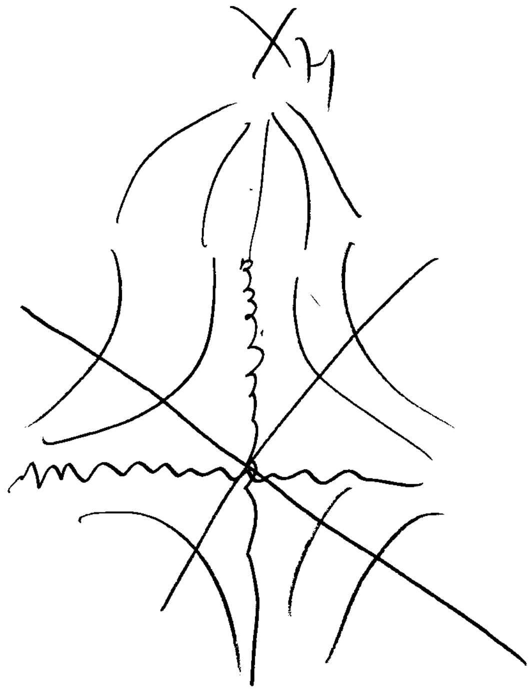
i.e. $\overline{\Sigma}_3$ is triangulated by 192 triangles



around a 16 triangles

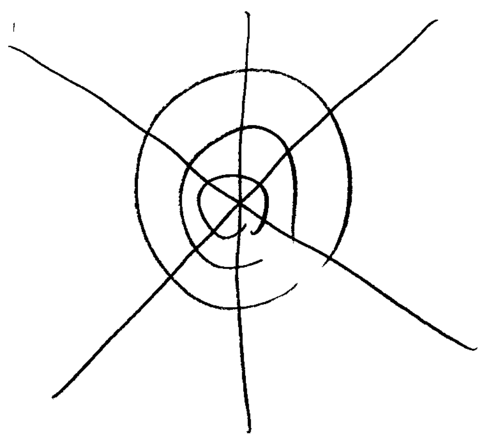


arond b



4 triangles

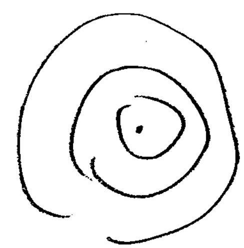
arond c



6 triangles

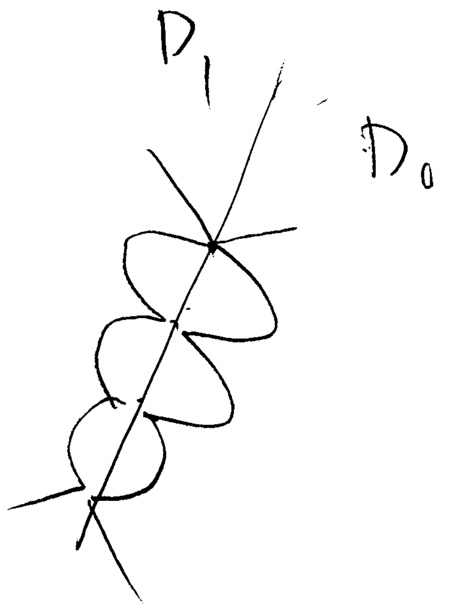
Neighborhood of $\bar{\Sigma}_3$ Disk bundle over $\bar{\Sigma}_3$

the dynamics of fiber direction



$\bar{\Sigma}_3$ direction X_h

Note at fixed points of type a
two of $\bar{Z}_3$ intersect transversally



$\bar{Z}_3^0$ nbd of D_0

$\bar{Z}_3^1$ nbd of D_1

$\bar{Z}_3^0 \cap \bar{Z}_3^1$ 4 pts of type a.

KAM holds in a nbd of those $\bar{\Sigma}_3$'s.

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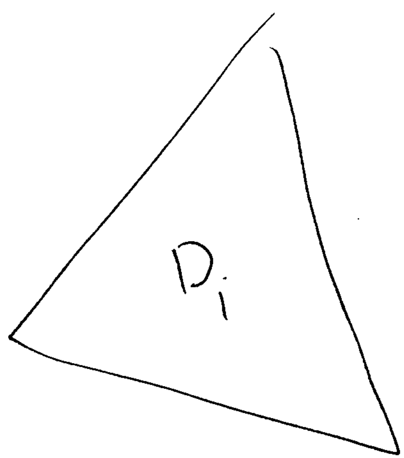
nbds of faces near $\partial D_i \cup B_i$

KAM holds.



But the dynamics is likely becomes Caotic inside.

Possible generalizations and open questions.

① Quintic 3-fold ?

$$\mathbb{P}^4 \longrightarrow \mathbb{C} \qquad \frac{z_0 z_1 z_2 z_3 z_4}{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5} = f$$

$\check{f} : \check{\mathbb{P}}^4 \longrightarrow \mathbb{C} \setminus \{0\}$ hol. map

Landau–Ginzburg

Mirror

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$$M_\varepsilon = \pi^{-1}(\varepsilon) \qquad z_0 z_1 z_2 z_3 z_4 = \varepsilon(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5)$$

Quintic 3 fold

$$H = |\hat{f}|$$

$\varphi_\varepsilon : M_\varepsilon \circlearrowright$ Poincaré map

<u>Problem</u> Find the number of fixed pts of φ_2 ?

More generally

Let X be a toric manifold

(may odd is ok.)

$D \subset X$ toric divisor $\quad (\mu^{-1}(\partial P) = D$

$P = \mu(X)$

$\mu : X \longrightarrow \mathbb{R}^n$

moment map)

$\mathcal{L} \to X$

cpx line bundle corresponding to D.

Assume $\mathcal{L}$ is effective

$s_0 : X \to \mathcal{L}$ section $s_0^{-1}(0) = D$

$s : X \to \mathcal{L}$ another section

$f : \frac{s_0}{s} : X \longrightarrow \mathbb{C}$ meromorphic

$\hat{f} : \check{X} \longrightarrow \mathbb{C} \cup \{\infty\}$ holo map

appropriate blow up

$M_\varepsilon = \hat{f}^{-1}(\varepsilon)$ CY hypersurface

(This is a typical construction of CY mfld and appears in the study of Mirror symmetry)

$H = |\hat{f}|$

$\varphi_\varepsilon : M_\varepsilon \to M_\varepsilon$ Poincaré map

Problem

Study φ_ε as Hamiltonian Dynamics

i.e calculate the number of fixed pts.

Rem

$$\frac{z_0 z_1 z_2 z_3}{g(z_0, z_1, z_2, z_3)}$$

is an example

g : homogeneous polynomial of order 4

We discussed the case

$$g = z_1^4 + z_1^4 + z_2^4 + z_3^4$$

But there are many other cases

Some part of the argument applies

But the short cut using symmetry

does not work