On Line Defects in Quantum Field Theory

Renormalization Group Flows and Applications

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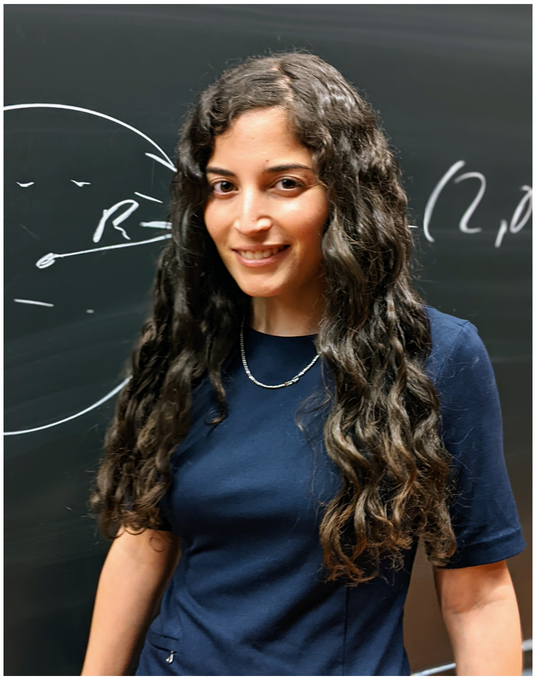


Quantum Field Theory (QFT) is a theoretical framework that has been successfully applied to diverse areas in physics ranging from high-energy particle physics to low energy condensed matter physics. It is the most accurate physical theory constructed to this day, being able to predict dozens of observables with outstanding accuracy and agreement with the measurements. Within this framework, the study of defects and boundaries in QFTs has attracted much scientific attention in recent years. The great relevance of such setups to physical systems in nature contributed to making it a forefront field of research these days, despite the various mathematical challenges involved in describing such systems. My research at the Simons Center for Geometry and Physics involved the study of line defects (one-dimensional defects) in QFTs and their applications to physical systems. I was fortunate to work in a collaboration of researchers from the SCGP on these topics, to whom I am grateful: Gabriel Cuomo, Zohar Komargodski and Márk Mezei. In this article, I will describe some of the progress that was achieved at the cutting-edge of this field of research.

Line Defects in QFTs

A natural question which arises in QFT is what are the implications of having defects inserted in some region of spacetime. In relativistic QFT, a defect can be thought of as an extended operator of the bulk QFT. Generally, expectation values of physical observables calculated in the presence of such operators can be modified due to their presence. Equivalently, the bulk and defect can be thought of as defining some new QFT which we refer to as the Defect Quantum Field Theory (DQFT). In this approach, it is convenient to consider a Lagrangian density $\mathcal{L}_{\text{DQFT}}$ which consists of the Lagrangian density of the bulk theory $\mathcal{L}_{\text{bulk}}$ and a Lagrangian density associated with the defect $\mathcal{L}_{\text{defect}}$ that is localized in some region of spacetime:

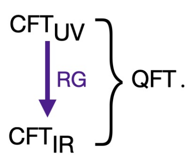
$$\mathcal{L}_{\text{DQFT}} = \mathcal{L}_{\text{bulk}} + \delta_D^{d-p}(x_\perp)\mathcal{L}_{\text{defect}}\,, \tag{1}$$

where p is the dimension of the defect and $\delta_D^{d-p}(x_\perp)$ is a Dirac-delta function localized on the p-dimensional defect. Various physical quantities can be calculated by analyzing $\mathcal{L}_{\text{DQFT}}$.

A somewhat simplified scenario of a DQFT, which nonetheless has many physical applications (as will soon be described), is when the bulk theory enjoys additional symmetries. In some cases, having additional symmetries in the bulk theory leads to nontrivial constraints on the physical behavior of the DQFT. This will be discussed in what follows.

Line Defects in CFTs

Our focus in this article will be on line defects in Conformal Field Theories (CFTs). Such systems are useful in describing many physical systems, for example they serve as frameworks to describe impurities in quantum critical models. These and other applications will be discussed in more detail later on in this article. Let us consider a d-dimensional flat space of Euclidean signature. A CFT is a quantum field theory that possesses an invariance under the group of conformal transformations, $SO(d+1,1)$. The conformal group consists of rescaling which do not change the relative angles between vectors. CFTs are of particular importance in physics, as they are present in a huge verity of physical systems, ranging from high-energy particle physics to astrophysics and to low energy condensed matter physics. Generically, QFTs become scale invariant in low energies (or long distances). It is very often the case[1] that scale invariance implies invariance under the full conformal group. The larger symmetry group provides useful tools in studying physical systems. Starting from some CFT in the UV (at high energies), one can trigger a renormalization group (RG) flow by introducing a mass scale to the theory, modifying it with a relevant perturbation. By that, one promotes the theory to some QFT. Note that physically, RG amounts at changing the resolution of the experiment. Generically, the flow is expected to end in a CFT in the deep IR (at low energies):



As in ordinary QFTs, in the case of DQFTs it is possible to define a Defect Conformal Field Theory (DCFT). For example, consider a p-dimensional flat defect inserted in a d-dimensional conformal bulk in a flat space of Euclidean signature. Generically, the presence of the defect will break the full conformal group $SO(d+1,1)$ to some subgroup. By using the term *DCFT* we refer to the case in which such a breaking leaves the system invariant under $SO(p+1,1)\times SO(d-p)$ subgroup of $SO(d+1,1)$. In other words, the DCFT is invariant under the maximal allowed subgroup that can be preserved by the defect. For instance, consider a straight line defect that extends along the d-th direction in space at $x^i=0$, where x^i is used to denote the coordinates in the directions orthogonal to the line (see figure 1).

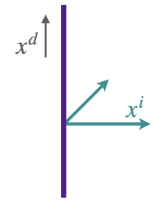


Figure 1. An illustration of a straight line defect that extends in the d-th direction in space.

We say that the defect is conformal and the theory is a DCFT when the system preserves the conformal group on the line, e.g. when it is invariant under $SL(2,\mathbb{R})\times SO(d-1)$, where the $SO(d-1)$ symmetry is associated with rotations along the orthogonal directions x^i, and the $SL(2,\mathbb{R})$ is the group of conformal transformations on the line, consists of translations along x^d, dilations and special conformal transformation. The latter three symmetries imply that the system at $x^i=0$ is invariant under transformations of the form:

$$x^{d'}=\frac{ax^d+b}{cx^d+d},$$

where $a,b,c,d\in\mathbb{R}$ and are subject to the constraint $ad-bc=1$.

RG Flows on Line Defects

Physical Applications

Line defects in conformal bulk theories provide a framework which is of relevance in describing various physical systems. In two-dimensions, examples include lattice systems with impurities, such as Kondo models [3, 4]. In three-dimensions, examples include symmetry or monodromy defects [5, 6] Well-known examples in four-dimensions include Wilson or 't Hooft lines in conformal gauge theories (see e.g. [7]).

In quantum critical systems, point-like impurities in space at zero temperature can be thought of as one-dimensional defects in spacetime. Consider a lattice which possesses an invariance under global $SO(3)$

[1]The question of whether and under which conditions scale symmetry implies invariance under the full conformal group is a fundamental question in QFT which is still under active study these days. Two and four dimensional unitary relativistic theories were analyzed in that context in [1, 2].

symmetry group, under which the atoms of the lattice transform (typically) in the spin $1/2$ representation. Replacing one of the atoms in the lattice with a doping atom that transforms in a spin S_{imp} representation (which is generally different than the representation of the other atoms in the lattice) under the global symmetry is an example of such a scenario, see figure 2 for an illustration. The impurity interacts with the degrees of freedom of the bulk in a way that preserves the global symmetry and allows for interesting dynamics to develop on the line theory. The Hamiltonian in such systems typically takes a form of $H = H_{\text{bulk}} + J_0 \vec{S}_{\text{imp}} \cdot \vec{\sigma}$, where the operators $\vec{S}_{\text{imp}}$ are in the spin S_{imp} representation of $SO(3)$, J_0 represents a coupling constant, and the operators $\vec{\sigma}$ represent the bulk spins and are in the spin $1/2$ representation. H_{bulk} is the bulk's Hamiltonian which is tuned to a quantum critical point. Note that this realization can be conveniently thought of in the form of eq. (1). Theories of these type were studied in various studies (see e.g. [8]-[10] and references therein). They are of particular importance due to their relation with magnets in three spacetime dimensions. Other $O(N)$ impurity models were analyzed e.g. in [11].
Another type of a DQFT with line defects which can be realized physically and have experimental applications is the external field defect. The external field defect can be constructed by taking a lattice and acting with a magnetic field which is localized in a particular direction in the fields space (as illustrated in figure 3). Such theories were recently studied in [12, 13]. In both types of examples, various physical predictions have been made, some of them were even recently supported by Monte-Carlo simulations [12], [14].

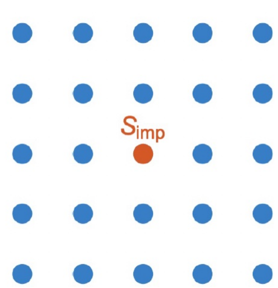


Figure 2. Illustration of a lattice with an impurity that transforms in the spin S_{imp} representation under a bulk's global $SO(3)$ symmetry.

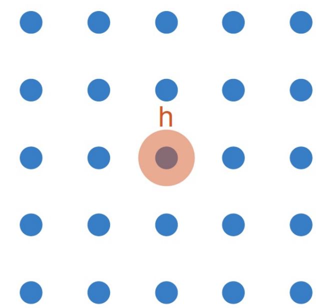


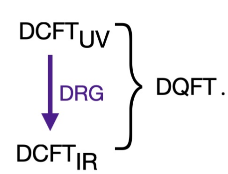
Figure 3. Illustration describing a magnetic field h localized on one of the atoms in a lattice.

Wilson lines are examples of line defects which are of broad interest to the study of high energy physics. Physically, a Wilson line describes the insertion of a probe particle that moves around some worldline. In the Hamiltonian formalism, the ground-state of the system changes due to the insertion of the probe particle. In the framework of Euclidean QFT, having a Wilson line amounts to adding a gauge-invariant line operator to the theory. Since their original introduction in [15] in the context of lattice gauge theories, Wilson lines have been broadly studied in many examples, and they possess various applications, one famous example includes Wilson lines in $\mathcal{N} = 4$ SYM (see e.g. [16, 17] and references therein).

In all the examples mentioned above, there is a nontrivial RG flow on the line defect, which is triggered while affecting the bulk very little away from the defect, such that the bulk remains conformal.

Monotonicity of the Defect Entropy

Let us consider a theory that is described by some DCFT in the UV. Upon turning on a RG flow on the defect (by introducing a mass scale), it will generically flow to a different theory. It is broadly general, though it does not always has to be the case, that at very long distances the flow will terminate in some DCFT in the deep IR:



A natural question which arises in that context is whether such a RG flow on the line defect exhibits monotonicity properties. Monotonicity theorems are of fundamental importance to the study of QFT: Interpolating between theories at high energies and low energies, they contribute to our understanding that the number of physical degrees of freedom reduces through the process of renormalization group flows. In QFT, monotonicity theorems (see e.g. [18]-[20]) are very useful in providing constraints on the low-energy physics. This can be done by, for example, eliminating the possibility of having RG limit cycles, in which upon turning a nontrivial RG flow the theory ends up in the same CFT in the IR as in the UV.

We now turn to focus on the following geometry: consider a circular line defect with a radius R that is placed in a bulk theory which is defined over a d-dimensional flat space of Euclidean signature. The bulk theory is assumed to be a CFT. One can also consider any other manifold $\mathcal{M}$ which is conformally equivalent (see figure 4 for examples).

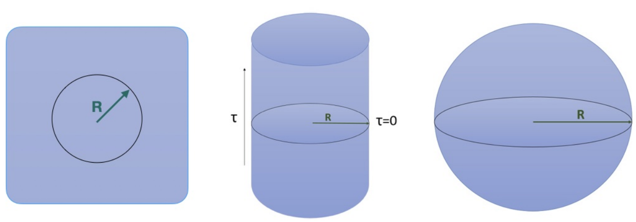


Figure 4. Left: an illustration of a circular line defect of radius R in a flat space of Euclidean signature $\mathbb{R}^d$. Center and right: illustrations of some conformally equivalent manifolds; the cylinder $\mathbb{R} \times S^{d-1}$ with a line defect placed on the equator of the S^{d-1} sphere with radius R at a fixed value of Euclidean time $\tau = 0$, and the S^d sphere with the line operator spanning a maximal circle of radius R.

In any of these geometries, one can define the following quantity s, which we refer to as the *defect entropy*:

(2) $$s(\mu R) \equiv \left(1 - R\frac{\partial}{\partial R}\right) \log g(\mu R),$$

where μ is the mass scale associated with the defect RG flow, and g is the defect contribution to the partition function, defined by:

(3) $$\log g \equiv \log Z_{\mathcal{M}} - \log Z_{\mathcal{M}}^{CFT}.$$

In the above definition, $Z_{\mathcal{M}}$ is the partition function of the theory in the presence of the defect, $Z_{\mathcal{M}}^{CFT}$ is the partition function without the defect (of the bulk theory alone), and both are defined over the same regime. $\log g$ (and therefore also the defect entropy s) depends only on the dimensionless product of μR. In particular, the defect entropy reduces to the radius-independent contributions to $\log g$ at the end points of the flow. We denote these values as $\log g_{UV}$ and $\log g_{IR}$. It has been shown in [21] that the defect entropy s is scheme-independent, free of any ambiguities, and satisfies the following gradient formula:

(4) $$\mu\frac{\partial s}{\partial \mu} = - R^2 \int_D d\phi_1 \int_D d\phi_2 \langle T_D(\phi_1) T_D(\phi_2)\rangle \left[1 - \cos(\phi_1 - \phi_2)\right],$$

where $\int_D$ stands for integration on the line defect, and T_D is the defect stress-tensor. In particular, it vanishes at the conformal fixed points. The latter is a result of the fact that line defects in DCFTs do not support any defect stress tensor. This follows from the simple argument that for a line defect the defect stress tensor has only a single component, and conformal invariance implies that it must vanish when the defect is conformal. The physical interpretation is that at a DCFT there is no possibility to localize energy on the line defect, and energy always ends up being smeared into the bulk. In the above formula, the $(1-\cos(\phi_1-\phi_2))$ which appears in the integrand guarantees that the resulting right-hand side of the equation is free of any divergences and ambiguities. In addition, the right-hand side is manifestly negative in a reflection positive theory. This implies that, in particular, in such theories $g_{UV} \geq g_{IR}$. Furthermore, the defect entropy s provides a monotonic function which decreases along the RG flow. The proof of the formula (4) is given in [21]. The main idea behind it is that by surrounding the defect with the conformal charges associated with $SL(2, \mathbb{R})$ (see figure 5), one gets nontrivial identifications once the defect is non-conformal.

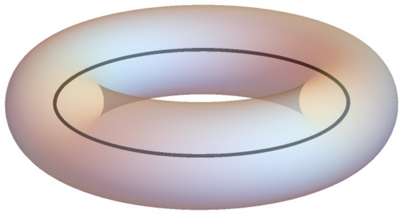


Figure 5. An illustration of a toroidal surface that wraps a circular defect.

These identifications relate certain theories with different space-dependent mass scales. They follow from the fact that the $SL(2,\mathbb{R})$ charges must vanish when integrated over a surface that wraps the defect without intersecting it (as in figure 5). The latter follows directly from the symmetries of the bulk, being a CFT. We see that the ambient CFT in the bulk places nontrivial constraints on the possible RG flows on the line defect. In particular, it guarantees that in reflection positive theories no limit cycles are allowed, and in general once there is a nontrivial RG flow on the line defect $DCFT_{IR} \neq DCFT_{UV}$. For example, note that from the definition (3) it follows that the trivial line (that is, if there is no defect at all) has $g = 1$, and $s = 0$. Therefore, upon turning on a nontrivial RG flow starting from the trivial line in the UV, one can eliminate the possibility that the flow will end with a trivial theory. In two dimensions, in the context of boundary conformal field theories, the inequality $g_{UV} \geq g_{IR}$ was famously conjectured to hold by Affleck and Ludwig [22]. In the regime in which the flow can be described in terms of finitely many beta functions, a gradient formula equivalent to eq. (4) was proposed in the context of string field theory in two dimensions [23, 24]. In [25], the authors established a proof for the two-dimensional claim, and an alternative proof was given in [26] using methods of quantum information. The formula (4) extends the well-known results in two dimensions to line defects in an arbitrary number of space-time dimensions.

Outlook

Line defects in CFTs provide an interesting physical setup which is of relevance in describing various physical systems. It is an open question whether one can find a proof for the irreversibility of the defect RG flows associated with line defects using methods of quantum information, in analogy to the proof of [26] in the two-dimensional case. Another interesting question is whether one can find bounds on the allowed values of the defect entropy s given a certain CFT in the bulk. Some progress in that direction has been made in [27] and more recently in [28].

Clearly, due to their importance as mathematical frameworks describing physical systems in nature, it is desirable to improve our understanding of higher dimensional defects and their implications in QFT. There are many interesting research directions to pursue. In the context of finding constraints over the space of RG flows, irreversibility of the defect RG flows associated with two- and four-dimensional defects (in bulk theories of dimensions $d \geq 3$ and $d \geq 5$ respectively) was proven in [29] and [30] via Weyl anomaly matching, using approaches similar to the one given in [19]. In both cases it was proven that $b_{UV} \geq b_{IR}$, where b is the coefficient of the Euler density in the defect's Weyl anomaly. Yet, a monotonic function that decreases through the defect RG flow remains unknown. In other cases, including the important setup of three-dimensional defects in bulk theories of dimensions $d \geq 4$, the question of whether the defect RG flows is reversible is an open question to this day.

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