Hamiltonian Dynamics of monodromy of the maximal degenerate family of CY manifolds 5

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SYZ fibration

Conjecture (Strominger-Yau-Zaslow)

$X \xrightarrow{\pi} D^2$ maximal degenerate family as before

$\varepsilon \in D^2 \setminus 0$ $\quad M_\varepsilon = \pi^{-1}(\varepsilon)$

$\exists$ $pr : M_\varepsilon \longrightarrow B$ s.t.

① $\Gamma \subset B$ codimension 2

s.t. π is a Lag. fibration on $B \setminus \Gamma$

Remarks SYZ conj that π is a special Lag. fibration outside $B \setminus \Gamma$ (Namely $T_b = p_r^{-1}(b)$ is special in particular minimal)

Nowadays the conjecture of this form is not believed. It is expected

$\exists\ U_\varepsilon(\Gamma) \supset \Gamma$ st

① $p_r^{-1}(b)$ is a special Lagrangian for $b \in U_\varepsilon(\Gamma)$

③ $\bigcap_{\varepsilon \to 0} U_\varepsilon(\Gamma) = \Gamma$.

There are various proposed construction of SYZ fibration. Probably the most successful one is by W. D. Ruan.

Reference: 0104010 2001

Lag. Torus fibration and Mirror Symmetry of CY mfds

: 990412, 0411264, —

Lagrangian torus fibrations of quintic Calabi-Yau hypersurfaces I, II, III

etc

(X, ω) toric manifold
$\circlearrowleft$
T^{n+1}

$\mu : X \longrightarrow \mathbb{R}^{n+1}$ moment map

$P = \mu(X)$

$D = \mu^{-1}(\partial P)$ toric divisor

$\mathcal{L} \to X$ The line bdl corresponding to D.

Ass. s_0 section (holomorphic) of $\mathcal{L}$

$s_0^{-1}(0) = D$

s another section of $\mathcal{L}$.

$f : \frac{s_0}{s} : X \longrightarrow \mathbb{C}$

$\Downarrow$

$\check{f} \qquad \check{X} \longrightarrow \mathbb{P}^1$ holomorphic

blow up of X.

Replace X by $f^{-1}(D^2)$ $\qquad D^2$ nbd of $0 \in \mathbb{C}$

$\pi : X \longrightarrow D^2 \qquad \varepsilon \in D^2$

$M_\varepsilon = \pi^{-1}(\varepsilon) \qquad$ CY mfld

Take

$$H = |\pi|$$

$$X \to \mathbb{R}$$

before we considered Hamilton vector field of H.

W.D. Ruan studies gradient vector field

$-\operatorname{grad} H$.

- Integration of grad H 'defines'

$$p \in M_\varepsilon \qquad l_p(t) : [0,\infty) \longrightarrow X \qquad \begin{cases} \frac{dl_p}{dt} = -\operatorname{grad} H \\ l_p(0) = p \end{cases}$$

$$\left[\begin{array}{l} t \to \infty \\ l_p(t) \rightsquigarrow D \end{array} \right.$$

$$\lim_{t\to\infty} \mu(l_p(t)) = \partial P \cong S^n$$

Theorem (a bit imprecisely stated) (W.D. Ruan)

$p \rightsquigarrow \lim_{t\to\infty} \mu(l_p(t)) \cong S^n$ defines a SYZ fibration

$M_\varepsilon \to S^n$ i.e. It is Lag fibration

outside $\Gamma \subset S^n$

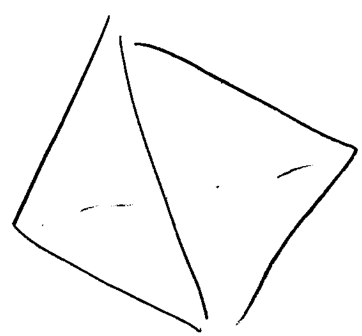
Example

Quartic surface

$X = \mathbb{P}^3 \qquad f = \dfrac{z_0 z_1 z_2 z_3}{z_0^4 + z_1^4 + z_2^4 + z_3^4}$

$M_\varepsilon = f^{-1}(\varepsilon) \qquad z_0 z_1 z_2 z_3 = \varepsilon (z_0^4 + z_1^4 + z_2^4 + z_3^4)$

$\mu(X) = P$

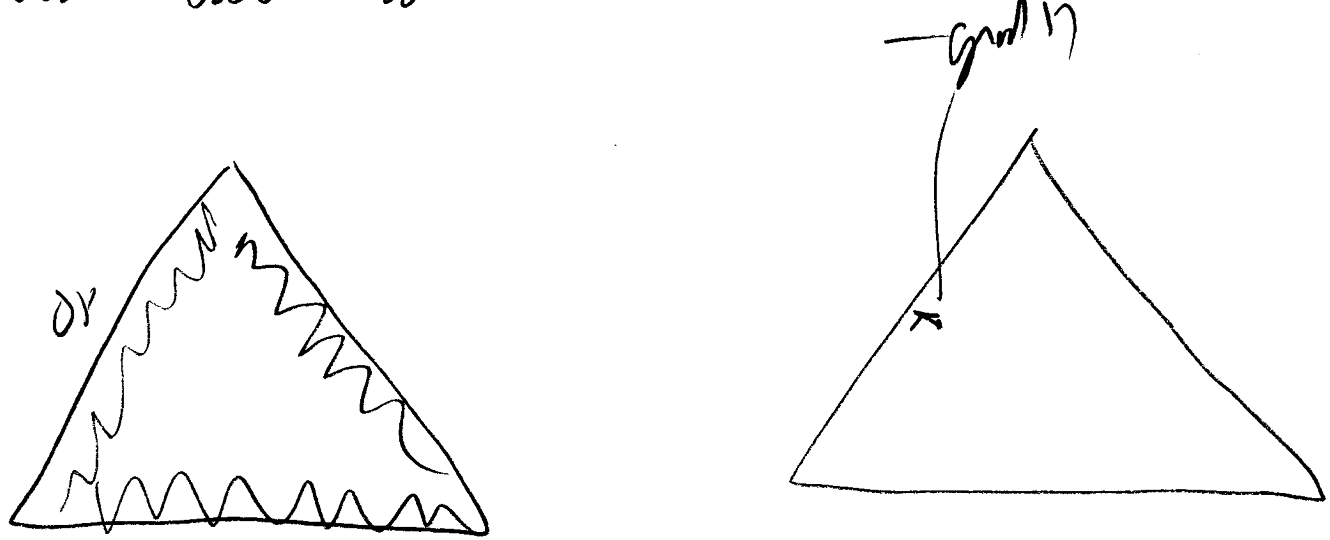


D : $z_0 z_1 z_2 z_3 = 0$

$\mu(D) = \partial P$

$-\operatorname{grad}|f|$: "direction" to boundary ∂P

Picture for cubic curve



P

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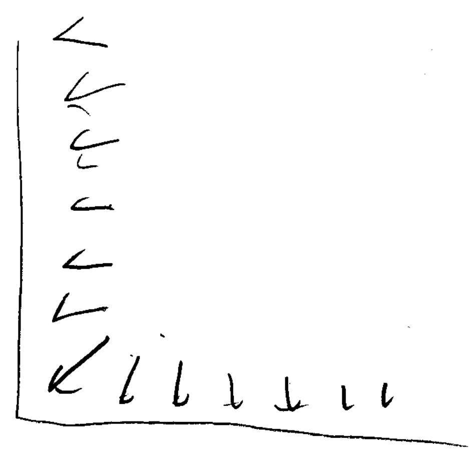
How the fiber looks like?

nbd of vertex

$f = z_1 z_2 z_3 \quad \mp \quad |f| \doteq r_1 r_2 r_3$

$-\operatorname{grad}|f| \doteq -r_2 r_3 \frac{\partial}{\partial r_1} - r_1 r_3 \frac{\partial}{\partial r_2} - r_1 r_2 \frac{\partial}{\partial r_3}$

(picture: again in curve case)



fiber of $\bar{\pi} : M_\Sigma \longrightarrow S^2$ is Lag tori

given by $r_1; r_2, r_3 =$ constant (approximately)

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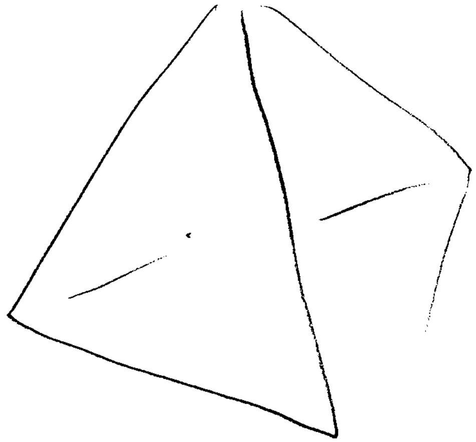
nbd of faces

for example $z_0 = 0$



$$|f| \doteq \underset{r}{|v|} \frac{|zw|\sqrt{1+|z|^2+|w|^2}}{|1+z^4+w^4|}$$

this direction is dominant

this part is mostly constant

$\operatorname{grad} f \sim -\frac{\partial}{\partial r}$

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Fiber is app. T^2 given by

$|z| = c_1, \ |w| = c_2$

Interesting thing happens near the edge.

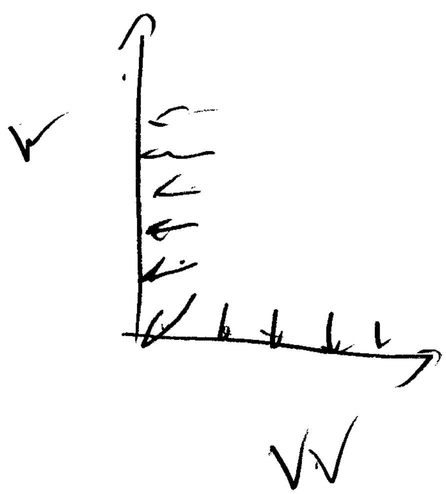
$$|f| = \underbrace{|v||w|}_{\text{this part is dominant}} \frac{|z|(1+|z|^2)}{\underbrace{|1+z^4|}_{h(z)}}$$

When $h \neq \infty$

— grad H is most determined by

$|V||W|\, h$ → the part

the other is



The fiber in V, W direction is S^1

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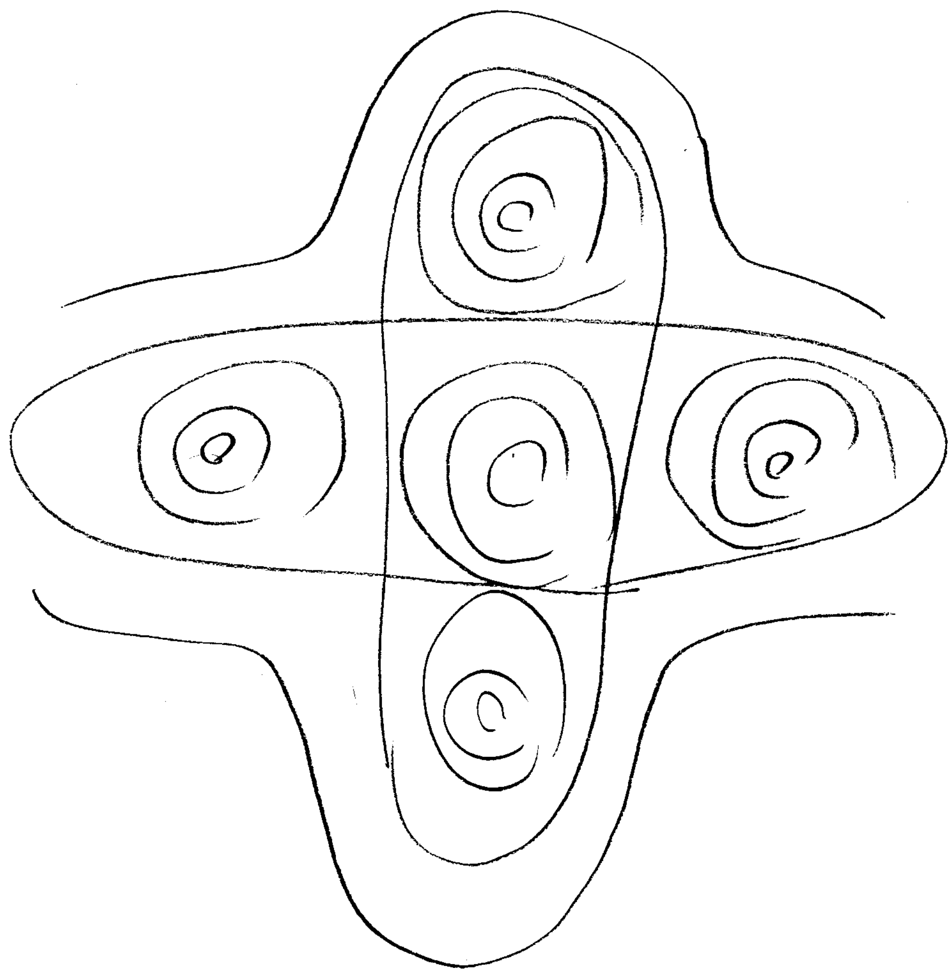
level set of h

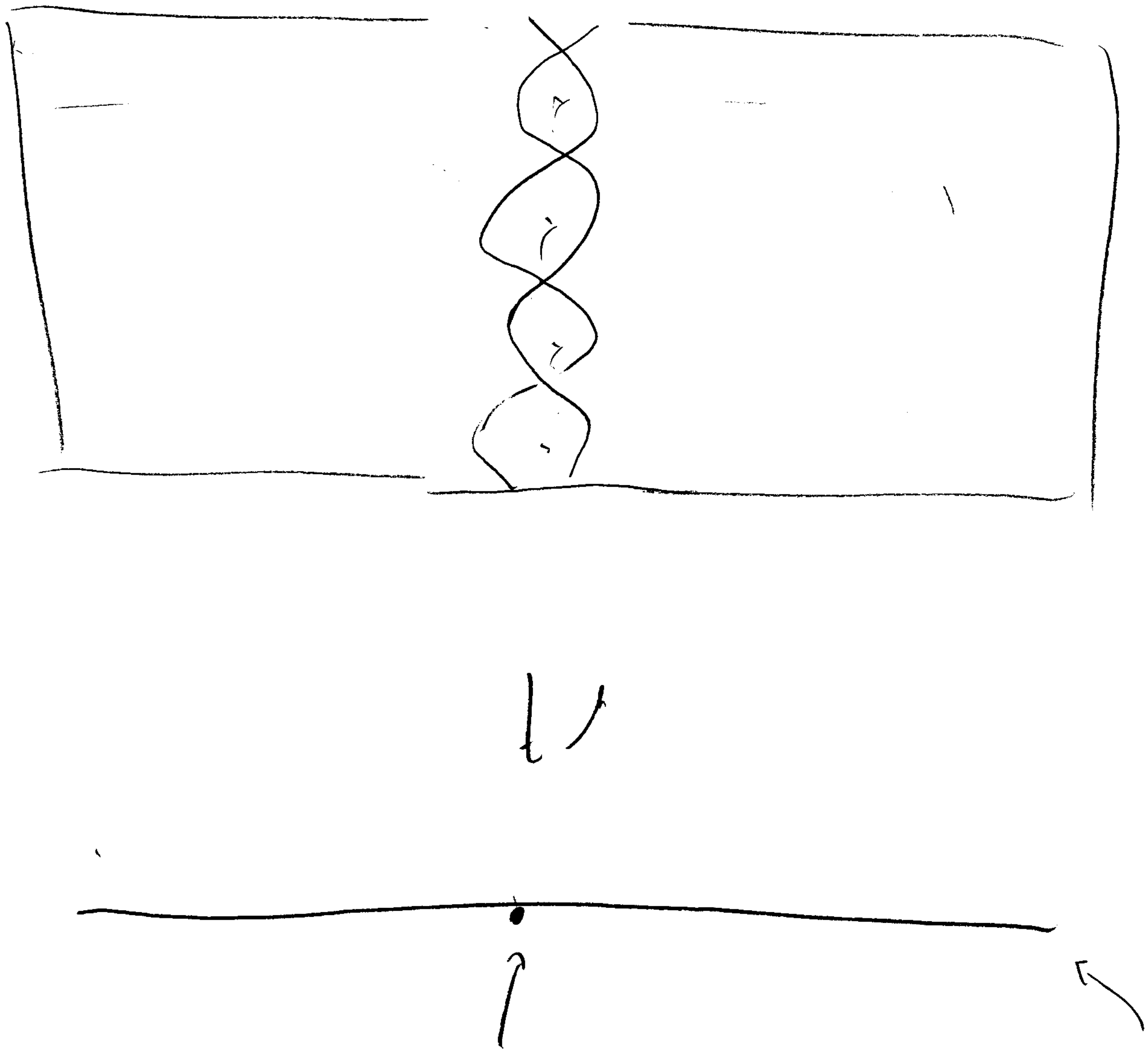


The fiber in z direction is also S^1

$$z \longmapsto \frac{\mu(z)}{|z|} \qquad S^1 \ni \operatorname{Arg} z.$$

So the fiber is again T^2 as expected

What happens near $h = \infty$



S^2

$[z,w]$

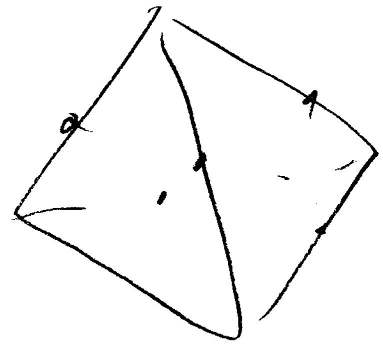
$\downarrow$

$\dfrac{|z|^2}{|z|^2+|w|^2}$

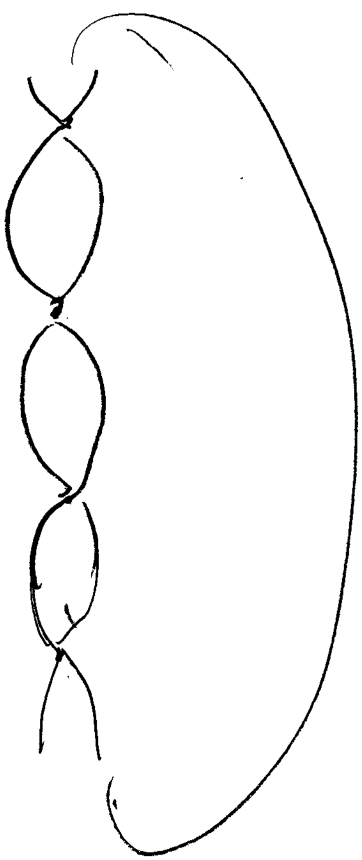
edge

4 twist lines or all above.

The fiber of this pt



are



Type I_4

singular fiber

Thus $M_2 \longrightarrow S^2$

is Lag. torus fibration with 4

singular fiber of A_4 Type.

<u>Rem</u> $f \rightsquigarrow \dfrac{z_0 z_1 z_2 z_3}{S(z_0, z_1, z_2, z_3)} \longleftarrow$ deg 4

non polynomial

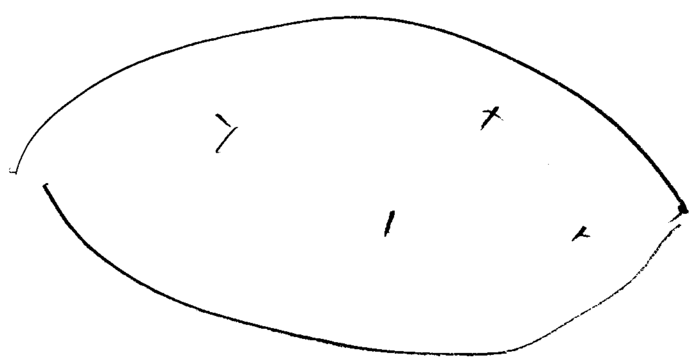
Do the same story

$g(0, 0, z_2, z_3) \rightsquigarrow$ The locus

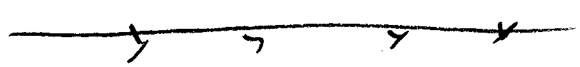
$g(0, 0, z_2, z_3) = 0$

are 4 pts on P^1

x_1, x_2, x_3, x_4



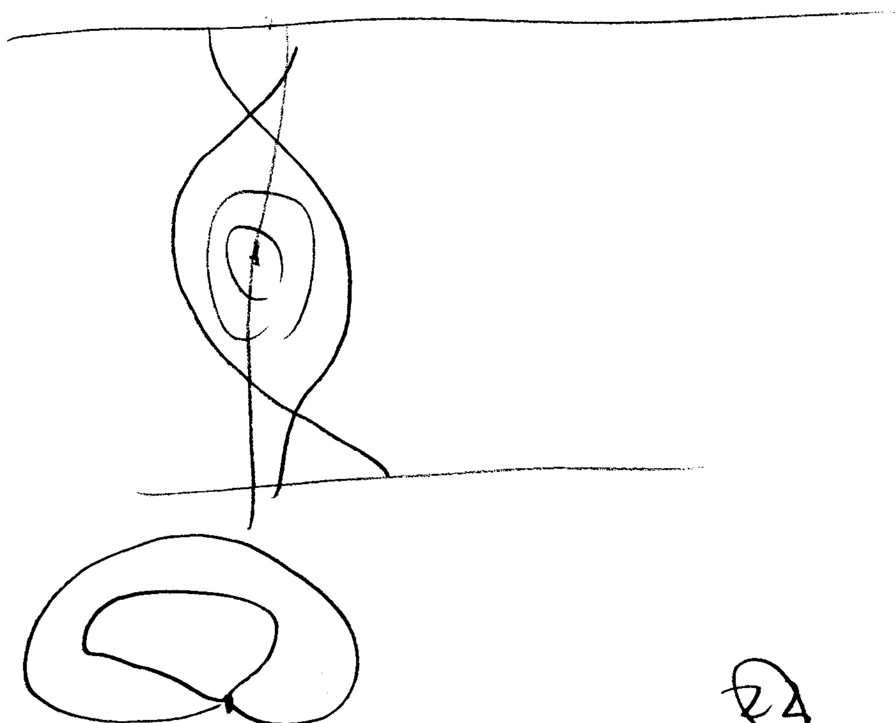
in general $\{x_i\}$ are distinct



So in this case

$$M_\varepsilon \longrightarrow S^2$$

is Lag. torus fibration with 16 A_1 singularity

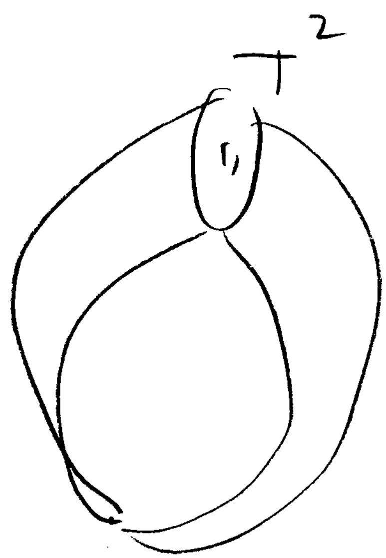


W. D. Ruan studies the case of quintic

$$z_0 z_1 z_2 z_3 z_4 = \varepsilon \sum z_i^5$$

in detail and find SYZ type fibration.

Some singular fiber he found is



type II

$\Longleftarrow$ This is not immersed

<u>Problem</u> Can one work out Lag. Floer theory for this case?

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<u>Problem</u> How picture for quintic has rel to Hamiltonian dynamics of our story?

SYZ fibration, homological Mirror symmetry of
$f_{\varepsilon} : M_{\varepsilon} \longrightarrow M_{\varepsilon}$,

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SYZ conjecture (continued)

$$\begin{array}{c} T^n \hookrightarrow M \\ \downarrow \\ B \end{array}$$

SYZ fibration

$\Gamma \subset B$ singular locus. $B_0 = B \setminus \Gamma$

$B_0 = B \setminus P$

$$\begin{tikzcd} T^n \arrow[r, hook] & M \setminus \pi^{-1}(P) \arrow[d] \\ & B_0 \end{tikzcd}$$

T^n (Lag torus fibration)

$$\begin{tikzcd} T^n \arrow[r, hook] & \check{M}_0 \arrow[d] \\ & B_0 \end{tikzcd}$$

More precisely

$$M_0 = \underbrace{T^* B_0}_{\text{cotangent bundle}} / \text{Lattice}$$

Lattice: $\mathbb{Z}^n$ bundle whose fiber is a lattice of $T_p^* B_0$

$\check{M}_0 = TB_0$ / Dual lattice bundle

<u>SYZ</u> $\check{M}$ (Mirror of M) is a compactification of $\check{M}_0$.

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M maximal degenerate family
⇕
$\check{M}$ maximal degenerate family

と予想されている

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$$\left.\begin{array}{ll} M = M_\varepsilon & \varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon \\ \check{M} = \check{M}_\varepsilon & \check{\varphi}_\varepsilon : \check{M}_\varepsilon \longrightarrow \check{M}_\varepsilon \end{array}\right\} \text{monodromy map}$$

It is <u>not</u> expected that $\check{\varphi}_\varepsilon$ is a mirror to φ_ε.

Then what are the mirrors of φ_ε & $\check{\varphi}_0$

Conjecture

$\varphi_\varepsilon : M_\varepsilon \to M_\varepsilon$ $\xleftrightarrow{\text{Mirror}}$ $\exists\, \mathcal{L}$: certain cpx line bundle

sym. diffeo

$\mathcal{E} \longmapsto \mathcal{L} \otimes \mathcal{E}$.

$\psi \in Gal(\Lambda^{\mathbb{Q}}/\mathbb{C})$ $\xleftrightarrow{\text{Mirror}}$ $\check{\varphi}_\varepsilon : \check{M}_\varepsilon \to \check{M}_\varepsilon$

act on

$Fuk(M_\varepsilon)$

I will explain the statement of this conjecture below.

$$\Lambda^{\mathbb{Q}} = \left\{ \sum c_i T^{\lambda_i} \;\middle|\; \begin{array}{ll} c_i \in \mathbb{C} & \lambda_i < \lambda_{i+1} < \cdots \\ \lambda_i \in \mathbb{Q} & \lim \lambda_i = \infty \end{array} \right\}$$

Novikov field $\supset \mathbb{C} = \{ c T^0 \}$

$$\psi\left(\sum c_i T^{\lambda_i}\right) = \sum_i c_i e^{2\pi\sqrt{-1}\lambda_i} T^{\lambda_i}$$

ψ is an Aut which fix elements of $\mathbb{C}$.

$$\psi \in \mathrm{Gal}(\Lambda/\mathbb{C})$$

Thm (F : Turkish J. Math 2003 Galois Symmetry on

Supp $\exists$ vol ω $\to$ $\Omega = \omega$ vanish Floer coh)

$\exists$ Automorphism of $\mathrm{Fuk}(M_\varepsilon)$ ψ

$\Lambda^{\mathbb{Q}}$ linear category

st

$$\begin{array}{ccc} HF(L, L') & \xrightarrow{\hat{\psi}} & HF(\psi(L), \psi(L')) \\ \circlearrowleft & & \circlearrowleft \\ \Lambda & \xrightarrow{\psi \text{ action}} & \Lambda \end{array}$$

$Fuk(M_\varepsilon)$ A_∞ category where an object is

$\mathbb{L} = (L, b)$ $L \in M_\varepsilon$ Lag. submfd (spin, Maslov $= 0$,

rational

($\cap\omega : \pi_2(M, L) \to \mathbb{R}$'s

image is in $\mathbb{Q}$)

$b \in H^1(L; \Lambda_0^{\mathbb{Q}})$ a bounding cochain

(solution of MC eq.)

$$\Lambda_0^{\mathbb{Q}} = \{ \sum c_i T^{\lambda_i} \mid \lambda_i \geq 0, \lambda_i \in \mathbb{Q}, \lambda_i \to \infty \}$$

Put

$b = b_0 + b_+$ $b_0 \in H^1(L; \mathbb{C})$ $b_+ \equiv 0 \mod T$

$b_0 \in H^1(L, \mathbb{C})$ determine a flat $\mathbb{C}$ bundle on L

$\mathbb{L} \sim ((L, \mathcal{L}^{b_0}), b_+)$ $\qquad b_+ \in H(L; \Lambda_+^{\oplus})$

$\mathcal{L}^{b_0}$

Maurer–Cartan equation for b_+

$\iff \sum m_k^{b_0}(b_+, \cdots, b_+) = 0$

$m_k^{b_0} : H^1(L; \Lambda_0)^{\otimes k} \longrightarrow H^2(L; \Lambda_0)$ A_∞-relation

$\Psi(\mathcal{L}_{b_0}) := \overset{\mathrm{def}}{=} \mathcal{L}_{b_0} \otimes \xi)|_L$

Note $F_\xi = c\omega$

$\Rightarrow \xi$ is flat on L.

Observation

$$\boxed{m_k^{\psi(\alpha_{b_0})} \circ \psi = \psi \circ m_k^{b_0}}$$

This implies

$$\overbrace{(L, \alpha_{b_0}, b_+)}^{\mathbb{L}} \text{ is an object}$$

$$\Longleftrightarrow \overbrace{(L, \alpha_{b_0} \otimes \xi, \psi(b_+))}^{\psi(\mathbb{L})} \text{ is an object}$$

Thus ψ acts on the set of objects

Sketch of the proof of the observation

I explain the equality

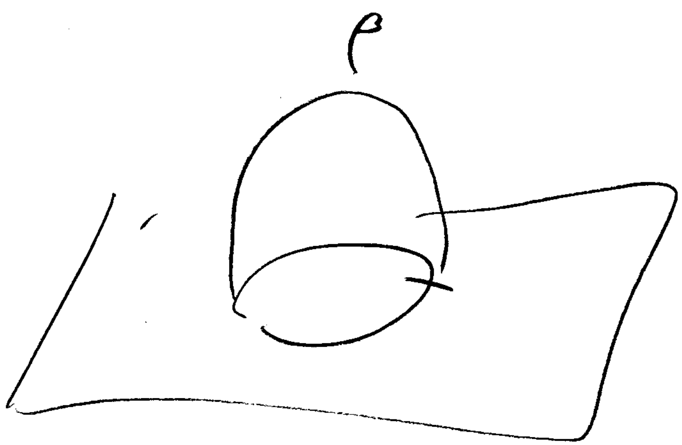
$T^{\lambda} \rightsquigarrow \exp(2\pi i \lambda)\, T^{\lambda}$

Galois action

$$\mathfrak{m}_0^{\psi(\mathcal{L}_{b_0})} = \psi \circ \mathfrak{m}_0^{\mathcal{L}_{b_0}}$$

$$\mathfrak{m}_0^{\mathcal{L}}(1) = \sum_{\beta} T^{\beta \cap \omega}\, \mathrm{hol}_{\mathcal{L}}(\partial\beta)\, \mathcal{M}_2(\beta)$$

$\beta \in \pi_2(X, L)$



holomorphic

$$\mathcal{M}_1(\beta) = \left\{ u : D^2 \to M \;\middle|\; \begin{array}{l} \bar\partial u = 0 \\ [u] = \beta \end{array} \right\} \Big/ \sim$$

$u \sim u' \iff u' = u \circ v \qquad v : D^2 \to D^2$ biholo

$v(1) = 1 \qquad 1 \in \partial D^2$

$\mathcal{M}_1(\beta) \longrightarrow L \qquad u \longmapsto u(1)$

$[\mathcal{M}_1(\beta)] \in C_*(L)$ chain of L

$hol_{\mathcal{L}}(\partial\beta)$ $\quad$ $\mathcal{L}$: flat line bundle on L

$\Uparrow$ $\quad$ $\partial\beta$ loop $(\beta \in \pi_2(M, L))$

holonomy of $\mathcal{L}$ along $\partial\beta$

$\beta \cap \omega \in \mathbb{R}_+$ $\qquad$ a chain on L with Λ coeff.

$$m_0^{\mathcal{L}}(1) = \sum_{\beta} T^{\beta\cap\omega}\, hol_{\mathcal{L}}(\partial\beta)\, [\mathcal{M}_1(\beta)]$$

We compare $m_0^{\mathcal{L}}$ and $m_0^{\mathcal{L}\otimes\xi}$

Recall $\Psi((L,\mathcal{L})) = (L, \mathcal{L}\otimes\xi)$

ξ prequantum bundle (curvature of $\xi = \omega$)

$$\mathrm{hol}_{\mathcal{L}\otimes\xi}(\partial\beta) = \mathrm{hol}_{\mathcal{L}}(\partial\beta)\, \mathrm{hol}_{\xi}(\partial\beta)$$

$$m_0^{\Psi(\mathcal{L})}(1) = \sum_{\beta} \mathrm{hol}_{\xi}(\partial\beta)\; T^{\beta\cap\omega}\, \mathrm{hol}_{\mathcal{L}}(\partial\beta)\,[\mathcal{M}_1(\beta)]$$

Note

$$\psi(m_0^{\mathcal{L}}(1)) = \psi\left(\sum_\beta T^{\beta\cap w}\, hol_{\mathcal{L}}(\partial\beta)\,[M_1(\beta)]\right)$$

$$= \sum_\beta e^{2\pi i\, \beta\cap w}\, T^{\beta\cap w}\, hol_{\mathcal{L}}(\partial\beta)[M_1(\beta)].$$

Since the curvature of $\mathcal{B} = w$

$$hol_{\mathcal{B}}(\partial\beta) = e^{2\pi i\, \beta\cap w}$$

Thus $m_0^{\psi(\mathcal{L})}(1) = \psi(m_0^{\mathcal{L}}(1))$ //

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One can write

Λ^{Φ} vector space $\Leftarrow \psi \in Gal(\Lambda^{\Phi}/\mathbb{Q})$ acts

$$\exists \psi_{\mathbb{L},\mathbb{L}'} \quad HF(\mathbb{L},\mathbb{L}') \xrightarrow[\cong]{\psi_{\mathbb{L},\mathbb{L}'}} HF(\psi(\mathbb{L}), \psi(\mathbb{L}'))$$

$$\psi \circ \psi_{\mathbb{L},\mathbb{L}'} \sim \psi_{\mathbb{L},\mathbb{L}'} \circ \psi$$

Compositions and higher compositions of A_∞ category are also preserved by ψ action

So ψ acts on $Fuk(M_0)$

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In complex geometry

$$\begin{tikzcd} X^\vee \arrow[d] \\ D^2(0) \end{tikzcd}$$

Maximal degenerate family of CY manifolds

its function space is $\mathbb{C}[[T]]$

$\mathrm{Func}(D^2(0))$

define

$$\begin{tikzcd} \check{X}^\vee_k \arrow[r] \arrow[d] & X^\vee \arrow[d] \\ D^2(0) \arrow[r] & D^2(0) \end{tikzcd}$$

$z \mapsto z^k$

Consider coh. sheaf for sm

$\check{X}_k$.

We denote the derived category of coherent sheaves on $\check{X}_k$ by $\mathbb{D}(X_k)$

$$X_k \longrightarrow D^2$$

$\circlearrowleft$ $\qquad$ $\circlearrowleft$

lifted $\leftarrow$ $\mathbb{Z}_k$ act $\quad z \mapsto \exp(2\pi i/k)z$

$\Rightarrow$ $\mathbb{Z}_k$ act on $\mathbb{D}(X_k)$

$X_{kk'} \to X_k$ $\quad$ k' fold cover

$$\begin{array}{ccc} \mathbb{D}(X_p) & \longrightarrow & \mathbb{D}(X_{pp'}) \\ \circlearrowleft & & \circlearrowleft \\ \mathbb{Z}_p \text{ act} & & \mathbb{Z}_{pp'} \text{ act} \\ 1 & \longleftarrow & 1 \end{array}$$

pull back

compatible

$$\lim_{p\to\infty} \mathbb{D}(X_p) = \mathbb{D}(\check{X})$$

on this category $\varprojlim \mathbb{Z}_p = \hat{\mathbb{Z}}$

(profinite completion of $\mathbb{Z}$)

act.

$$\Lambda_0^{\mathbb{Q}} = \left\{ \sum c_i T^{\lambda_i} \,\middle|\, c_i \in \mathbb{C},\ \lambda_i \in \mathbb{Q},\ \lambda_i \geq 0,\ \lambda_i \to \infty \right.$$

Function of $D^2(0) = \mathbb{C}[[T]]$

$$D^2(0) \longrightarrow D^2(0)$$
$$z \longmapsto z^k$$

$\mathbb{C}[[T^{1/k}]]$ $\qquad$ $\mathbb{C}[[T]]$

$\Lambda_0^{\mathbb{Q}} = \underleftarrow{\lim}$ Function on $D^2(0)$ ...

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Therefore Λ_i^{Φ} action on $\mathbb{D}(\check{X})$

$\hat{\mathbb{Z}} \curvearrowright \mathbb{D}(\check{X})$ this is twisted Λ_i^{Φ} linear

namely $1 \in \hat{\mathbb{Z}}$ acts by $T^{\lambda} \longrightarrow \exp(2\pi i \lambda) T^{\lambda}$

$\Downarrow$

then $\psi \curvearrowright \mathbb{D}(\check{X})$ is com with the Λ_i^{Φ} linear structure.

the same as appeared in sym. side

I think that this is a way to describe monodromy of $\check{X} \longrightarrow D^2$ in algebraic geometry. (SGA 7) ??

Remark that $\check{\varphi}_\varepsilon : \check{M}_\varepsilon \longrightarrow \check{M}_\varepsilon$ is <u>not</u> a holomorphic map (H is symplectic.) So it does <u>not</u> induce $\mathbb{D}(\check{M}_\varepsilon) \circlearrowleft$

Second Conjecture

$$\begin{tikzcd} \mathrm{Fuk}(X_\varepsilon) \arrow[r, hook, "\text{Mirror}"] \arrow[d, phantom, "\circlearrowright"] & \lim\limits_{k\to\infty}(\mathbb{D}(\check{X}_k)) \arrow[d, phantom, "\circlearrowright"] \\ \psi \in \mathrm{Gal}(\Lambda^{\oplus}/\mathbb{C}) & \psi \in \mathrm{Gal}(\Lambda^{\oplus}/\mathbb{C}) \end{tikzcd}$$

action $\qquad\qquad\qquad\qquad$ action

Let me explain the first conjecture

$\varphi_\Sigma : M_\Sigma \to M_\Sigma \longleftrightarrow$ Automorphism of

symplectic $\qquad\qquad\qquad \mathbb{D}(M_\Sigma^\vee)$,

differ

$\Downarrow \qquad\qquad\qquad\qquad\qquad \vdots$

Induces

$\varphi_\Sigma^* : \mathrm{Fun}(M_\Sigma) \circlearrowleft \qquad\qquad ?$

Conjectures

$$\varphi_\varepsilon \longmapsto \otimes \mathcal{L}$$

i.e. $\exists$ a holomorphic line bundle $\mathcal{L}$ on $\check{M}_\varepsilon$

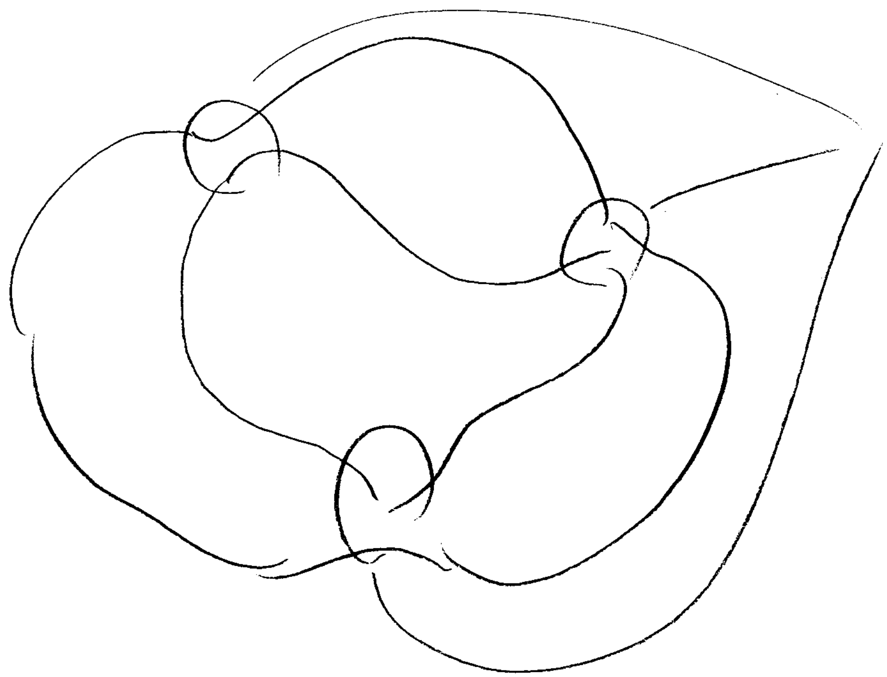
s.t. via HMS

$$\begin{tikzcd}
\mathrm{Fuk}(M_0) \arrow[r, "\wedge"] \arrow[d, "\varphi_\varepsilon"'] & \mathbb{D}(\check{M}_0) \arrow[d] & \varepsilon \arrow[d, mapsto] \\
\mathrm{Fuk}(M_\varepsilon) \arrow[r, "\sim"] & \mathbb{D}(\check{M}_\varepsilon) & \varepsilon \otimes \mathcal{L}
\end{tikzcd}$$

Example

$t_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon$, M_ε is a cubic curve

t_ε is a comp. of 3 Dehn twist



Dehn twist

$$\check{\Psi}_\varepsilon : \check{M}_\varepsilon \longrightarrow \check{M}_\varepsilon \qquad \otimes\, \mathcal{O}(3)$$

$M_\varepsilon \subset \mathbb{P}^{n+1}$ case $\check{\Psi}_\varepsilon : \check{M}_\varepsilon \circlearrowleft$ is $\otimes\, \mathcal{O}(n+1)$.

I do not know how much this is proved

I next want to use this conjecture to guess how Ψ_ε looks like

Note two conjectures seems to be related

Symplectic

$(L, \mathcal{L}) \mapsto (L, \mathcal{L} \otimes \xi)$

$\mathrm{Gal}(\Lambda^{\mathbb{Q}}/\mathbb{C})$ symmetry $\Longleftrightarrow$

similar

$\varphi_\Sigma : M_\Sigma$ ⟲ index

$\mathrm{Fuk}(M_\Sigma)$ ⟲

(PL theory) $\Longleftrightarrow$

Complex

$\varphi_\Sigma : M^\vee_\Sigma$ ⟲

"action" Picard-Lefschetz theory

similar

$\Sigma \mapsto \Sigma \otimes \mathcal{L}$

Namely

$\otimes$ Line bundle

on one side

induced by

$\varphi_\Sigma : M_\Sigma \to M_2$

on the other side.

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