Vladimir Dragovic (The University of Texas at Dallas) "Integrable Billiards, Extremal Polynomials, and Isoharmonic Deformations" (WEEK 4: October 30 – November 3, 2023

Abstract. The aim of the course is to present a recently established synergy of integrable billiards, extremal polynomials, Riemann surfaces, combinatorics, potential theory, and isomonodromic deformations. The cross-fertilization between ideas coming from these distinct fields has been leading to new results in each of them.

To make the course reasonably self-contained, we first review the basic notions related to integrable billiards in the plane and in higher dimensions, Poncelet Theorem, and elliptic curves and functions.

A comprehensive study of periodic trajectories of the billiards within ellipsoids in the d-dimensional Euclidean space is then presented. The novelty of our approach is based on a relationship established between the periodic billiard trajectories and extremal polynomials of the Chebyshev type on systems of d intervals on the real line. The positive answer to all three conjectures of Ramirez Ros from 2014 is given. The classification of periodic trajectories has led to a new combinatorial object: billiard partitions.

The case studies for d=2 as well as of trajectories of small periods n, $d \le n \le 2d$ for $d>2$ are given. A complete catalog of billiard trajectories with small periods is provided for $d = 3$.

A method of construction of algebro-geometric solutions of Schlesinger systems is presented. For an elliptic curve, a meromorphic differential is constructed with the following property: the common projection of its two zeros on the base of the covering, regarded as a function of the only moving branch point of the covering, is a solution of a Painleve VI equation. This differential provides an invariant formulation of a classical Okamoto transformation for the Painleve VI equations. A generalization of this differential to hyperelliptic curves is also constructed. The corresponding solutions of the rank two Schlesinger systems associated with elliptic and hyperelliptic curves are constructed in terms of these differentials. The initial data for the construction of the meromorphic differentials include a point in the Jacobian of the curve, under the assumption that this point has non-variable coordinates with respect to the lattice of the Jacobian while the branch points vary. This method is motivated by an observation of Hitchin, who related the Poncelet polygons to algebraic solutions of a Painleve VI equation.

Developing these ideas further, we establish dynamics of Zolotarev polynomials as extremal polynomials on the union of two intervals which satisfy a polynomial Pell equation. We introduce a new notion of isoharmonic deformations. We study their isomonodromic properties in the first nontrivial examples, and indicate the genesis of a new class of isomonodromic deformations. We interpret the results in terms of integrable billiards.

The course is based partially on results obtained jointly with Milena Radnovic and partially on results obtained jointly with Vasilisa Shramchenko. The following references will be used:

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3. V. Dragovic, V . Shramchenko, Algebro-geometric solutions of the Schlesinger systems and the Poncelet-type polygons in higher dimensions, International Math. Research Notices, 2018, Vol. 2018, No 13, p. 4229-4259.

4. V. Dragovic, V. Shramchenko, Algebro-geometric approach to an Okamoto transformation, the Painleve VI and Schlesinger equations, Annales Henri Poincare, 2019, Vol. 20, No. 4, 1121–1148.
5. V. Dragovic, M. Radnovic, Caustics of Poncelet polygons and classical extremal polynomials, Regular and Chaotic Dynamics, 2019, Vol. 24, No. 1, p. 1-35.
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