On algebraically integrable planar, dual and projective billiards

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A *caustic* of a strictly convex planar bounded billiard is a smooth curve whose tangent lines are reflected from the billiard boundary to its tangent lines. The famous Birkhoff Conjecture states that if the billiard boundary has an inner neighborhood foliated by closed caustics (such a billiard is called *integrable*), then the billiard is an ellipse. It was studied by many mathematicians, including H.Poritsky, M.Bialy, S.Bolotin, A.Mironov, V.Kaloshin, A.Sorrentino and others. It is a well-known folklore fact that integrability of a billiard Ω is equivalent to Liouville integrability near the boundary of the *billiard flow:* the geodesic flow on $T\mathbb{R}^2|_\Omega$ with reflections from the boundary.

In 1997 Sergei Tabachnikov [5] introduced *projective billiards:* planar curves equipped with a transversal line field, which defines a reflection acting on oriented lines. They are common generalization of billiards on surfaces of constant curvature. Tabachnikov suggested a generalization of the Birkhoff Conjecture for projective billiards. It would imply the original Birkhoff Conjecture and its versions for billiards on surfaces of costant curvature.

In the minicours we present a survey of the Birkhoff Conjecture and solutions of its different algebraic versions:

1) Bolotins Conjecture stating that each billiard with C^2-smooth connected boundary whose flow admits a non-trivial first integral polynomial in velocity is bounded by a conic, and the integral can be chosen quadratic (joint theorem with Misha Bialy and Andrey Mironov [1, 2]).

2) Its corollary stating that if the billiard on a C^2-smooth connected planar curve γ has a complex algebraic caustic, then γ is a conic [3].

3) Rational dual version of the Tabachnikov Conjecture. Consider a closed smooth strictly convex curve $\gamma \subset \mathbb{RP}^2$ equipped with a *dual billiard structure:* a family of non-trivial projective involutions acting on its projective tangent lines and fixing the tangency points. *Suppose that its outer neighborhood admits a foliation by closed curves (including γ) such that the involution of each tangent line permutes its intersection points with every leaf. Suppose in addition that the latter foliation has a* **rational first integral.** *Then γ and the leaves are* **conics** *forming a* **pencil** [3].

4) Classification of germs of C^4-smooth planar curves γ equipped with a *rationally integrable* dual billiard structure: this means that there exists a non-constant rational function $R(x, y)$ whose restrictions to the tangent lines to γ are invariant under the corresponding involutions [3].

5) Dual result: classification of germs of C^4-smooth planar curves equipped with a *rationally 0-homogeneously integrable* projective billiard structure,

i.e., having a non-constant rational 0-homogeneous first integral in the velocity [3].

6) Classification of *piecewise smooth* rationally 0-homogeneously integrable projective billiards and their dual objects: rationally integrable dual multibilliards [4].

In 4) and 5) we show that the underlying curve is always conic. But unexpectedly, the list of rationally integrable dual (or projective) billiard structures on a conic contains not only the standard examples given by pencils of conics, but also *two infinite series of exotic examples* with integrals of *arbitrary even degrees* greater than two, and five other exotic examples.

Unexpectedly, we show that there exist rationally integrable piecewise smooth projective billiards associated to dual pencils of conics with minimal degree of integral being equal to **12**, while for polynomially integrable piecewise smooth Euclidean billiards (which are associated to confocal conical pencils) the minimal degree can be equal only to 1, 2, or 4.

We present proofs of results 2) and 4) by methods of complex algebraic geometry and singularity theory. We also explain how to retrieve results 1), 3) and 5) from result 4).

References

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