Hamiltonian Dynamics of the maximal degenerate family of CY manifolds ?

(Kyoto Univ. 2021 Jan)

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SCGP

$\pi : M \longrightarrow B$ SYZ fibration

$\Gamma \subset B$

$\pi^{-1}(B \setminus \Gamma) = M_0$ $B_0 = B \setminus \Gamma$

$\pi : M_0 \longrightarrow B_0$ Lagrangian (torus) fibration

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Assume $\exists\, S : B \longrightarrow M$ Lagrangian section

(Mirror to trivial line bundle $\mathbb{C} \times M^\vee \to M^\vee$.)

Conjecture (explained yesterday) 1

$\xi \in P$ $\quad T_\xi^{reg} = \left\{ x \in M \;\middle|\; \begin{array}{l} \pi(x) = \xi, \\ D_x\pi : T_xM \to T_\xi B \\ \text{is surjective} \end{array} \right\}$

$\exists$ a structure of abelian group on T_ξ^{reg}

s.t. $\xi_i \in B \setminus P$ $\quad x_i \in T_{\xi_i}$, $y_i \in T_{\xi_i}$

$\lim \xi_i = \xi$ $\quad \lim x_i = x$ $\quad \lim y_i = y$ $\quad x, y \in T_\xi^{reg}$

$\Rightarrow \lim (x_i + y_i) = x + y$.

group str. on T_{ξ_i} $\quad$ group structure on T_ξ

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Conjecture 2

$L_1, L_2 \subset M$ Lag. submanifold

$\Rightarrow$ after generic perturbation the closure of

$$\overset{\circ}{L} = \{ (x+y) \mid x \in L_1,\ y \in L_2 \quad \pi(x) = \pi(y) = B \setminus D \}$$

is an immersed Lagrangian submanifold

We write it $L_1 \otimes L_2$

Conjective 1 $\Rightarrow$ Conjecture 2 is OK

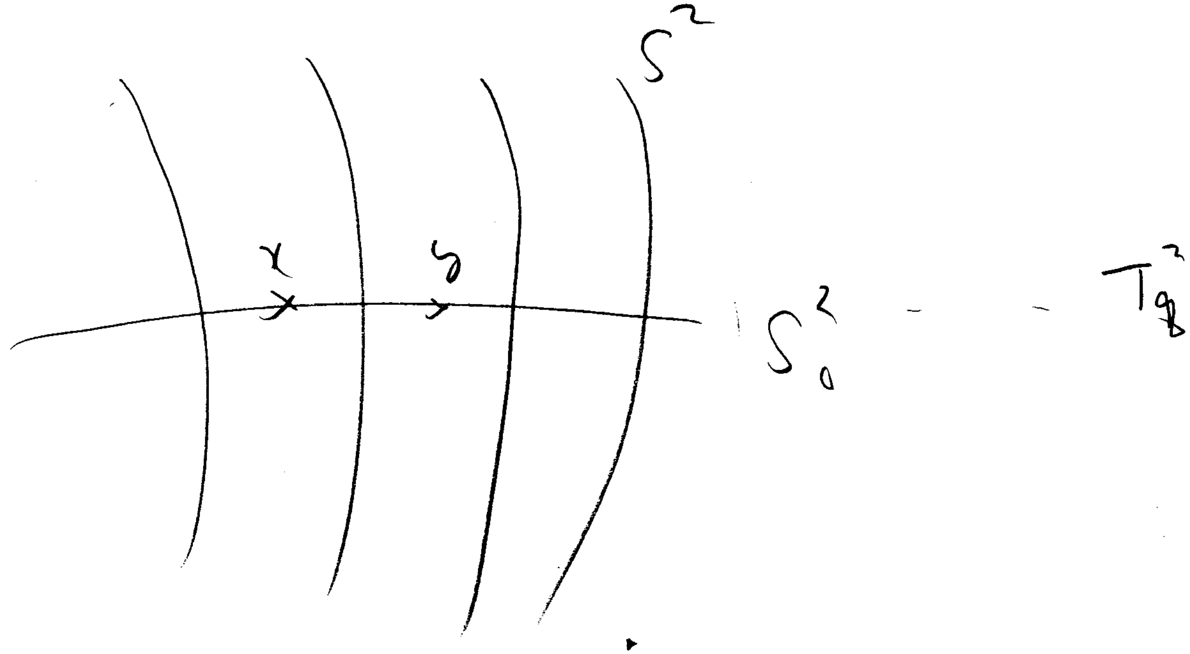
if $T_\xi \setminus T_\xi^{reg}$ has positive dimension

(we can perturb st $L_i \cap (T_\xi \setminus T_\xi^{reg}) = \phi$)

So A_k singularity $(n=2)$

III_k singularity $(n=3)$ are OK

The conjecture 2 is OK for D_4 singularity



$V_x \pitchfork T_q^2$ at x

$V_y \pitchfork T_q^2$ at y

$q_i \to q$

$$V_x \cap T_{q_i} = \{x_i^1, x_i^2\}$$

$$V_y \cap T_{q_i} = \{y_i^1, y_i^2\}$$

Two points $x +^1 y$, $x +^2 y$

$$\lim (x_i^1 + y_i^1) = x +^1 y = \lim (x_i^2 + y_i^2)$$

$$\lim (x_i^1 + y_i^2) = x +^2 y = \lim (x_i^2 + y_i^1)$$

So the closure intersects T_q at two points

$x +^1 y$, $x +^2 y$

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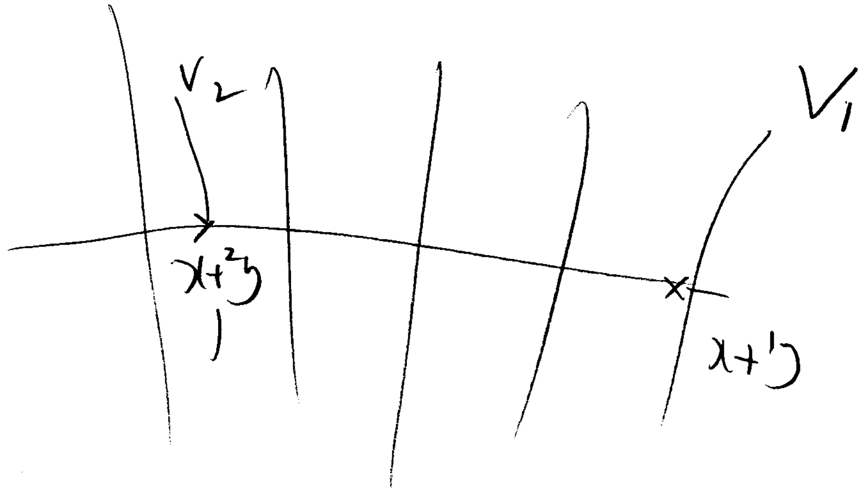


$V_1 \cap T_{\xi_i} = 2$ pts $\qquad V_2 \cap T_{\xi_i} = 2$ pts. //

<u>Exercise</u> check the conjecture for all ADE singularity of Kodaira classification.

Conjecture 3

$L_1 \otimes L_2$: closure of $\left\{ x+y \;\middle|\; \begin{array}{l} x \in L_1,\ y \in L_2 \\ \pi(x) = \pi(y) \in B \cap P \end{array} \right\}$

① If b_i is a bounding cochain of L_i

$\Rightarrow \exists$ bounding cochain $b_1 \otimes b_2$ of $L_1 \otimes L_2$

② $(L_1, b_1) \times (L_2, b_2) \mapsto (L_1 \otimes L_2, b_1 \otimes b_2)$ is a object part of an A_∞-bifunctor

$$\mathrm{Fuk}(M) \times \mathrm{Fuk}(M) \longrightarrow \mathrm{Fuk}(M)$$

Problem

$$\mathring{\Delta} = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x,y,z \in M \quad \pi(x)=\pi(y)=\pi(z) \in B\setminus P \\ x+y=z \end{array} \right\}$$

$\subset M \times M \times M$ Lag. submanifold

$(-pr_1^* \omega - pr_2^* \omega + pr_3^* \omega$ is the sym form$)$

- How much its $\underbrace{\text{closure}}_{\Delta}$ is singular?
- Can one define a Lagrangian HF for Δ.

If Δ is immersed Leg submanifold

$\Rightarrow$ Conj. 3 is OK (Lagrangian correspondence)

A reason why one can believe Δ has a Floer homology.

HMS

$$\Delta \longleftrightarrow \text{diagonal in}$$

$$M^3 \qquad\qquad\qquad M^{\vee 3}$$

$\{(x,x,x) \mid x \in M^\vee\}$ is a complex submanifold of $M^{\vee 3}$

$\parallel$

$\Delta_{M^\vee}$

so it gives an object of $\mathbb{D}(M^{\vee 3})$.

HMS $\quad \mathrm{Fuk}(M^3) \longleftrightarrow \mathbb{D}(M^{\vee 3})$

the Mirror should be Δ.

Remark

The case $M = T^{2n}$ $\quad \Delta \longleftrightarrow \Delta_{M^\vee}$

is actually true

$\pi : (T^{2n})^3 \longrightarrow (T^n)^3$ direct product of SYZ fibration

$\pi^\vee : (\check{T}^{2n})^3 \longrightarrow (\check{T}^n)^3$ dual torus fibration

We use then and family HF picture of HMS

to find the Mirror of $\Delta_{T^{2n}}$

$$\Delta_{T^{2n}} = \{(x,y,z) \in (T^{2n})^3 \mid x=y=z\}$$

Its image in $(T^n)^3$ is the Diagonal

$$\Delta_{T^n} \hookrightarrow (T^n)^3$$

The fiber of $\Delta_{T^{\vee 2n}} \longrightarrow \Delta_{T^n}$

at (q, q, q) is

$\{((q, \xi_1), (q, \xi_2), (q, \xi_3)) \mid \xi_1 = \xi_2 = \xi_3\} \hookleftarrow \star$

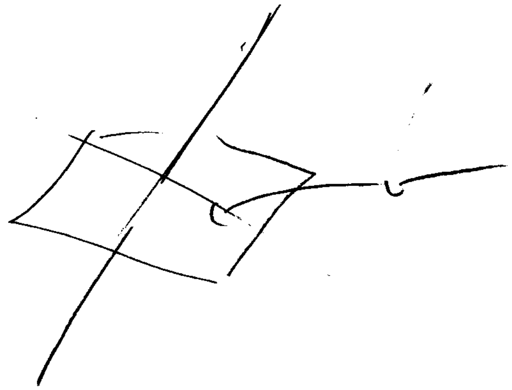
Suppose $L \subset (T^{2n})^3$ is Mirror to $\Delta_{T^{\vee 2n}}$

the image of $L \longrightarrow T^{2n}$ should be Diagonal

Δ_T is actually conormal bundle of the diagonal $\Delta_{T^n} \subset (T^n)^3$. $\star$ is known to its mirror.

In fact

$$T^n \subset (T^n)^3$$

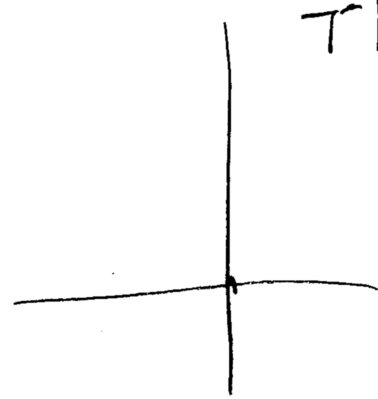


normal dir.

locally $\quad \mathbb{R}^n \times \mathbb{R}^{2m}$

$\quad\quad\quad T^*\mathbb{R}^n \times T^*\mathbb{R}^{2m}$

$\Delta \quad \begin{array}{c}\mathbb{R}^n \\ \times \\ 0\end{array} \quad \begin{array}{c}0 \\ \times \\ \mathbb{R}^{2m}\end{array}$



$T^*\mathbb{R}^n$

$\mathbb{R}^3$

Mirror to a cotangent fiber $\Longleftrightarrow$ a point

(sky scraper)

Mirror to 0 section $\Longleftrightarrow$ trivial bundle

$U \times K^n \longleftrightarrow$ pt

$\mathbb{R}^n \times U \longrightarrow \check{\mathbb{R}}^{2n} \longleftrightarrow$ 2n dim

$\uparrow$

$\Delta_{(K^n)^3}$

Δ

//

<u>Problem</u> Can one generalize this picture to the case of $K3 \longrightarrow \mathbb{P}^1$ (with ADE singularity?)

<u>Note</u> Fourier-Mukai correspondence by $\mathcal{O}_{\mu^\vee} \subset M^3$ is tensor product functor. So $\mathcal{O}_L \longmapsto \mathcal{O}_{\mu^\vee}$ "shows" $\otimes \longmapsto$ tensor product via HMS.

A side

A possible app of symmetric monoidal str. on $\mathrm{Fuk}(M)$.

<u>Conj</u> $\exists$ Adams operation on $K(\mathrm{Fuk}(M)) \otimes \mathbb{Q}$ split closure

$L = \mathrm{obj}(\mathrm{Fuk}(M))$

$\underbrace{L \otimes \cdots \otimes L}_{n}$ defined by sym monoidal structure.

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$S_k \curvearrowright L \otimes \cdots \otimes L$

One can decompose $L \otimes \cdots \otimes L$ using this action in the same way as the case of usual tensor product and may obtain

$\Lambda^k L$ exterior product in the cat $\mathrm{Fuk}^+(M)$: the split closure of $\mathrm{Fuk}^+(M)$

$Q_k(\sigma_1 - \sigma_r)$ newton polynomial

Now σ_i i-th sym function $\sigma_i(x_1 - x_r)$

$(\sigma_1(x_1, x_2 \cdots x_n) = x_1 + x_2 + \cdots + x_n$ etc $)$

st.

$$Q_k(\sigma_1 \cdots \sigma_r) = x_1^k + \cdots + x_n^k$$

Then we "define"

$$\psi^k([L]) = Q_k(\Lambda^1 L, \Lambda^2 L, \cdots, \Lambda^k L)$$

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On $K^{top}(M^\vee)$, one application of Adams operation is the following:

chern character defines an iso.

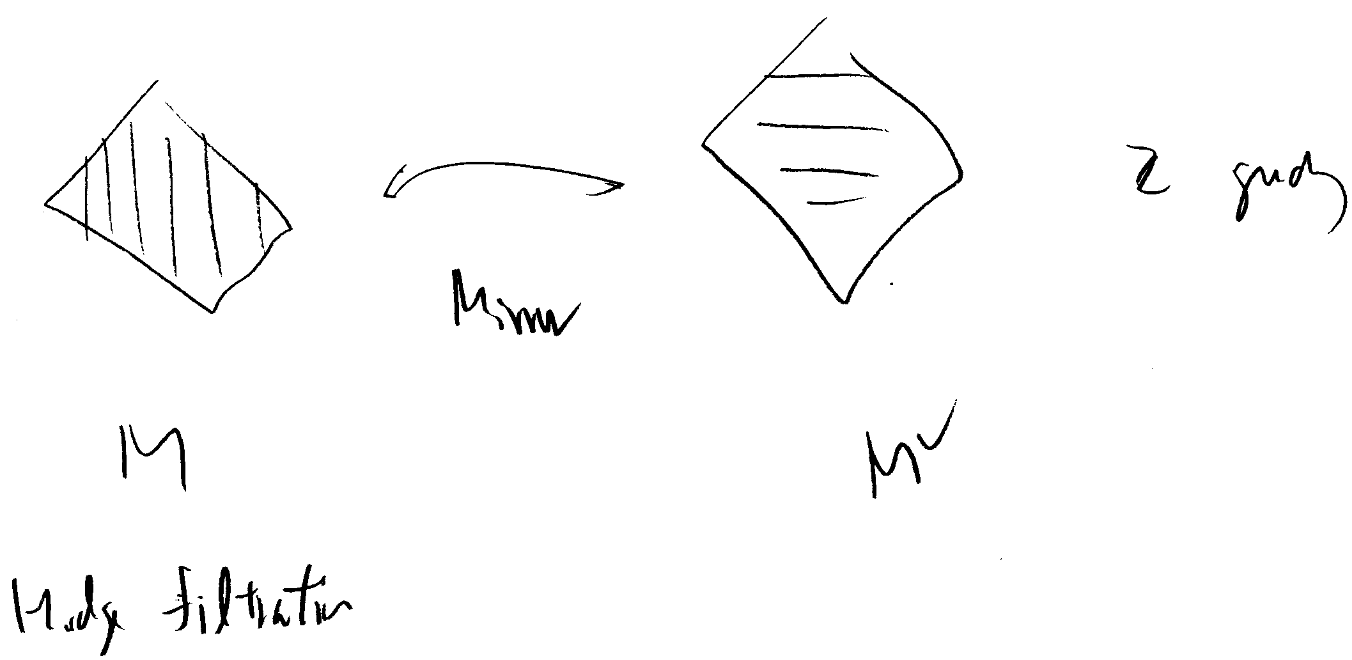
$$ch: K^{top}(M^\vee) \longrightarrow H(M^\vee; \mathbb{Q})$$

However $\mathbb{Z}$ grading is not preserved

It is known that using ψ^n (Adams operation) and its eigenspaces we can recover

$\mathbb{Z}$ grading of $H(M^\vee;\mathbb{Q})$ or $K^{top}(M^\vee)$

(See Atiyah K-theory Prop. 3.2.7.)



Prob

Can one see the "Hodge filtration" of M by Adams operations ψ^k or $K(\mathrm{Fuk}^+(M))$?

Going back to our situation

$\pi : X \longrightarrow D^2$ maximal degenerate family

$M_\varepsilon = \pi^{-1}(\varepsilon) \qquad \varepsilon \neq 0$

$\varphi_\varepsilon : M_\varepsilon \longrightarrow M_\varepsilon$ Poincaré map

$\pi : M_\varepsilon \longrightarrow B$ SYZ fibration

$\check{\pi} : M_0^\vee \longrightarrow B$ SYZ Mirror

We conjectured

$\varphi_\Sigma \longleftrightarrow \otimes \mathcal{L}$ $\quad$ $\mathcal{L}$ line bundle

$\exists$

$s_1 : B \to M_\Sigma$ Lagrangian section $\xleftrightarrow{\text{Mirror}}$ $\mathcal{L}$

fiber wise sum $\Longleftrightarrow$ $\otimes$

(Not $\exists$ $s_0 : M \to M_\Sigma$ Lagrangian section to trivial bundle.)

Conjectures

$\exists F : \varphi_\varepsilon \to \varphi_\varepsilon$ Hamiltonian iso s.t.

$\varphi_\varepsilon' = F \circ \varphi_\varepsilon$ is defined on tillons

$x \in M_\varepsilon \qquad \underbrace{\pi(x)}_{q} \in B \setminus P \qquad s(\varepsilon) \in T_\varepsilon$

Then

$$\varphi_\varepsilon'(x) = x + s(\varepsilon)$$

(Here $+$ is a structure of abelian group)

Remark

The map $x \mapsto x + S(\varepsilon)$ on $\pi^{-1}(B \setminus P) \subset M_\varepsilon$

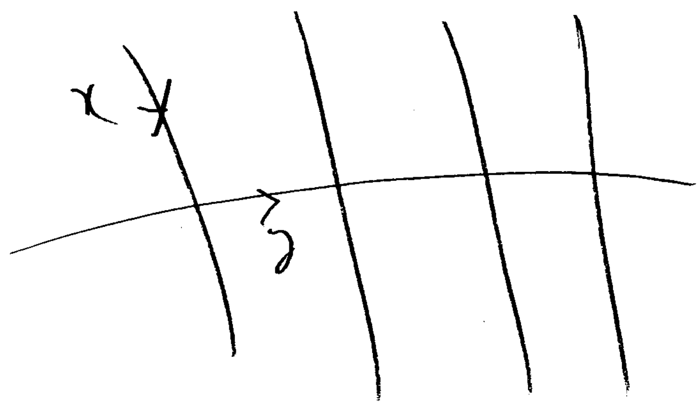
is likely extended to M_ε. Actually

Cor The product $T_\varepsilon^{reg} \times T_\varepsilon^{reg} \to T_\varepsilon^{reg}$

is extended to $T_\varepsilon \times T_\varepsilon^{reg} \to T_\varepsilon$

which is also a limit of $T_{\varepsilon_i} \times T_{\varepsilon_i} \to T_{\varepsilon_i}$ for $\varepsilon_i \in B \setminus P$.

Ex Again D_4 sing fiber



Let $x \in T_q^{reg}$

$y \in S_0^2 \setminus 4pts$

$q_i \to q \quad q_i \in B \setminus D$

$V_x \pitchfork T_q$ at x $\quad V_y \pitchfork T_2$ at y

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$V_x \cap T_{q_i} = x_i$ one point

$V_y \cap T_{q_i} = \{y_i^1, y_i^2\}$ 2 pts.

Note $T_{q_i} \longrightarrow S^2$ double cover

(cf yesterday's construction)

$\tau(y_i^1) = y_i^2$

$\left\{ \begin{array}{l} \tau \circlearrowright T_{q_i} \quad \text{Involution} \\ w_1, w_2, w_3, w_4 \text{ (nodal pt or } S^2_0) \end{array} \right.$

are fixed pts of τ.

$$\tau(x_i + y_i^1) = x_i + y_i^2$$

Hence $\lim_{i\to\infty}(x_i + y_i^1) = \lim_{i\to\infty}(x_i + y_i^2)$ //

By construction we have

$$\varphi_0'((L,b)) = (L,b) \otimes s(B)$$

Image of Lcs sect. Mirror to $\mathcal{L}$.

If $\varphi_\Sigma \longmapsto \mathcal{L}\Theta$

the φ'_Σ and φ_Σ induces the same map on

$Fuk(M_\Sigma)$.

This is an evidence of Conjecture.

—

Not φ'_Σ is a complete integrable system

in the usual sense

Observation

φ_ε' satisfies non-degeneracy condition of KAM theory

$\because)$ $\xi \longmapsto S(\xi)$ can be seen

$\mathbb{R}^n \longrightarrow \mathbb{T}^n.$

This is the frequency map of our φ_ε'

So it is sufficient to show that

$DS : T\mathbb{R}^n \longrightarrow T_{S(\xi)} T^n$ is invertible.

S is graph of df locally

$\Rightarrow$ DS is Hess f

In the story of "local HM" it is calculated that

Hess $f \Longleftrightarrow$ curvature of $\mathcal{L}$

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(Note $S \longmapsto \mathcal{L}$ — a line bundle)

Also $\mathcal{L}$ is expected to be an ample line bundle (In case of quintic 3 fold it is $\mathcal{O}(5)$ ...)

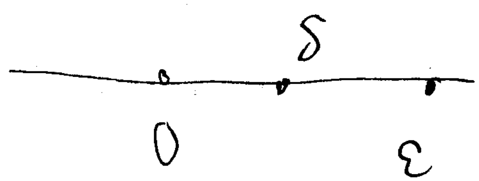
So positivity of curvature.

Actually the conj. $\varphi_\varepsilon = F\varphi'_\varepsilon$ F Hamiltonian is not so satisfactory. Since Dynamics of $F\varphi'_\varepsilon$ depend much of F.

A better Conjecture is:

Conj $\exists\, \omega^0$ a complete Kähler form on $X \setminus M_0$ (complete means, complete in a neighborhood of M_0) st if X^0_H is a Hamiltonian vector field of $H = |\pi|$ wrt sym structure ω^0

Then it "converges" to φ_ε' in the following sense

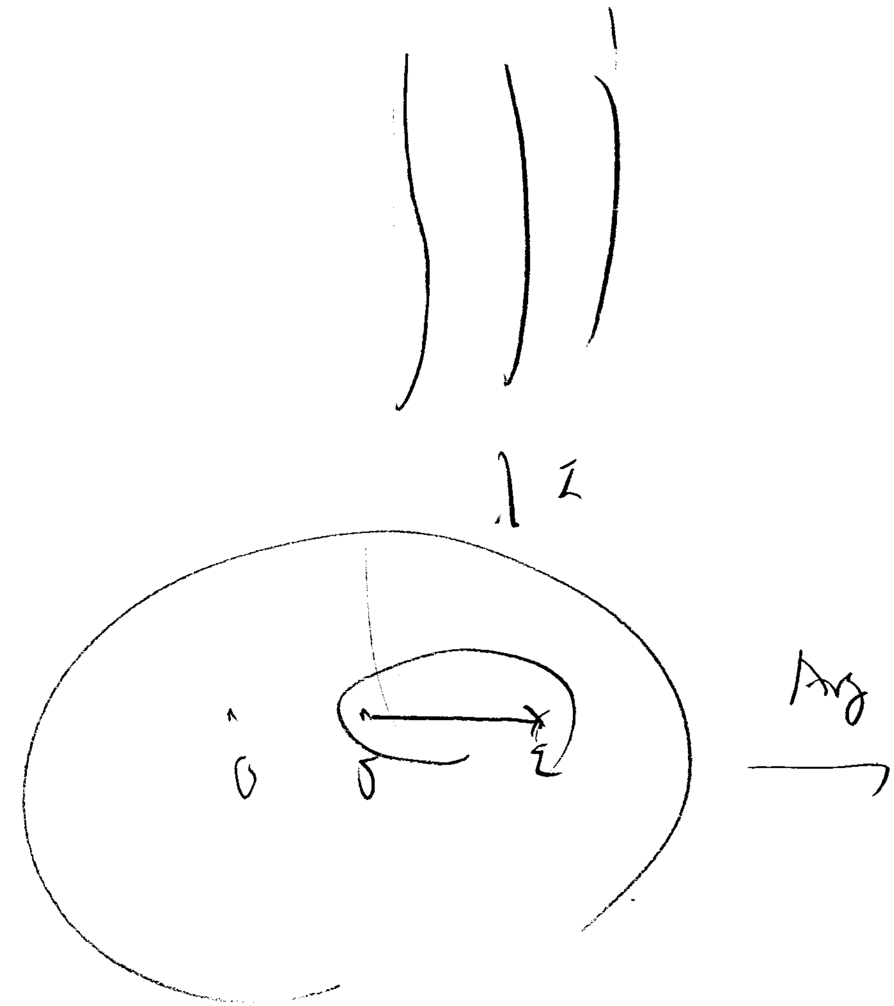


Note $\varepsilon \in \mathbb{R}_+$. Let $\delta \in (0,\varepsilon)$.

We define

$$\psi_{\varepsilon,\delta} : M_\varepsilon \to M_\delta \quad \text{as follows}$$

Let $G = \operatorname{Arg}\circ\pi$: nbd of $\pi^{-1}([\delta,\varepsilon]) \to \mathbb{R}$

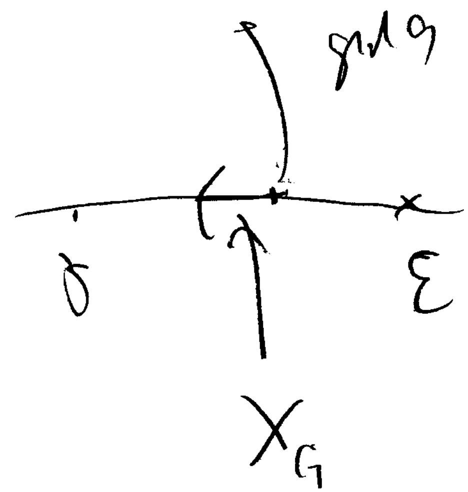


$$\mathrm{Arg}(re^{2\pi i\theta}) = \theta$$

X'_G Hamiltonian vector field of G

$p \in M_\varepsilon$ $\qquad \gamma_p(t)$ $\qquad \begin{cases} \dfrac{d\gamma_p}{dt} = X_G \circ \gamma_p \\ \gamma_p(0) = p \end{cases}$

$\exists!\ t_p > 0$ st $\gamma_p(t_p) \in M_\delta$



$$\Psi_{\varepsilon\delta}(p) = \gamma_p(t_p)$$

$\Psi_{\varepsilon\delta} : M_\varepsilon \longrightarrow M_\delta$ is a symplectic diffeo

$\varphi^0_\delta : M_\delta \to M_\delta$ Poincaré map of X_H wrt the sym. form ω_0

$$\boxed{\underline{\text{Conj}} \quad \lim_{\delta\to 0} \Psi^{-1}_{\varepsilon\delta}\varphi^0_\delta\Psi_{\varepsilon\delta} = \varphi'_\varepsilon}$$

$x \longmapsto x + S(\pi(x))$

integrable system

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If conjecture is ok then

KAM holds for $\psi_{\varepsilon\delta}^{-1} \varphi_{\delta}^{\circ} \psi_{\varepsilon\delta}$ for

sufficiently small δ.

<u>Note</u> $X_G = \operatorname{grad} H$

Hamiltonian vector field of G

gradient vector field of H

Therefore the map $\lim_{\delta \to 0} \Psi_{\varepsilon\delta}$ is "the same" as the map W.D. Ruan used for his construction of SYZ fibration

Actually there is a "slight" difference.

W.D. Ruan used $\operatorname{grad} H = X_H$ wrt the Kähler form ω (which is defined on X (not only on $X \setminus M_0$)).

Let we define

$\varphi^{W}_{\varepsilon\delta} : M_\varepsilon \to M_\delta$ using X_q - grad H wrt W.

Then

$\lim_{\delta \to 0} (\varphi^{W}_{\varepsilon\delta})^{-1} \varphi_\delta^{n_\delta} \varphi^{W}_{\varepsilon\delta}$ $(n_\delta \to \infty)$ is exactly the map

which we studied in the first 3 days)

M_ε quantize this limit is the time C map

of X_h $\quad h = \dfrac{|ZW| \sqrt{1+|z|^2+|w|^2}}{(1+z^4+w^4)}$ etc.

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So it does **not** converge to a complete integrable system.

I will finally explain why I believe that the conjecture to hold for ω_0 certain Kähler form ω_0 on $X \setminus M_0$

I first recall a conjectured relation between

SYZ fibration, Calabi-Yau metric & Gromov-Hausdorff convergence

Let $\pi : X \longrightarrow D^2$ be a maximal degenerate family of Calabi-Yau manifolds.

Let $M_\varepsilon = \pi^{-1}(\varepsilon)$. We take its Ricci flat Kähler metric g_ε st $\mathrm{Diam}(M_\varepsilon, g_\varepsilon) = 1$.

Conj $\lim_{\varepsilon \to 0} (M_\varepsilon, g_\varepsilon) = (B, d_B)$ ← Base space of CYZ fibration

↑ GH limit

Moreover $\exists\, d_\varepsilon$ a metric on $M_\varepsilon \cup B$

s.t. 1) $d_\varepsilon|_M = d_M$, $d_\varepsilon|_B = d_B$

2) M_ε, B are ε dense in $M_\varepsilon \cup B$.

3) $\exists x \in M_\varepsilon$, $\exists y \in B$ $d(x, y) < \varepsilon$

(Note $\exists\, d_\varepsilon \iff$ "GH dist. between $M_\varepsilon, B < \varepsilon$")

moreover $d(\pi(x), y) < \varepsilon$ $\pi : M_\varepsilon \to B$ SYZ.

Furthermore it is conjectured that

$U_\varepsilon(P)$ ε nbd of $P \subset B$

then the sectional curvature of M_ε on $\pi^{-1}(B \setminus U_\varepsilon(P))$

is uniformly bdd.

Namely sectional curvature blow up at the singular fibers

In this situation $B \setminus U_\varepsilon(P)$ is a Rie. mfd

and $\exists$ Rie. sub $\pi^{-1}(B \setminus U_\varepsilon(P)) \longrightarrow B \setminus U_\varepsilon(P)$

(F, 1982)

This Rie sub is expected to coincides with $\pi : M \to B$.

$=$

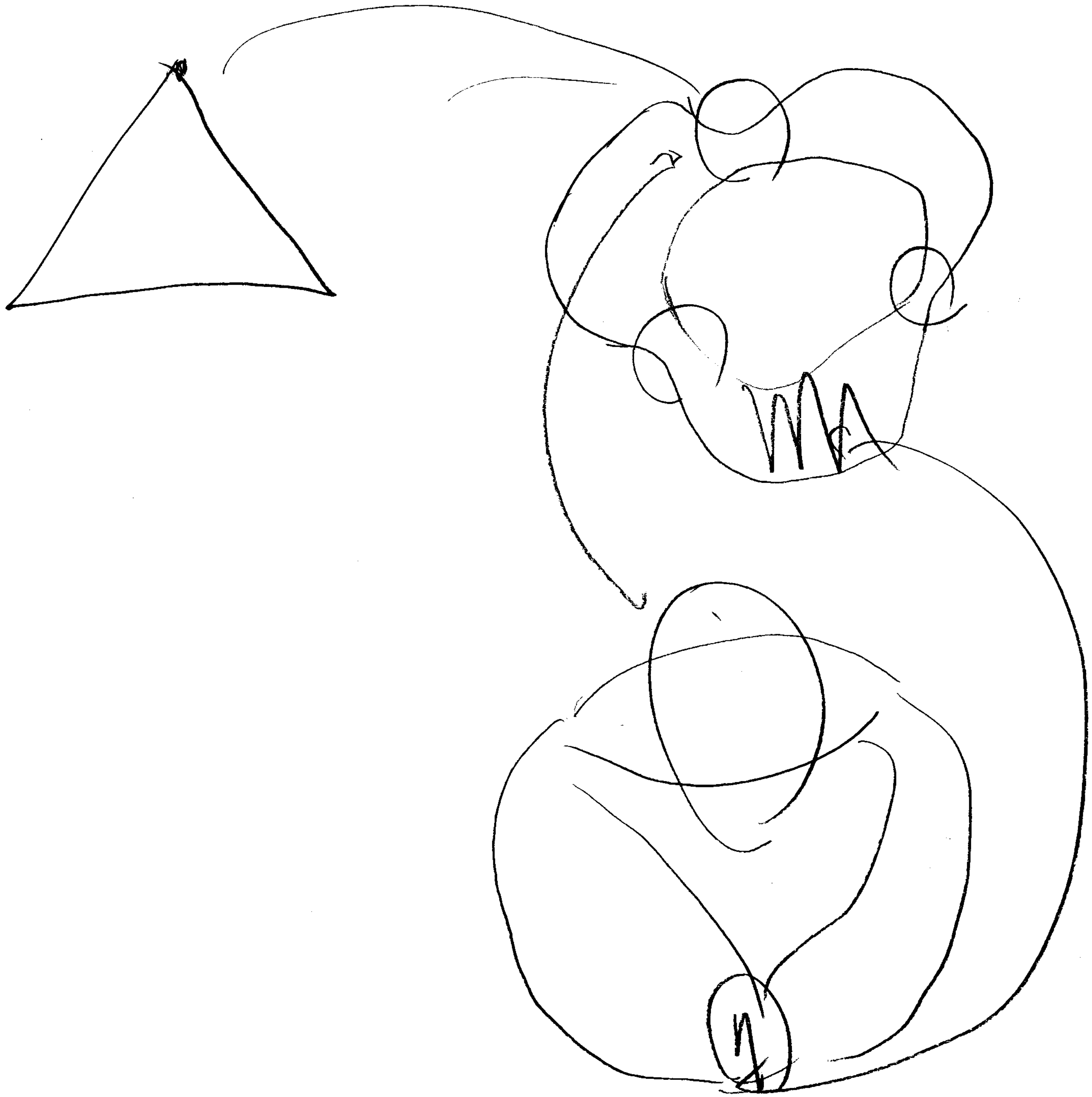
Now we consider the case $M_\varepsilon = \pi^{-1}(\varepsilon)$

$\pi : X \to D^2$

$p \in M_0$ $p \in$ pt1 int of irr. components

It is also expected

$$\lim_{\varepsilon \to 0} \mathrm{Vol}\left(M_\varepsilon \cap \bigcup_p B_\varepsilon(p)\right) / \mathrm{Vol}\, M_\varepsilon = 1$$



cpx coordinate

Riemann geometry

The part which looks "small" in complex coordinate of X

is "big" in the Riemannian geometry of M_ε

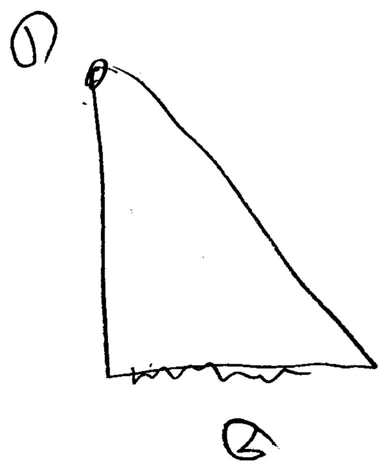
☆

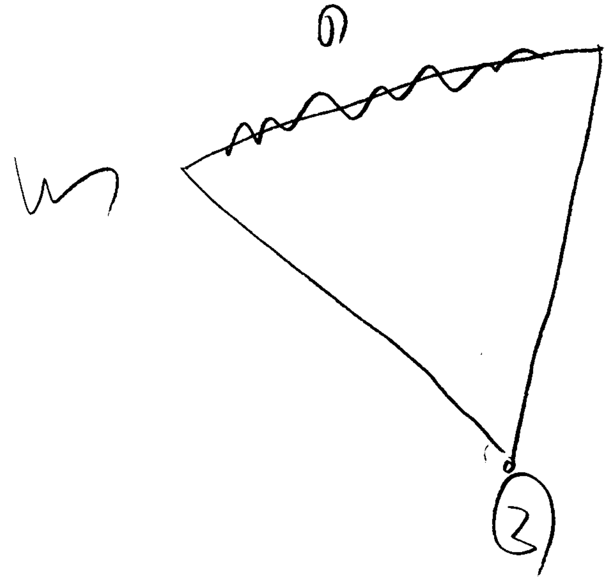
When we regard B as the GH lim of M_ε

it is $\partial P^\vee$ of dual Polygon $P^\vee$ of P

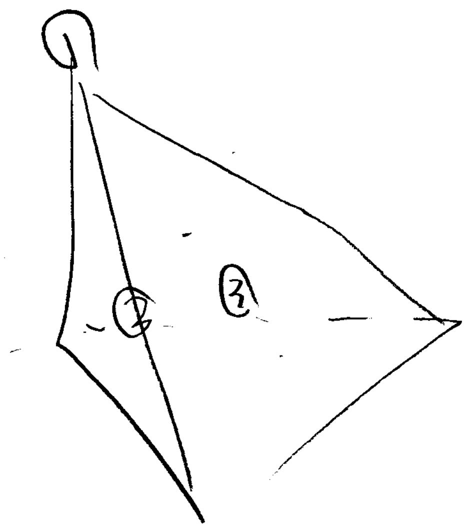
rather than ∂P

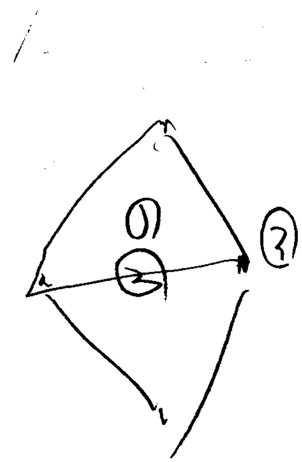
$\mathbb{P}^2 \supset$ cubic curve





$\mathbb{P}^3 \supset$ quartic surface





Now we consider W^{a}.

<u>Example</u> $\mathbb{P}^{n+1} \xrightarrow{\pi} \mathbb{P}^1$

$D = \pi^{-1}(0)$ D is a blow up of

$n+2$ copies of $\mathbb{P}^n \subset \mathbb{P}^{n+1}$

$$\underbrace{z_0=0}_{} \cup \underbrace{z_1=0}_{} \cdots \cup \underbrace{z_{n+1}=0}_{}$$

$$D = \bigcup_{i=0}^{n+1} D_i$$

$\mathbb{P}^{n+1} \setminus D_i \cong \mathbb{C}^{n+1}$ $\qquad$ $\mathbb{C}^{n+1} \setminus 0 = S^{2n+1} \times \mathbb{R}_\tau$

θ contact form on S^{2n+1} $\qquad$ $r = e^\tau$ $\;(r = |\vec{z}|)$

ω_i Kähler form on $\mathbb{P}^{n+1} \setminus D_i$ st

$\omega_i = d(r\theta) = dr \wedge \theta + r\, d\theta$ outside compact set.

(This is the standard choice in ST-T.)

Put $\omega_0 = \sum_{i=0}^{n} \omega_i$

Actually we might need to modify it near the basic locus.

We calculate grad $H = X_G$ wrt the w'.

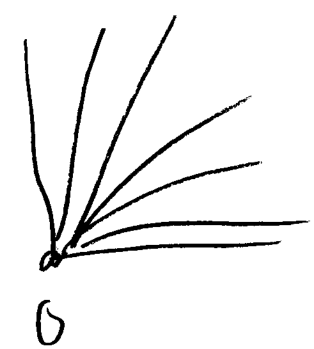
Note for w grad H

This is what W.D. Kuhn used

In a neighborhood of $[1:0:\cdots:0] = 0$

$$\omega_0 = \sum_{i=1}^{n} \frac{1}{r_i^2} (dr_i \wedge d\theta_i) + \text{smaller}$$

($z_1 \cdots z_n$ is homogeneous coordinate of $\mathbb{C}^n \subset \mathbb{C}P^{n+1}$, $0 \in \mathbb{C}^n$, $z_i = r_i e^{2\pi i \theta_i}$)



$$-\operatorname{grad} H \doteq \sum \operatorname{grad} r_i \, r_1 \cdots \hat{r}_i \cdots r_n$$

$$\doteq \sum_i r_1 \cdots r_{i-1} r_i^2 r_{i+1} \cdots r_n$$

$\frac{dr_i}{dr_j} \sim \frac{r_i}{r_j} \Rightarrow (r_1(t), \cdots, r_n(t))$ is asymptotic to straight lines.

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Thus

Observation

For almost all pts $p \in M_\varepsilon$

$\gamma_p(t)$ $\quad\left(\frac{d\gamma_p}{dt} = \operatorname{grad} H \right)$

converges as $t \to \infty$ to one of the vertex point

1st consequence

$\psi_{\varepsilon\delta}^{-1}\,\varphi_{\varepsilon}'\,\psi_{\varepsilon\delta}$ satisfies conclusion of KAM.

i) We can check KAM holds near each vertex point.

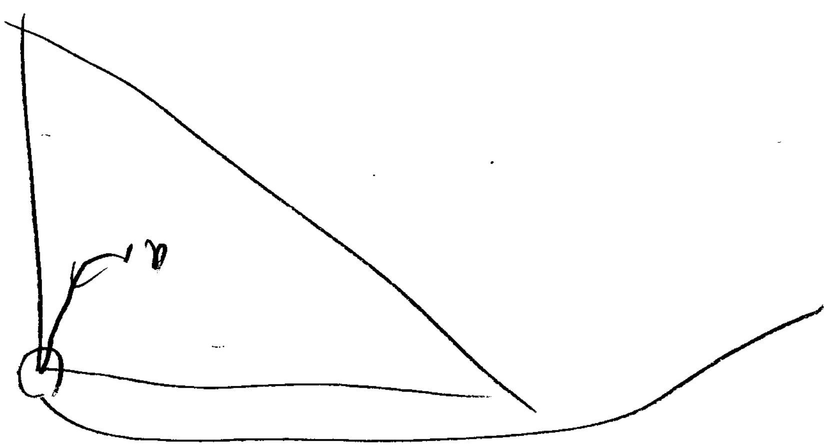
2nd Consequence

It seems possible we obtain a map

$$M_\varepsilon \longrightarrow \partial(P^\vee) \quad \text{dual polygon}$$

by $\lim_{t\to\infty} \gamma_p(t)$

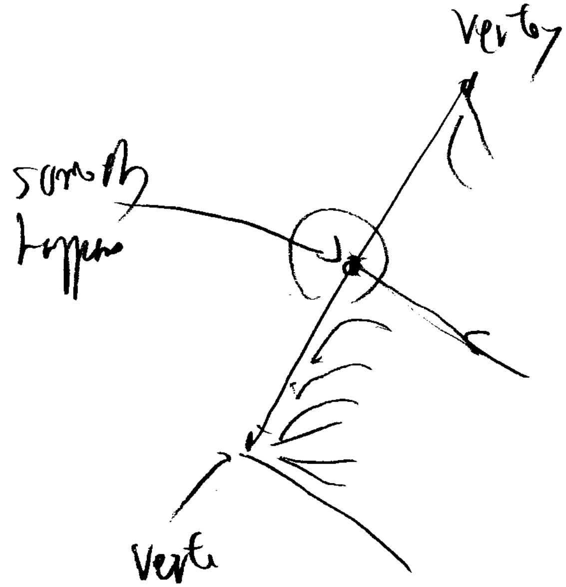
$\operatorname{grad} H \doteq r \sum r_i \frac{\partial}{\partial r_i}$

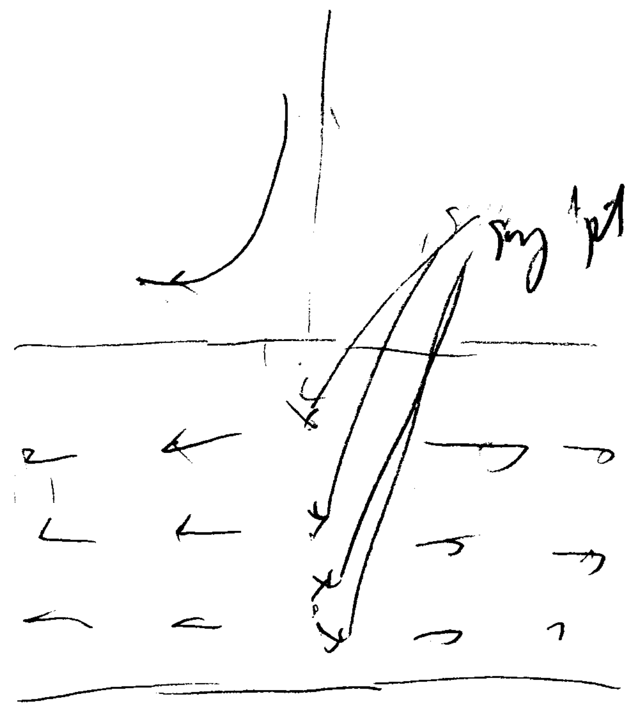


associate "angle as $t\to\infty$"

What happens in the case of quartic.

Nbd of edges

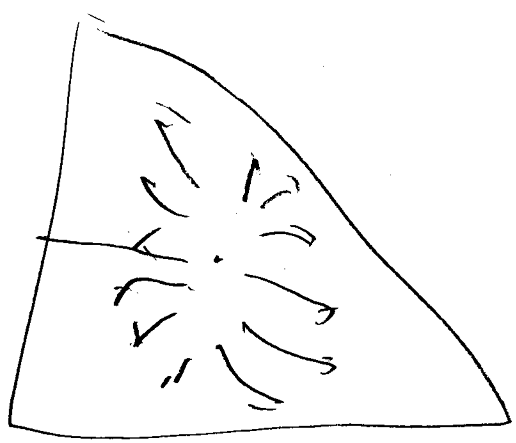


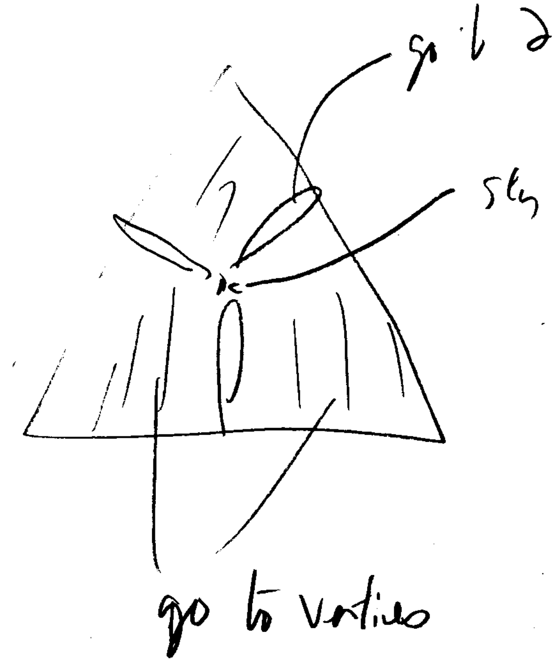


So near the edge outside sing fibe A_4

everything goes to the edges by $\gamma_p(t) \xrightarrow[t\to\infty]{}$

Near the faces





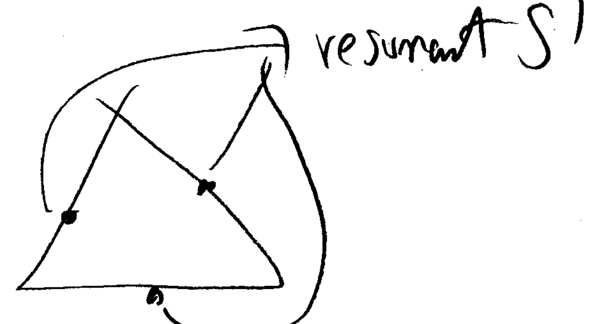
Something around barycenter does not go to vertex.

This does not correspond to singular fiber.

But a resonant T^2.

($\varphi'_\varepsilon : \mathbb{C}^3 \to \mathbb{C}^3$ the fiber T^2 is fixed by φ'_ε)

A simple case $T^2 \to T^2$



Thus we might see that

Chaos $\varphi_2 : K3 \to K2$ burn near singular fibers + resonant fibers.

We might be analyse how it occurs in a way similar to the first 3 days of lecture.

To make these observation rigorous we need to understand w^0 near the basic locus, which I do not know yet