Celebrating John Milnor

By Simon Donaldson,
Permanent Faculty Member
Simons Center for Geometry and Physics

Photo courtesy Institute for Mathematical Sciences, Stony Brook University

On February 20th. 2021, John Milnor celebrated his 90th. birthday. Milnor is a legendary figure in mathematics. A volume to mark his 60th. birthday contained surveys by different experts on Milnor's work in: Algebraic Topology, Differential Topology, Differential Geometry, Algebra, Singularity Theory and Dynamics. It would require the whole space of this *Newsletter* and, more important, a whole team of writers, to give any kind of real account of his work—stretching over seven decades— and its impact on these diverse branches of mathematics. What follows is some small partial substitute for that.

It is hard not to begin with Milnor's first paper—written while he was an undergraduate. Let Γ be a closed space curve. The curvature of Γ is the rate of change of its unit tangent vector with respect to arc length along the curve. In the case when Γ is a convex plane curve the integral of the curvature is exactly 2π—the tangent vector goes around exactly once. Milnor's result is that for a *knotted* space curve the integral must exceed 4π. Thus it makes a connection between differential geometry and topology. The proof is based on a beautiful formula. For each unit vector z define N_z to be the number of points on Γ where the velocity vector is orthogonal to z. Milnor's formula states that the integral of the curvature is π times the *average value* of N_z, as z ranges over the unit sphere. Now N_z is at least 2. For z defines a "height" function $x \mapsto \langle x, z \rangle$ and N_z is the number of stationary points of the height function on Γ. This number must be at least two—because the function has a maximum and a minimum—and (for almost all z)it is an even number. If there is any z with $N_z = 2$ then one can see that the curve is unknotted. So for a knotted curve N_z is at least 4, and some further arguments show that it cannot be equal to 4 almost everywhere, giving the result.

Best-known of all Milnor's results is his discovery of exotic spheres. From the beginning of the study of manifolds there were different definitions and directions, based on triangulations, differentiable maps or general continuous maps. Much of 20th. century topology was taken up with unravelling the distinctions between these and Milnor's result was a key, and unexpected, step in that. He showed that there are manifolds Σ_k which are homeomorphic to the seven dimensional sphere but not "diffeomorphic". That is, it makes a real difference whether one works with continuous or differentiable maps.

The integral of the Gauss curvature of a closed surface, divided by 2π, is an integer topological invariant —the Euler characteristic. In higher dimensions, for a closed oriented smooth manifold M of dimension $4m$ there is another integer topological invariant, the signature $\sigma(M)$. A formula of Hirzebruch expresses this as the integral of a certain expression in other topological constructs called the Pontrayagin classes $p_1, p_2, \dots$.

$$\sigma(M) = \int_M L(p_1, p_2, \dots).$$

The key point is that $L(p_1, p_2 \dots)$ is a polynomial with *rational* co-efficients so that *a priori* the right hand side is a rational number while the left hand side is an integer. In dimension 8 the polynomial is $L = \frac{7}{45}p_2 - \frac{1}{45}p_1^2$. Milnor's 7-dimensional manifolds Σ_k are easy to describe. The integer k is required to be equal to 2 modulo 4. Each Σ_k comes as the boundary of an 8-manifold with boundary. Adding a cone over Σ_k gives a space M_k which is in fact a closed *topological* 8-manifold. The signature of M_k is equal to 1 and the integral of p_1^2 is equal to k^2. So if M_k is a smooth 8-manifold there must be an integer $\mathbf{p}_2$ satisfying the Hirzebruch formula

$$1 = \frac{7}{45}\mathbf{p}_2 - \frac{1}{45}k^2,$$

which is only possible if $45 = -k^2 \bmod 7$. If $k = 6$ (say) this is not true, so M cannot be made into a smooth 8-manifold, which implies that Σ_k is not the standard sphere. This discovery was soon developed into a comprehensive theory by Milnor and Kervaire, which among other things began a new interaction between number theory and topology, both in connection with exotic spheres and also in associated questions in homotopy theory. The general recipe for the L-polynomials involves the power series expansion of the function $x/\tanh x$ whose coefficients are multiples by elementary factors of the *Bernoulli numbers*, well-known in number theory.

Milnor is renowned for the perfection of his writing, both of research papers and a series of books which have educated generations of mathematicians. Many of these contain gems of exposition slightly auxiliary to the main purpose of the volume. The student who wants to learn Riemannian geometry should take the "rapid course" in the last part of *Morse Theory*; to learn about connections and the Chern-Weil theory, or find information about Bernoulli numbers, there is no better place than the Appendices in *Characteristic classes*. In geometric topology it is easy to make mistakes, to be lead astray by "hand-waving" arguments. In Milnor's books there is no hand-waving: everything is carefully written down but in a completely lucid and readable style.

Milnor's book *Singular points of complex hypersurfaces* has been particularly influential. It develops a beautiful mathematical picture. Let f be a polynomial function of $n+1$ complex variables $z_0, \dots z_n$ which vanishes, along with its derivatives, at the origin. Then the origin is a singular point of the complex subvariety V defined by the equation $f(z_0, \dots, z_n) = 0$. Milnor studies the "link" K, which is the intersection of V with a small sphere S centred at the origin. Even the simplest examples produce interesting topology. If $n = 1$ and f is $z_0^2 + z_1^3$ then K is a trefoil knot (thinking of the 3-sphere S as the union of $\mathbf{R}^3$ and a point at infinity). If $n = 4$ then suitable polynomials give exotic 7-spheres K. In his book, Milnor introduced what is since called the *Milnor fibration*, which is the map $g = f/|f|$, from the complement $S \setminus K$ to the unit circle in $\mathbf{C}$. There is a delightful interplay between the algebraic geometry of the polynomial f and the topological properties of this map and this has been the starting point for many further developments. In one direction this structure gives an example of an *open book decomposition* with "pages" the fibres $g^{-1}(\theta)$, meeting along the "binding" K. Such decompositions have become very important in contact geometry.

There is so much more than one could write. To pluck out some further examples; Milnor has seminal work on Ricci curvature and eigenvalues of the Laplacian, on homotopy theory, on algebraic K-theory and of course over the past 40 years on dynamical systems. But my space is used up, so let us end by sending our congratulations, our admiration and our wishes for many happy returns from all in the Simons Center.

Abel Laureate John Milnor and Ragni Piene, chair of the Abel committee, at the award ceremony in Oslo, Norway
Photo: Kyrre Lien/The Abel Prize/The Norwegian Academy of Science and Letters.