The Work of Peter Scholze

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The work of Peter Scholze has created a revolution in the field of arithmetic geometry by introducing, in a very short time, a panoply of new ideas, tools and objects and by proving wonderful theorems, some unexpected, and some that have been longstanding central questions. It is clear that these new tools are fundamental and that even more beautiful mathematics has yet to emerge from his work and ideas.

Buzzwords. Here are two partial lists related to his already monumental body of work. *Objects and tools*: perfectoid spaces; pro-étale topology for adic spaces; diamonds; pro-étale topology for schemes with Bhargav Bhatt; v-topology for perfectoid spaces; prismatic cohomology with Bhargav Bhatt, with important precursors integral p-adic Hodge Theory with Bhargav Bhatt and Matthew Morrow. *Results*: new proof of the Local Langlands Correspondence for the general linear group; constructions of Galois representations in the Langlands Program; proof of important special cases of the Monodromy Weight Conjecture; Hodge Theory of smooth proper rigid analytic spaces; period maps for families of smooth proper rigid analytic spaces; prismatic cohomology for schemes over p-adic rings and ensuing comparison theorems. I will only touch upon some of these buzzwords.

Algebraic, Arithmetic and p-adic geometry

Taxonomy. Algebraic geometry studies solutions to polynomial equations, often over algebraically closed fields. Arithmetic geometry studies solutions over special fields and rings: the integers, the p-adic integers, the rational and p-adic fields, their finite extensions, finite fields, perfect fields, etc.

Algebraic varieties. Solutions are organized into spaces: algebraic varieties. One then studies the properties of these varieties: non-emptiness, connectedness, irreducible components, dimension, smoothness, singularities, different kind of cohomology groups: étale, algebraic de Rham, crystalline, etc.

Field extensions. Unique prime factorization implies that the equation $x^2 = 2$ has no rational solution. If we take the 2-adic field and we apply the 2-adic norm to both sides of the equation, then we reach the same conclusion. We thus learn that looking for solutions over the extensions of a field can tell us something about solutions over the field. The system of finite extensions of a given field, encoded in its absolute Galois group, is an essential tool in studying varieties over non algebraically closed fields.

Why p-adic fields? Here are three classical results that point to a partial answer. *Ostrowsky Theorem*: every non-trivial absolute value on the field of rational numbers is equivalent to the usual real absolute value or to a p-adic absolute value. Completion of the rationals with respect to these absolute values yields the reals and the fields of p-adic numbers. *Hasse Local to Global Principle*: certain types of equations have rational solutions if they have real solutions as well as p-adic solutions for every prime p. *Hensel Lemma*: if a univariate polynomial has a simple root modulo the prime p, then it has a solution over the p-adic integers.

Rings of integers, residue fields, mixed and equal characteristic. By asking for solutions over unital commutative rings, we can do algebraic geometry over such rings. A p-adic field, which is an example of a non-Archimedean local field, contains the subring of p-adic integers (defined by the p-adic norm being at most 1) which, divided by p, gives us a finite residue field with p elements. This kind of picture remains true for all the non-Archimedean local fields, i.e. the finite extensions of p-adic fields and the fields of Laurent series over finite fields, these latter necessarily of cardinality a power of a prime p. In the former case, we say we are in a situation of mixed characteristic 0 and p, in the latter, of equal characteristic p. If we do algebraic geometry over the resulting ring of integers, we can then take the "general" geometry over the local field (i.e. the quotient field), or the "special" one over the finite residue field. The two are related in a subtle way, and this is a thread of fundamental importance, woven into the very fabric of modern algebraic and arithmetic geometry.

Rigid analytic spaces. By viewing polynomials as holomorphic functions, we can view complex algebraic varieties as complex analytic spaces and then bring into the fold the tools of complex geometry and topology, e.g. singular and de Rham cohomology. John Tate extended this paradigm to varieties over non-Archimedean local fields, by using their non-Archimedean norms, and he built what we call rigid analytic spaces. Rigid analytic geometry has become relevant in various areas: Local Langlands Correspondence; Abyhankar's Conjecture concerning fundamental groups of curves in positive characteristic; modularity of Galois representations; p-adic Hodge decomposition and comparison of algebraic de Rham cohomology with étale cohomology for varieties over p-adic fields.

Étale and crystalline cohomology. A Grothendieck topology is governed by a class of morphism into a space as a means to define coverings, vs merely inclusions of open sets. Such a notion makes it possible to introduce sheaf and cohomology theory. The spectacularly successful étale cohomology, envisioned by Alexander Grothendieck and brought to fruition by Pierre Deligne's proof of the Weil Conjectures, singles out the class of étale morphisms. We can think of étale morphisms as analogues of local homeomorphisms with fibers finite sets (this is true over the complex numbers, if we take the classical topology). There are many Grothendieck topologies, and comparing the resulting cohomology theories is often a difficult, but important task, for example when trying to understand when a principal bundle is locally trivial in some topology. Étale cohomology with coefficients p-adic integers is pathological when working with varieties over fields of characteristic p. Crystalline cohomology is Grothendieck's remedy to this pathology.

A Glimpse into the Work of Scholze

Fontaine-Wintenberger equivalence. Consider the characteristic zero field L obtained from the p-adic field by extracting for every n the p^n-th roots of p. Consider the characteristic p field $L^\flat$ obtained from the field of Laurent series over the finite field with p elements by extracting all the p^n-th roots of the variable t. Then the absolute Galois groups of L and $L^\flat$ are canonically isomorphic: the corresponding systems of finite field extensions for the two fields correspond to each other. For example, the splitting fields of x^2-p and x^2-t correspond to each other via literally replacing p with t. This is a shadow of an important and deeply magical principle underlying the perfectoid techniques, namely that we can somehow treat the prime number p as a variable. The isomorphism of Galois groups allows to compute certain invariants for L by computing them for $L^\flat$. For example, the cohomological dimension with coefficients in the finite field with p elements of the field $L^\flat$ is classically known to be ≤ 1, so that the same is true for L. The isomorphism of Galois groups also implies that the étale topologies of the two fields are identified.

Tilting. In algebraic geometry, fields give rise to zero dimensional varieties; their étale topology is not trivial as it is governed by their absolute Galois group. One early, deep and seminal intuition of Scholze's is to view the Fontaine-Wintenberger equivalence as the zero dimensional instance of a far more general phenomenon, called tilting equivalence, relating geometries in mixed and in zero characteristic to ones in characteristic p, with little loss of information.

Perfectoid fields and spaces. A characteristic p field is said to be perfect if the Frobenius operation of raising to the p-th power is surjective on it. Fields of characteristic zero are defined to be perfect. The fields L and $L^\flat$ are examples of what Scholze calls perfectoid fields (for the given prime p): in particular, they contain a certain subring that divided by the prime p gives a ring where Frobenius is surjective. Given a perfectoid field K of characteristic zero, there is a canonical perfectoid field $K^\flat$ of characteristic p associated with it, called its tilt. Given a perfectoid field κ, one constructs the category of perfectoid κ-spaces by using certain algebras, called perfectoid affinoid, as building blocks. The resulting spaces are special (Roland Huber) adic spaces, a generalization of Tate's rigid analytic spaces.

Tilting Equivalence. Scholze proves that there is an equivalence of categories, called tilting, between perfectoid K-spaces and perfectoid $K^\flat$-spaces. Something even deeper is true: the equivalence preserves the étale topology, which is key to the transfer of cohomological information from one side to the other.

The Monodromy Weight Conjecture. Let us start with a nonsingular projective variety over one of our local fields (p-adic, or formal Laurent series). We pass to the algebraic closure and now the étale cohomology of the variety carries an action of the absolute Galois group of the field. From this set-up, we extract two operators on the étale cohomology: a logarithm of monodromy, and a Frobenius-like operator coming from Frobenius on the finite residue field. These operators induce two filtrations on the cohomology of the variety. Pierre Deligne's *Monodromy Weight Conjecture*, perhaps the single most important open question in the étale cohomology of algebraic varieties, asserts that these two filtrations coincide. Deligne proved this conjecture in the equal characteristic case by viewing the field of formal Laurent series as arising geometrically from a point on a curve over a finite field, and by using his machinery of weights (i.e. the eigenvalues of Frobenius). In mixed characteristic this conjecture has strong consequences in the context of Artin's Analyticity Conjecture on L-functions: the failure of analyticity may occur at worst in some vertical strip in the complex plane. Scholze has proved this conjecture in the mixed characteristic case of set-theoretic complete intersections of hypersurfaces in toric varieties (e.g. hypersurfaces in projective space) by using the theory of perfectoid spaces that he developed specifically for this purpose. There are no known examples of nonsingular projective varieties that are not of that kind. He constructs a perfectoid version of projective space, and, in it, a perfectoid version of the complete intersection—essentially an ϵ-neighborhood of the pre-image of the complete intersection from the ordinary projective space. He then uses his tilting equivalence to reduce to the known case of equal characteristic. Note that in the context of the tilting equivalence, there is no general prescription to produce perfectoids in characteristic zero starting from an algebraic variety: the tilting works, if you can produce one.

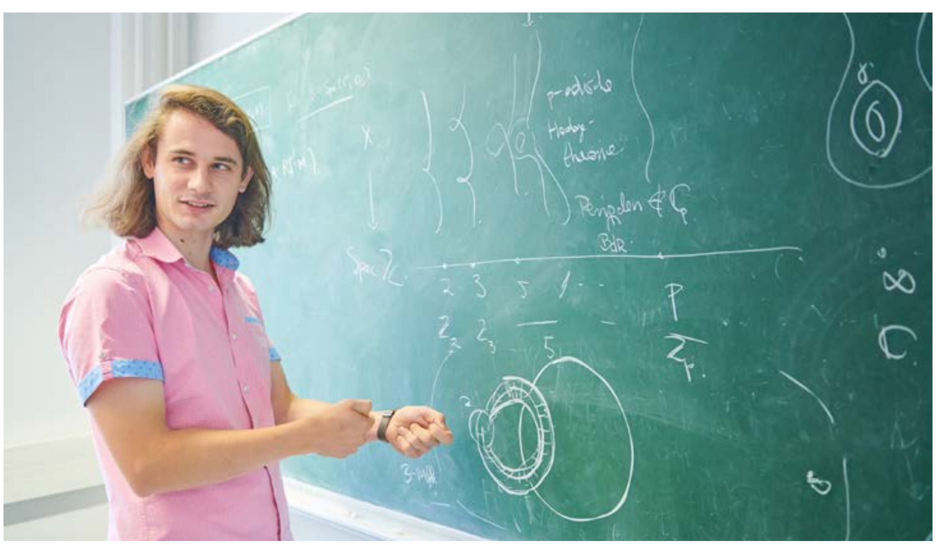
p-adic Hodge Theory. In the case of complex projective manifolds, we have the degeneration of the Hodge-to-de Rham spectral sequence: (p, q)-forms calculate a purely algebraic version of de Rham cohomology, which in turn coincides with the usual de Rham/singular cohomology. In other words, we have comparison theorems between different cohomology theories on complex varieties. Gerd Faltings had famously proved these kinds of results for varieties over p-adic fields. Scholze shows that we have similar comparison theorems on rigid analytic spaces. To do so, he uses his perfectoid theory and his new pro-étale topology: a key step is that he shows that any rigid-analytic space may be covered, with respect to the pro-étale topology, by affinoid perfectoids which have trivial higher cohomology. It is remarkable that, contrary to the complex case, Scholze's comparison theorems hold without any Kähler-type hypothesis.

Integral p-adic Hodge theory. These comparison theorems have been greatly refined by Scholze, with Bhatt and Morrow, when the rigid analytic space over the local field is already defined over the subring of its integers (norm ≤ 1). In this case, we have the spaces over the ring of integers (total space), over the p-adic field (general fiber) and over the residue field (special fiber). They prove comparison theorems relating the various ensuing cohomology groups (de Rham, étale, crystalline). Importantly, they build a complex of modules over a suitable ring that, through a combination of various homological algebra operations–e.g. tensoring with other modules over the ring–yields the different cohomology theories. In essence, they prove that these different theories all come from one formerly secret place.

Pro-étale topology of schemes. The étale topology and cohomology of schemes has been enormously successful, e.g. in the proof of the Weil conjectures, and up to very recently, it has been the theory to go to when one wants to use the intuition coming from the classical singular cohomology of complex varieties. This theory has a fundamental drawback: if we want to work with cohomology groups with coefficients in a field of characteristic zero, we must first develop the theory with finite coefficients and then take inverse limits: in short, we do not work with sheaves but with inverse systems of sheaves. The pro-étale topology and cohomology of Bhatt and Scholze is a fundamental and foundational re-working of étale cohomology, where one replaces the étale topology with the pro-étale topology. In short, one takes inverse limits of étale morphisms. Then sheaves are actual sheaves (not inverse systems of sheaves), the cohomology is the cohomology of these sheaves (and not an inverse limit of cohomology groups) and, in geometric situations, the pro-étale cohomology agrees with étale cohomology, and, finally, locally constant sheaves arise as representations of the pro-étale fundamental group. This change of topology, from étale to pro-étale, has brought enormous conceptual simplifications and these may lead to surprising developments.

Prismatic cohomology. White light enters a prism and we see colors exiting. The analogue of white light is a complex of modules over a special ring (the prism) constructed starting from a rigid analytic space over a kind of p-adic ring; the exiting colors are the various cohomology groups we can construct in this situation. These groups are extracted from the complex of modules by means of algebraic operations, such as tensoring with various modules over the ring. Bhatt and Scholze's Prismatic cohomology is the recently developed cohomology theory for schemes over p-adic rings that realizes this vision. It generalizes, unifies and illuminates p-adic, étale and de Rham/crystalline cohomology, as well as their relations to each other via p-adic Hodge theory.

In conclusion, let me quote Michael Rapoport's words from his Laudatio Lecture delivered on the occasion of the Fields Medal being conferred to his former student Scholze: "What is remarkable about Scholze's approach to mathematics is the ultimate simplicity of his ideas. Even though the execution of these ideas demands great technical power (of which Scholze has an extraordinary command), it is still true that the initial key idea and the final result have the appeal of inevitability of the classics, and their elegance." ♦



Peter Scholze
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