Hamiltonian Dynamics of monodromy of the maximal degenerate family of CY manifolds 6

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$\pi : X \longrightarrow D^2$ maximal degenerate family of CY models

$\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon \qquad M_\varepsilon = \pi^{-1}(0)$

Poincare map of $X_H \qquad H = |\pi|$

A Conjecture I mentioned yesterday

$\varphi_\varepsilon : Fuk(M_0) \circlearrowleft$ is Mirror to

$\otimes \mathcal{L} \qquad D(M_0^\vee) \circlearrowleft$ tensor product functor with a line bdle $\mathcal{L}$

(2)

To use this conjecture to understand φ_ε we study the mirror of the functor $\mathcal{L} \otimes$

I) Mirror to a line bundle $\mathcal{L}$ on $M_\Sigma^\vee$

$\Longleftrightarrow$ A Lagrangian section $\mathcal{S}$ to $M_\Sigma \to B$
(SYZ fibration)

Why?

We recall a project to obtain homological Mirror symmetry functor

$$\mathrm{Fuk}(M_\Sigma) \hookrightarrow \mathbb{D}(M_\Sigma^\vee)$$

by family Floer homology

Let (L_v, b) and object of $Fuk(M_\Sigma)$

$M_\Sigma \to B$ $S^1 \times 2$ fibration

$M_\Sigma^\vee$ and elb of $M_\Sigma^\vee$

Note $M_\Sigma^\vee$ is a compactification of

$M_{\Sigma,0}^\vee \to B \setminus \Gamma$ that is a dual torus fibration to

$$\begin{array}{c} M_{\varepsilon U} \longrightarrow B \setminus D = B_0 \\ \shortparallel \\ \pi^{-1}(B \setminus D) \end{array}$$

A point p in $M^{\vee}_{\varepsilon U}$ $\Longleftrightarrow$ $q \in B \setminus D$

$+ \ \rho : \pi_1(\pi^{-1}(q)) \longrightarrow U(1)$

T_q

$\xi \in H^1(T_q ; \mathbb{C}) \Big/ 2\pi i \, H^1(T_q ; \mathbb{Z})$

This determine $\mathbb{L}_p = (L, \mathcal{L}^{\rho})$

$\mathcal{L}^{\rho}$: flat line bundle ass to ρ

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. We use Floer homology

$HF(\mathbb{L}, \mathbb{L}_p)$ parametrized by $p \in \check{M}_\varepsilon$.

This is a family of vector spaces parametrized by $p \in \check{M}_\varepsilon$

This becomes a "vector bundle"

$E(\mathbb{L}) \to \check{M}_\varepsilon$ on $\check{M}_\varepsilon$.

Cor $\mathcal{E}(\mathbb{L})$ has a "str. of holomorphic vector bundle" and it become a mirror to $\mathbb{L}$

Rem Actually $HF(\mathbb{L}, \mathbb{L}_p)$ may jump while we move p. So honest object we obtain is an "object of derived category of coherent sheaves", rather than a vector bundle.

In sum

$$\mathbb{L} \longmapsto \left(p \longmapsto \underset{\mathcal{E}(\mathbb{L})}{\overset{\shortparallel}{HF(\mathbb{L}, \mathbb{L}_p)}} \right)$$

is the HMS functor.

Now suppose $\mathcal{E}(\mathbb{L}) = \mathcal{L}$ a line bundle.

Then $\mathrm{rk}\, HF(\mathbb{L}, \mathbb{L}_p) = 1$

This occurs if L $(\mathbb{L} = (L, b))$

intersect fibers of $M_\varepsilon \to B$ transversally

at one pts.

$$L \pitchfork L_q \quad \& \quad \#(L \cap L_q) = 1$$

$$(L_q = \pi^{-1}(q) \quad q \in B)$$

Note $q \mapsto$ The point $L \cap L_q$ is a section

of $\pi : M_\varepsilon \longrightarrow B$.

In sum.

$\mathcal{L}$ line bdle on $M_0^\vee$

$\updownarrow$

Lagrangian section of $M_\Sigma \to B$. //

$$\mathcal{P}_\Sigma \xleftrightarrow{\text{Mirror}} \otimes \mathcal{L}$$

What is a Mirror to $\otimes$.

Symmetric monoidal category.

Operation $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

$C_1 \otimes C_2 \cong C_2 \otimes C_1$ etc (associativity)

Cor

$M_\varepsilon \to B$ SYZ fibration

$\exists$ a Lag section

$\Rightarrow Fuk(M_\varepsilon)$ is a symmetric monoidal category

$\otimes$ on $Fuk(M_\varepsilon)$ is

Mirror to $\otimes$ on $\mathbb{D}(M_0^\vee)$.

cf. The notion of model structure for A_∞ category is not so established.

This is proved by Aleksandar Subotic

in case $M_\varepsilon \to B$ has no singular fiber.

$(M_\varepsilon = T^{2n} \quad B = T^n)$

A Monoidal Structure for the Fukaya Category
Thesis Harvard 2010

Let me explain his argument.

First we assume

$$\begin{array}{c} T^{2n} \\ \downarrow \uparrow \vec{s} \\ T^n \end{array}$$

Lagrangian section

Fiber $= T^n = \mathbb{R}^n/\mathbb{Z}^n$

Note $\mathbb{R}^n = \tilde{T}^n$'s structure as flat affine space is canonically given.

In fact $q_1, \ldots, q_n$ local coordinate of base $B = T^n$

$X_{q_1}, \ldots, X_{q_n} \leftarrow$ Hamilton vector field of q_i

define a vector field of T^n.

$[X_{q_i}, X_{q_j}] = 0 \quad \Rightarrow \mathbb{R}^n$'s affine structure

Section

$$\begin{tikzcd} T^n \arrow[r, hook] & T^{2n} \arrow[d] \\ & B \arrow[u, bend right, "s"'] \end{tikzcd}$$

gives $s(b) \subset T_b = \pi^{-1}(b)$

$\exists$ canonical group str. on T_b

1) compatible with affine str. of $\tilde{T}_b$

2) $s(b)$ is the unit.

This each fiber T_b is an abelian group

Let $L, L' \subset T^{2n}$ Lagrangian submanifold

<u>Define</u>

$$L \otimes L' = \left\{ x+y \;\middle|\; \begin{array}{l} x \in L,\ x' \in L' \\ \pi(x) = \pi(x') \end{array} \right\}$$

Not $\pi(x) = \pi(x') = b \Rightarrow x, y \in T_b$

T_b is an abelian group so $x+y$ is defined $\in T_b$

<u>Lemma</u> After generic perturbation of L

$L \otimes L'$ is an immersed Lagrangian submanifold.

$L, L' \mapsto L \otimes L'$ is the symmetric

monoidal structure

To define a functor

$$\mathrm{Fuk}(T^{2n}) \times \mathrm{Fuk}(T^{2n}) \longrightarrow \mathrm{Fuk}(T^{2n})$$

which gives $L, L' \mapsto L \otimes L'$

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On object, we use the idea from Lagrangian correspondence.

$$M = (T^{2m}, \omega) \qquad \begin{tikzcd} T^{2m} \arrow[r] & T^{b} \arrow[l, bend right, "s"'] \end{tikzcd}$$

$T_p = \pi^{-1}(b)$ has a str of abelian group.

We consider

$$M \times M \times M \quad \text{with sy. str} \quad -pr_1^*\omega - pr_2^*\omega + pr_3^*\omega \; \overset{\sim}{\omega}$$

Let $\Delta := \left\{ (x, y, z) \in M^3 \;\middle|\; \begin{array}{l} \pi(x) = \pi(y) = \pi(z) \\ z = x + y \end{array} \right\} \subset M^3$

Note $\pi(x) = \pi(y) = \pi(z) = b$

$\Rightarrow$ $x, y, z \in T_b$ an abelian group

$\Rightarrow$ $z = x + y$ makes sense

Lemma $\Delta \subset (M^3, \hat{\omega})$ is a Lagrangian submanifold

$\because$ Easy calculation

We review Lagrangian Correspondence.

(M_1, ω_{M_1}), (M_2, ω_{M_2}) symplectic manifolds

Defn $L \subset M_1 \times M_2$ is a Lag. correspondence

$$\Leftrightarrow \left(-Pr_1^* \omega_{M_1} + Pr_2^* \omega_{M_2}\right)\Big|_L = 0$$

—

Th'm (cf arXive 1706.02131 generalizing earlier works by Wehrheim-Woodward, Ma'u).

Let $L_{12} \subset M_1 \times M_2$ be Lag correspondence (L_{12}, M_1, M_2 spin). $\mathbb{L}_{12} := (L_{12}, b) \in \mathrm{Fuk}(M_1 \times M_2)$

It induces an A_∞ functor

$$\mathcal{W}_{\mathbb{L}_{12}} : \mathrm{Fuk}(M_1) \longrightarrow \mathrm{Fuk}(M_2)$$

$$\begin{array}{c} \psi \\ \mathbb{L} = (L_1, b_1) \mapsto (L_2, b_2) \end{array}$$

$$L_2 = L_1 \times_{M_1} L_{12} \longrightarrow M_2 \quad \frac{\text{Immersed}}{\text{Lag. submanifold}}$$

—

Now we apply it to

$$\Delta \subset T^{2n} \times T^{2n} \times T^{2n}$$

It induced a A_∞-functor

$$\mathrm{Fuk}(T^{2m} \times T^{2n}) \longrightarrow \mathrm{Fuk}(T^{2n})$$

This is $\otimes$.

Note

$$(x, y, z) \in \Delta \iff (y, x, z) \in \Delta$$

So $L_1 \otimes L_2 \cong L_2 \otimes L_1$ seems

geometrically obvious

How much can one generalize it to a general SYZ fibration?

Let $\pi : X \longrightarrow B$ be a SYZ fibration.

<u>Remark</u> There is no rigorous Mathematical definition of SYZ fibration yet.

$\Gamma \subset B \qquad X_0 = \pi^{-1}(B \setminus \Gamma)$

$\pi : X_0 \longrightarrow B \setminus \Gamma$ is a Lagrangian torus fibration

$s : B \longrightarrow X$ Lagrangian section

$b \in B \setminus \Gamma \implies T_b = \pi^{-1}(b)$ has a structure of abelian group ($s(b)$ is 0.)

Conjecture

$q \in \Gamma \qquad T_q = \pi^{-1}(q) \qquad T_q^{reg} \subset T_q$

(defined below)

$q_i \in B \setminus \Gamma \qquad \lim q_i = q$

$\exists$ a structure of abelian group on T_q^{reg}

such that the structures abelian groups on

T_{q_i} converges to it

A more precise statement is given below.

Def $x \in T_\xi$ $\xi \in P$

$x \in T_\xi^{reg} \iff \pi : M \to B$ is a submersion at x.

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Str. on T_ξ converges on T_ξ^{reg} means the following

1) $S(\xi) \in T_\xi^{reg}$

2) $x, y \in T_\xi^{reg}$

Take V_x, V_y

s.t. $V_x \pitchfork T_q^{reg}$ at x

$V_y \pitchfork T_q^{reg}$ at y

$\pi : V_x \longrightarrow B$ diff. onto an open set

$\pi : V_y \longrightarrow B$ "

(since V_x, V_y exists since $x, y \in T_q^{reg}$)

$x_i := V_x \cap T_{q_i}$ $y_i = V_y \cap T_{q_i}$ (one points)

$\circledast$ $x_i + y_i \xrightarrow[i \to \infty]{} x + y$ ← group structure on T_q

group str. on T_{q_i}

Conjecture is OK in case $\dim_{\mathbb{C}} M = 2$

$\pi : M \to B$ SYZ fibration

It is known that such SYZ fibration is a hyper Kähler rotation of elliptic fibration

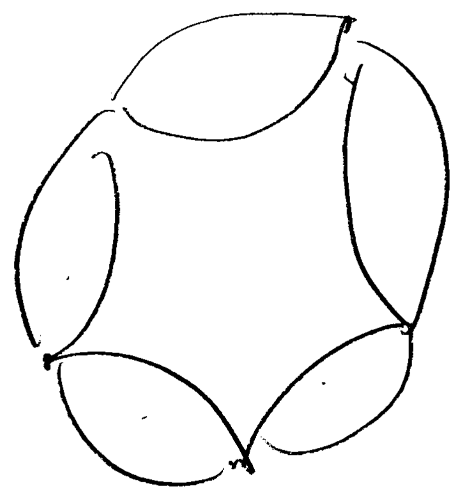
(In other words by "rotating" cpx str. of M π becomes holomorphic)

We can use Kodaira's classification of elliptic fibration to check the conjecture.

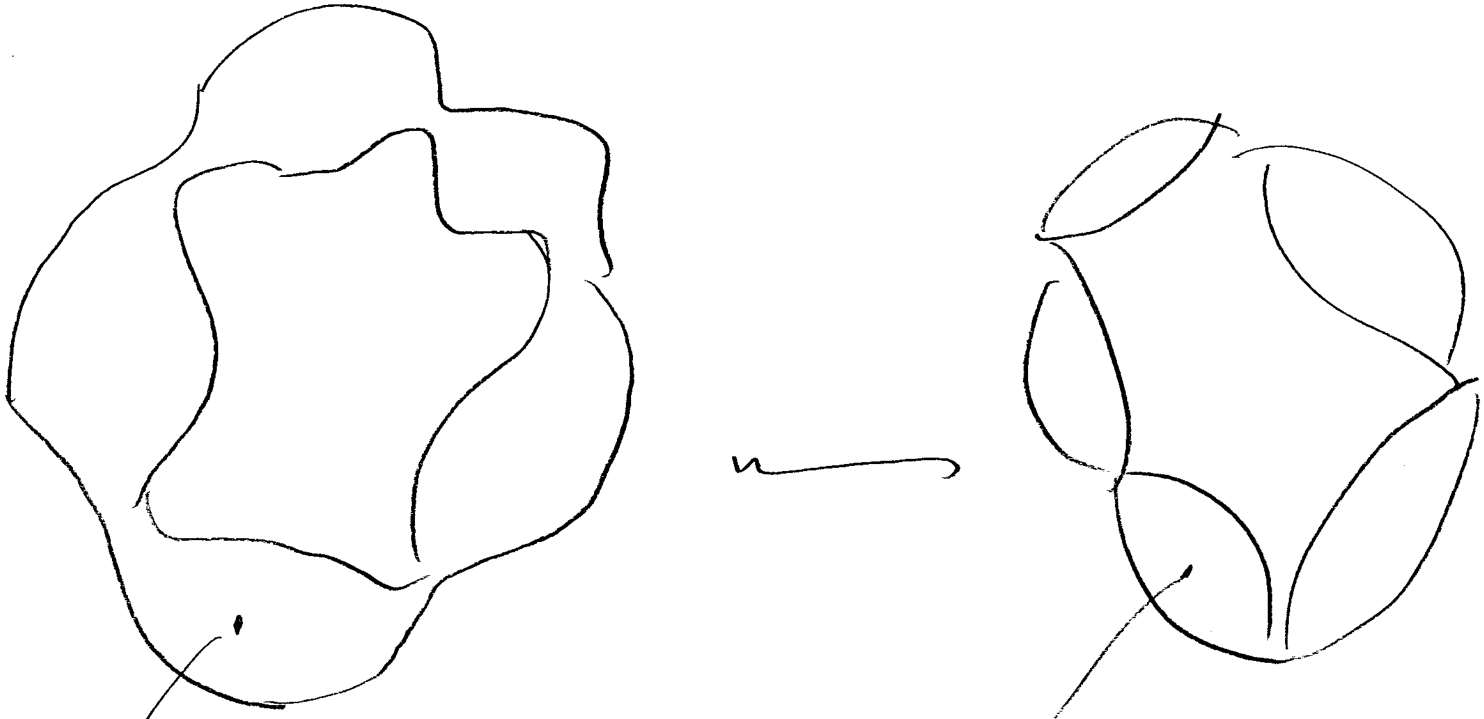
This is actually a part of the theory of Neron Model.

<u>Example 1</u>

A_k fiber



$k=5$



$S(\Gamma_1)$ $\quad T_{\Sigma_1}$ $\qquad\qquad S(\Gamma)$ $\quad T_{\Sigma}$

$$\|$$

$$\mathbb{R}^2/\mathbb{Z}\oplus c_1\mathbb{Z}$$

$c_1 \to \infty$

$$T_\xi^{reg} = T_\xi \setminus 5 \text{ points}$$

grap structure $\quad (S^1 \times \mathbb{R}) \times \frac{\mathbb{Z}}{5\mathbb{Z}}$

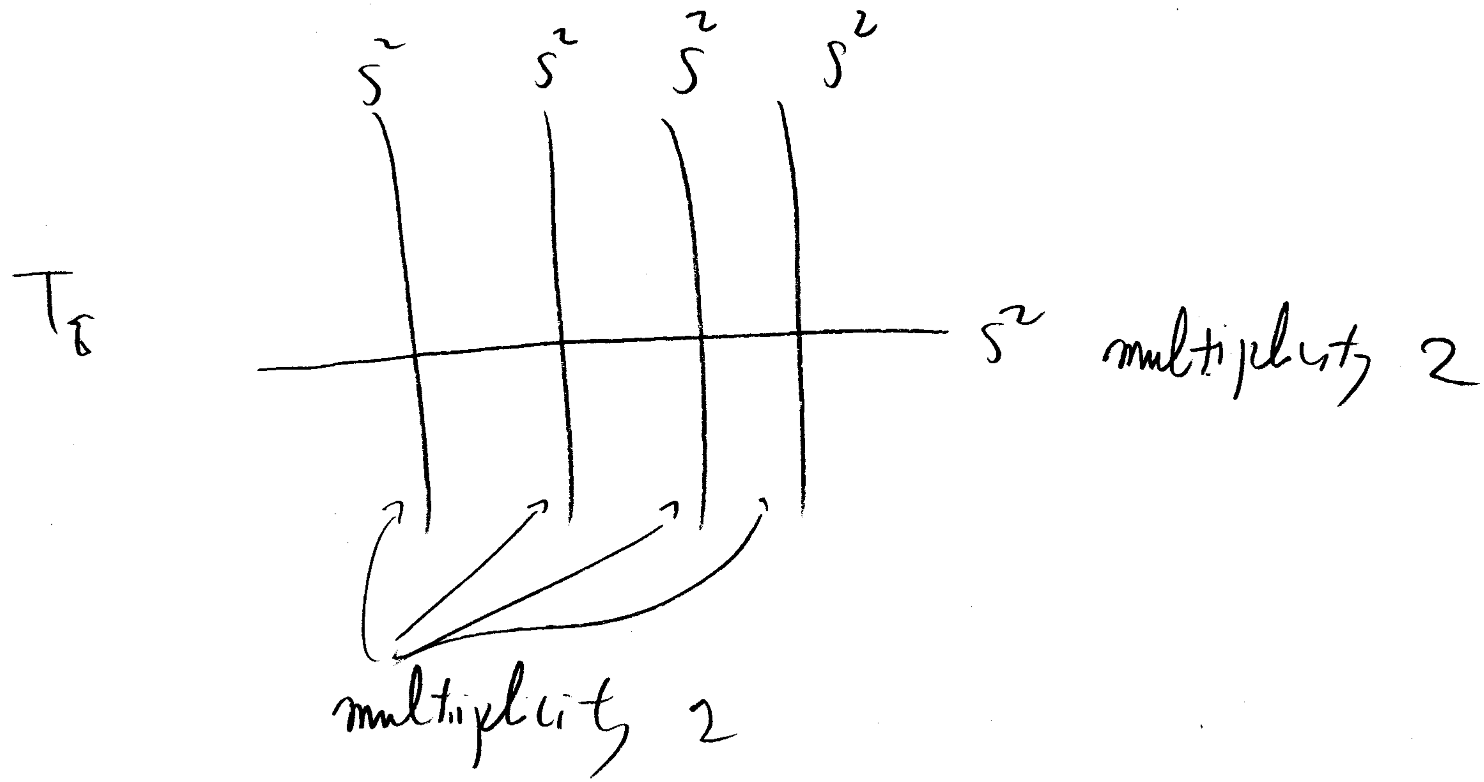
—

<u>Note</u> In the usual limit

$$\mathbb{R}^2 / \mathbb{Z} \oplus c_1 \mathbb{Z} \rightsquigarrow \mathbb{R}^2/\mathbb{Z} = S^1 \times \mathbb{R}$$

$c_1 \to \infty$ so this limit is a bit different

Example D_4 singular fiber



T_ξ^{reg} = 4 copies of $S^2 \setminus$ (one pt)

$$T_{\xi}^{reg} \cong (\mathbb{C} \setminus U) \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

as groups

Note S^2 with multiplicity is <u>not</u> a part of T_{ξ}^{reg}.

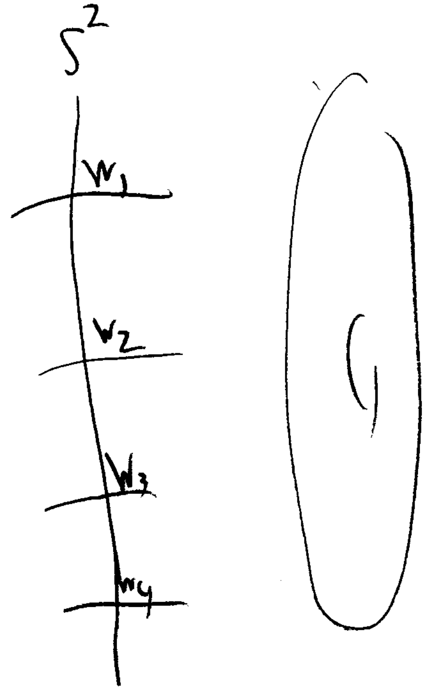
Let us recall how the singular fiber occures.

$$\begin{array}{c} T^2 \\ \downarrow \pi \\ S^2 \end{array}$$ branched double cover

4 branch point $\quad w_1, w_2, w_3, w_4 \in S^1$

$(z \neq w_i \Rightarrow \pi^{-1}(z)$ two points $)$

$\exists$ $\mathbb{Z}_2$ action $\circlearrowleft T^2$ $\quad T^2/\mathbb{Z}_2 = S^2$



$T_{\mathbb{Z}}$ $\quad$ $T_{\mathbb{Z}_v}$

$(T^2 \times D^2) \circlearrowleft \mathbb{Z}_2$ action τ

$\tau : T^2 \to T^2$ as above $\qquad \tau : D^2 \to D^2 \quad z \mapsto -z$

$$\frac{T^2 \times D^3}{\mathbb{Z}_2} = X \xrightarrow{\pi} D^2/\mathbb{Z}_2 = D^2$$

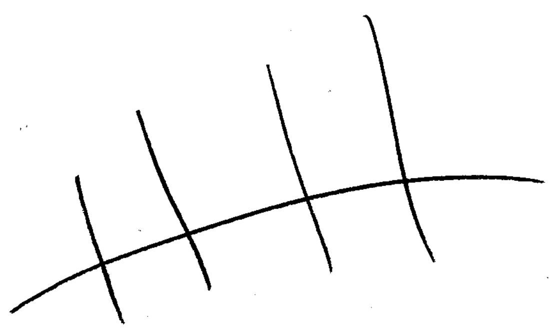
$\pi^{-1}(\varepsilon) = T^2$ $\quad \pi^{-1}(0) \simeq S^2$ homeo

with 4 cusps $w_1\ w_3\ w_2\ w_4$

Blow up X at $w_1 \ldots w_4$ $\quad \check{X}$

$\check{\pi} : \check{X} \longrightarrow D^2$

$\check{\pi}^{-1}(0) =$



S^2 $\quad D_4$ singular fiber

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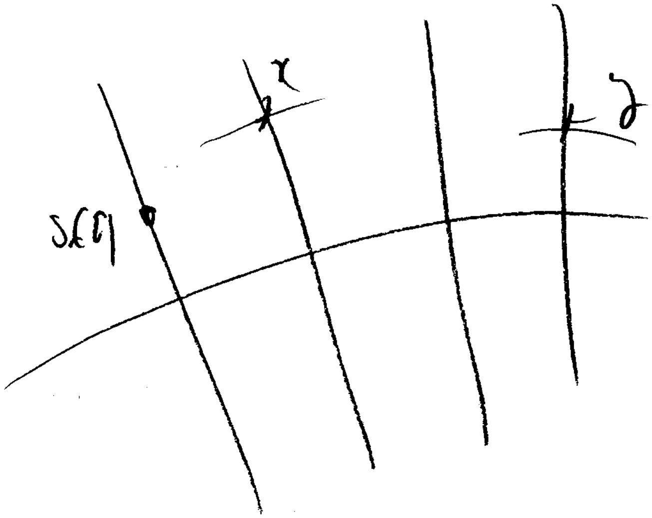
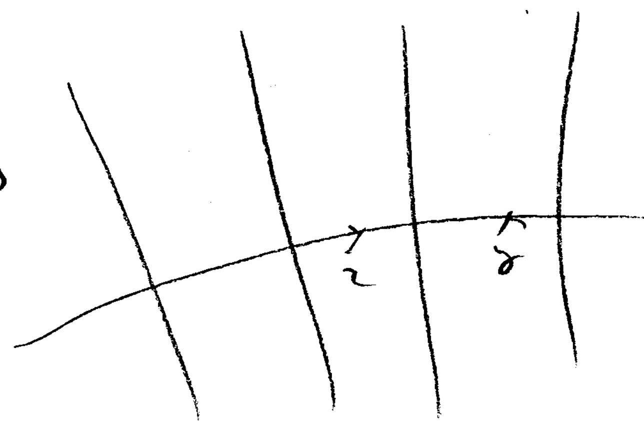
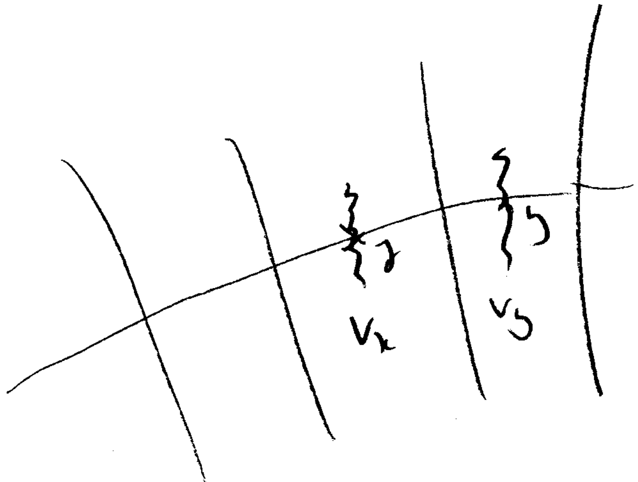


$x + y$ well defined

If $x, y \in S^2 \setminus 4$ pts



V_x, V_y $\quad$ $V_x \pitchfork S^2$ at x, $V_y \pitchfork S^2$ at y



$$V_x \cap T_{\varepsilon_i} = 2 \text{ pts} \quad x_i^1, x_i^2$$

$$V_y \cap T_{\varepsilon_i} = 2 \text{ pts} \quad y_i^1, y_i^2$$

$$\lim (x_i^1 + y_i^1) = \lim (x_i^2 + y_i^2)$$

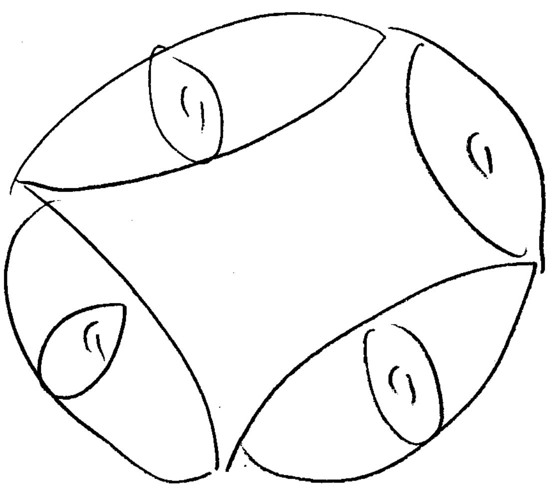
But $\quad \lim (x_i^1 + y_i^1) \neq \lim (x_i^1 + y_i^2)$

So $x+y$ is not defined.

$\equiv$

<u>Example</u> $X \to B^3$ generic

Type III n singular fiber (cf W.D. Ruan's paper)



T_q^{reg} = k copies of $\mathbb{R} \times T^2$

So the group structure seems to be

$\mathbb{Z}_k \times (\mathbb{R} \times T^2)$

I do not check yet that this group str. is compatible to the limit $q_i \to q$.

$\pi : X \longrightarrow B$

Let us assume the conj.

let $L_1, L_2 \subset X$ Lag. submanifold

We consider

$$\tilde{L} = \left\{ x + y \;\middle|\; \begin{array}{l} x \in L_1,\ y \in L_2 \\ \pi(x) = \pi(y) \in B \setminus P \end{array} \right\}$$

Conj After the generic perturbation the chain L of C is an immersed Lagrangian submanifold

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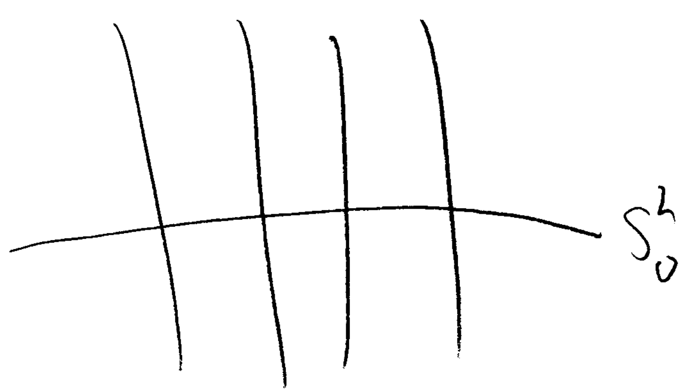
OK if $X \to B$ $\quad T_q \setminus T_q^{reg}$ has positive codimension (A_n type or D_n
$\quad$ 2 dim $\quad$ 3 dim)

Since by perturbing we may assume

$$L_i \cap T_b \subset L_i \cap T_b^{reg}.$$

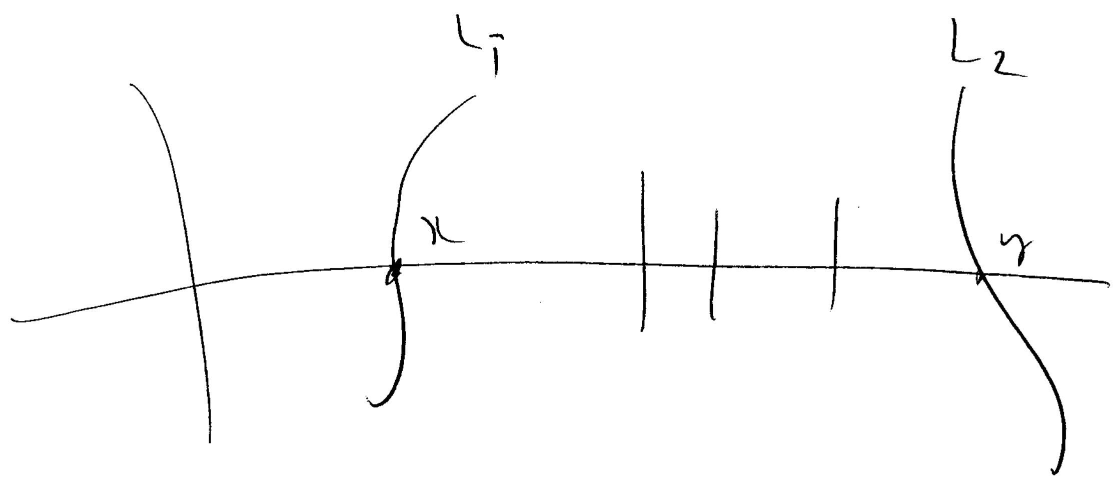
$$=$$

Suppose T_q is D_4 singularity



We cannot perturb $L_i \cap T_q \subset T_\Sigma^{reg}$

Since $L_1 \pitchfork S^2_a$ may be non empty



As explained before

$x+y$ is not well defined

but I think it is described as

2 points

$$\left(x_i^1, x_i^2 \to x \qquad y_i^1, y_i^2 \to y \right.$$

$$x_i^1 + y_i^1 \longrightarrow (x+y)^2$$

$$x_i^1 + y_i^2 \longrightarrow (x+y)^2$$

So $\mathring{C}$'s closure has 2 extra pts and it seems to define a Lag. sub

$L_1 + L_2 \subset X$.

<u>Problem</u> $X \to S^2$ $\times$ K3

Hyper Kähler twist of elliptic fibration

1) $L_1, L_2 \subset X$ show $L_1 + L_2$ is defined along this line

2) Construct an A_∞ functor which is $L_1, L_2 \mapsto L_1 + L_2$ for objects