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WHAT WE THINK ABOUT THE HIGHER DIMENSIONAL CHERN-SIMONS THEORIES: Preprint ITEP 91-70/

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The higher dimensional generalization of the 3d Chern-Simons gauge theory is considered. Connections with 4d gravity, moduli space of foliations and generalized links are discussed.

Fig. - , ref. - 5

1 Introduction

Recently one of the most interesting developments in mathematical physics was the investigation of the so-called topological field theories i.e. such theories which do not need a metric on the manifold for their definition and hence the observables of which are topologically invariant. The Chern–Simons (CS) functionals considered as actions give us examples the theories of such a type. The CS theory on a 3d manifold was firstly considered in the abelian case by A.S.Schwartz [1] and then after papers of E.Witten [2] there has been an explosive process of publications on this subject (see i.g. [4] and references therein). We are not going to give here a review instead we just recapitulate some of the main topics related to 3d CS theories:

topological invariants of the manifolds (like the Ray–Singer torsion) computed by the quantum field theory methods;

conformal blocks of 2d conformal field theories as vectors in the CS theory Hilbert space;

correlators of Wilson loop and the invariants of 1d links in 3d manifolds;

braid group;

unusual relations between spin and statistics;

Here we would like to consider the generalization of a part of the outlined ideas to the CS theories on higher dimensional manifolds. Some of our results intersect with [5].

2 Hamiltonian approach

According to the standart correspondence between the Lagrangian and the Hamiltonian approaches the Chern-Simons action on a 5d manifold M with the boundary ∂M induces a closed 2-form Ω on the boundary values of the fields i.e. on the space of g-connections A on ∂M.

$$\Omega = tr \int_{\partial M} \delta A \wedge \delta A \wedge F, \tag{1}$$

where $F = dA + A \wedge A$ is the curvature form.

The form Ω is obviously degenerate and therefore the classical phase space corresponding to the theory is not the space of connections on ∂M itself but the quotient space of it with respect to the distribution of kernels of the form Ω.

The action of the gauge group on the connections on M preserves the Lagrangian up to full derivatives and therefore the 4d gauge group acting naturally on connections on ∂M preserves the form Ω. The corresponding Hamiltonian functions are

$$H_w = tr \int_{\partial M} wF \wedge F, \tag{2}$$

where w is an element of the gauge algebra i.e. the g-valued function on ∂M.

Consider the Hamiltonian reduction of the space of connections by the gauge group over the zero value of of the momentum mapping. At the first step we restrict the space to the subspace of connections, satisfying $H_W(A) = 0$ for all w i.e. satisfying

$$F \wedge F \perp g \tag{3}$$

i.e. $tr\, wF \wedge F = 0$.

The form Ω restricted to this subspace is degenerate. So in order to obtain the symplectic manifold we have to factorize it along the kernel of it.

In the standard Hamiltonian reduction of the symplectic manifold the kernel of the restricted symplectic form is spanned by the vector fields realizing the action of the group and therefore the factorization along the kernel of the symplectic form is equivalent to the factorization by the group action.

In our case the form Ω is itself degenerate. Its kernel is spanned by the vectors $\int tr\alpha \frac{\delta}{\delta A}$ satisfying

$$\alpha \wedge F + F \wedge \alpha \perp g \tag{4}$$

i.e. $tr\; w(F \wedge \alpha + \alpha \wedge F) = 0$ for all $w \in g$.

Now let us restrict ourselves to the case of one-dimensional abelian Lie algebra g. In this case the equation (3) will be

$$F \wedge F = 0. \tag{5}$$

The kernel of the form Ω restricted to the subspace defined by (5) is spanned by both the vectors $\alpha = df$ corresponding to the gauge algebra action and the vectors satisfying $\alpha \wedge F = 0$ belonging to the kernel of the unrestricted form Ω. Note that this space is spanned also by $\alpha = df$ and by $\alpha = L_v A$ where v is a tangent vector field on ∂M, because the Lie derivative $L_v A$ can be decomposed as $i_v F + di_v A$. The first term in this decomposition is the general solution for (4) and the second one is a gauge transformation. The vector fields $\alpha = L_v A$ realize the action of the diffeomorphism algebra on the space of connections and therefore we obtain a description of the reduced space as *a space of all connections A satisfying $F \wedge F = 0$ modulo gauge transformations and diffeomorphisms*.

Now consider the geometrical meaning of the condition (5). This condition means that the form F has at least two dimensional kernel defining a distribution on ∂M which is integrable due to the closeness of the form F.

Unfortunately the structure of this space is rather complicated. For example consider the tangent space to it at the given point A. The tangent space at this point $\mathbf{T}_A$ is isomorphic to the space of 1-forms a such that the infinitesimal transformation $A \to A + a$ preserves the condition (5) modulo closed forms and the forms satisfying (4) and thus

$$\mathbf{T}_A = \frac{\{a \mid da \wedge dA = 0\}}{\{a \mid a \wedge dA = 0\} + \{a \mid a = df\}}. \tag{6}$$

i.e. it is isomorphic to the first homology group of the factorcomplex of the de Rham complex.

Note that this space can be infinite dimensional. Let us consider the case when the distribution given by $\ker F$ defines a fibration over a two dimensional base with non simply connected fibers. It is evident that the corresponding cohomology group consists of functions on the base taking values in the first cohomology groups of fibers. The only hope for the reduced space to be finite dimensional is that generically the foliation is not a fibration and the general leaf of the foliation is dense in ∂M. Note also that the form $a \wedge F$ is closed and defines a mapping $\mathbf{T}_A \to H^3(\partial M)$.

The situation slightly simplifies in the case of the complex abelian group. Consider the complex abelian connection A with the curvature satisfying (5) and an additional condition $F\wedge\overline{F}\neq 0$. The distribution defined by $\ker F$ is complex and defines the complex structure on ∂M. In other words this complex structure is defined by the requirement on the form F to be holomorphic form of type (2,0). The space of such forms up to diffeomorphisms is known to be isomorphic by the Yau theorem to the moduli space of self-dual Kähler metrics. Unfortunately for compact manifolds the space of holomorphic integer 2-forms is always trivial.

Above we have mentioned that the structure of the nonabelian 5d Chern-Simons theory is much more complicated then the abelian one. However it seems to be probable that the CS theory with the SL(4) gauge group on a 5d manifold with a boundary is closely connected with the self-dual gravity on the boundary like the CS theory in three dimensions is connected with the WZW conformal field theory. In order to illustrate this connection let us consider the Riemann curvature tensor R on a four dimensional Riemann manifold M. The self-duality of this tensor implies that the conformal Weyl tensor C coincides with R. The Weyl tensor is known to satisfy

$$C_{abcd}C^{abce}=\frac{1}{4}C_{abcf}C^{abcf}\delta^e_d \qquad (7)$$

The Weil tensor C can be interpreted as an $sl(4)$-valued 2-form $C=C_{abcd}dx^c\wedge dx^d$ and the above relation can be rewritten as

$$C\wedge *C=\frac{1}{4}\mathrm{tr}(C\wedge *C) \qquad (8)$$

where we have used the symmetry of the Weyl tensor $C_{abcd}=C_{cdab}$. Now, if the form C is self-dual, the Riemann curvature tensor satisfies the CS equations of motion (3).

3 Lagrangian approach

In this section we would like to show how it is possible to interprete the path integral

$$<\Phi(A)>=\int\mu(A)\Phi(A)\exp(\frac{ik}{3}\int_M A(dA)^2) \qquad (9)$$

which arises in the Lagrangian approach to quantization of the 5d CS theory.(In the formula $\mu(A)$ is the shift invariant integration measure and $\Phi(A)$ is some functional of A supposed to be gauge invariant.) One could try, in principle, to define the path integral (9) as usual in terms of perturbation theory if one added a quadratic term to the action, either by hands or in the background field method (then it is possible to come back to the original theory by a limiting procedure). However instead of this routine approach we shall discuss here the perturbation theory directly for the path integral (9). It turns out to be possible due to the very general properties of the path integral like the shift invariance of the measure and the Ward identities.

Let us discuss which expressions we are going to average by means of the path integration.The natural observables are the polynomials of $F(x)$ and $F(x)\wedge F(x)$ ($F=dA$). Another type of observables is given by Wilson loops and by generalized Wilson loops:

$$\exp(iW_1(A))=\exp(i\int_{C_1}A) \qquad (10)$$

$$\exp(iW_3(A))=\exp(i\int_{C_3}AdA) \qquad (11)$$

where C_1 and C_3 are 1- and 3-dimensional submanifolds of M respectively. The correlators of Wilson loops can be obviously expressed in terms of that of F's, so we shall first discuss them.

The shift invariance of the measure is expressed by the equation

$$0=\int\mu(A)\frac{\delta}{\delta A}\Phi(A) \qquad (12)$$

or in other words one can integrate by parts under the sign of path integral. Notice that

$$F(x)\wedge F(x)=\frac{1}{3}\frac{\delta}{\delta A}\int_M A\wedge dA\wedge dA \qquad (13)$$

what,along with (12), allows us to express the correlators of $F(x)\wedge F(x)$ in terms of those of $F(x)$ only. The correlators of $F(x)$

$$<F(x_1)\dots F(x_n)> \qquad (14)$$

as one can easily see from the Ward identities should be invariant under diagonal diffeomorphisms of the product

$$\underbrace{M\times\ldots\times M}_{n\ \text{times}} \qquad (15)$$

This condition implies that all such correlators vanish anywhere except at the coincident points, so it is reasonable to suppose that such correlators can be constructed only from δ-forms and their external derivatives. However, by dimensional arguments, it can be easily verified that it is not possible to construct such forms for the correlators of the type (14). So we conclude that all the correlators of the type (14) are zero, at least in the sense of distributions, or in other words these correlators give zero when integrated with currents.

It is worth mentioning that our definition of $F(x)\wedge F(x)$ in the correlators is different from that one can obtain from $F(x_1)\wedge F(x_2)$ for $x_1\to x_2$. The latter implies operations with the product of two operators at one point. It is hard – but very interesting – to define such a product in the theory with the cubic action.

Now let us consider the correlators of Wilson loops (10),(11). The correlators of 1d Wilson loops being expressed in terms of the correlators of F's only (not of $F\wedge F$'s) are trivial. It is quite natural for the reason that there is no linking of 1d curves on a five dimensional manifold. We note also that vanishing of such correlators, being a consequence of the invariance condition, is a reminiscent of the fact that the $U(1)$ Wilson loops operator for $N=2$ superstring is BRST equivalent to the identity [3].

As to the correlators of 3d Wilson loops and the mixed correlators of 1d and 3d Wilson loops these have no reason to vanish since 1d and 3d curves on a 5d manifold can be linked nontrivially. The same is true for three 3d manifolds. A precise definition of such linking numbers will be given in sect. 4. As an example consider the simplest case, the correlator of W_1 and W_3. Neglecting the self-linking of the submanifold C_3 one easily gets

$$< \exp(im_1 \int_{C_1} A) \exp(im_3 \int_{C_3} AdA) >= \exp(-\frac{im_1 m_3}{k} \int_{C_1 \times D_4} \delta) \qquad (16)$$

where $\partial D_4 = C_3$ and δ is a delta-form on $M \times M$ of degree 5 dual to the diagonal cycle. The integral in the r.h.s. of (16) is by definition the linking number of C_1 and C_3 on M. Note that in the expression for the correlators of Wilson loops in 3d CS theory there is a term like (16) responsible for the Bohm–Aharonov interaction of two charged particles (Wilson loops are interpreted as world lines of the particles). In the 5d case there exists such a type of interaction for a membrane (3d world volume) and a particle (1d world line).For the membranes only (without the particles) there exists "Bohm–Aharonov" interaction of "three body" type. Note also that while in 5d CS theory it is easy to express the correlators of 3d Wilson loops in the form of some series, it is difficult to convert it to elementary functions.

Unfortunately, the nonabelian case of 5d CS theory can not be worked out in the same manner as the abelian one because in the nonabelian case one has to develop more serious investigation including gauge fixing, ghosts, ghosts for ghosts and so on. Even in the abelian case such a type of investigation may be useful for uncovering relations between 5d CS theories and $N = 2$ superstrings, which, we think, are very close.

4 Generalizied linking numbers

Here we are going to describe some generalizations of usual linking number which probably will be useful for calculations of correlators in our theory.

Consider a set of submanifolds $M_1^{n_1}, \ldots, M_N^{n_N}$ imbedded into R^D. Here n_i is a dimension of M_i and N is the total number of them. For the sake of simplicity we shall consider two opposite situations. The first one is when none of submanifolds intersect each other. In general it requires that

$$\forall i \neq j \;\; n_i + n_j < D \qquad (17)$$

(for example, circles in R^3 do not intersect in general position).

For further development it is necessary to define a "space of configurations" $\Sigma_{N,D}$. It is a space of all ordered sets of N distinguished points in R^D, or (which is obviously the same) $R^D \times R^D \times \ldots \times R^D$ (N times) without diagonals $\Delta_{ij} = \{(\overline{x}_1, \ldots, \overline{x}_N)$ where $\overline{x}_i = \overline{x}_j\}$.

Now we are able to define a mapping Φ from the cross-product of all our submanifolds to the $\Sigma_{N,D}$. It is essentially a cross-product of imbeddings. Since manifolds do not intersect the image will be a submanifold of the "space of configurations" and thus it will represent some homological class of it. In another words, let us consider the generators of the cohomology ring $H^*(\Sigma_{N,D}; Z)$, the pullback of them defines differential forms on $M_1, \ldots, M_N$. If we integrate them we should obtain (in a proper normalization)

integer numbers which are invariant under the action of the group of diffeomorphisms of R^D since the induced action on cohomologies is trivial.

It is worth mentioning that since the cohomologies of the "configuration space" are non-zero only in dimensions $l(D-1)$, it is necessary for some integer $l < N$

$$\sum_{k=1}^{N} n_k = l(D-1) \qquad (18)$$

Of course, this condition does not provide non-vanishing of our integrals. However in the case of two submanifolds we shall obtain the generalized Gauss formula. The precise result is that the invariants described above are nothing, but the linking numbers of couples of the submanifolds.

Another possibility realizes when the submanifolds can intersect each other but intersections of all of them at one point is prohibited. An appropriate configuration space will have a simple topological structure. It is diffeomorphic to the cross-product of the $(N-1)D-1$ - sphere and R^{D+1}, which has a single homology generator ω in dimension $(N-1)D-1$ and if dimensions n_i satisfy the condition

$$\sum_{k=1}^{N} n_k = (N-1)D - 1 \qquad (19)$$

we shall obtain a linking number by integrating pullback $\Phi^*\omega$ over the cross-product of submanifolds (Φ is the same mapping that we have described above but now its target space is the space of all ordered k-point subsets in the R^D for all k such that $1 < k < N+1$). For example three 3-submanifolds in R^5 link in the sense that two of them have one-dimensional intersection which in the usual way links with the third one.

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