CORRIGENDUM OF "*CONSTRUCTION OF KURANISHI STRUCTURES ON THE MODULI SPACES OF PSEUDO HOLOMORPHIC DISKS I*, SURVEYS IN DIFFERENTIAL GEOMETRY XXII (2018), 133–190"

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This note is the corrigendum of [FOOO3, Lemma 9.1]. It concerns a coordinate system near the boundary and corners of a smooth structure of the compactified moduli space $\mathcal{M}^{\rm d}_{k+1,\ell}$ of stable marked disk with $k+1$ boundary marked points and ℓ interior marked point. (See [FOOO3, Definition 2.3] for the definition of $\mathcal{M}^{\rm d}_{k+1,\ell}$.) The statement of the lemma has been also used in the applications appearing in our several other related articles. This corrigendum will not affect them and exactly the same proofs as therein apply if one replaces the statement of [FOOO3, Lemma 9.1] by that of Lemma 2 below. (The authors thank A. Daemi and a referee of [DF], who pointed out this error.)

1. Overview of the corrigendum

Let us recall the notation we used in [FOOO3, Lemma 3.4]. We study the map

$$\Phi : \prod_{a \in \mathcal{A}^{\rm s}_{\mathbf{p}} \cup \mathcal{A}^{\rm d}_{\mathbf{p}}} \mathcal{V}_a \times [0,c)^{m_{\rm d}} \times (D^2_\circ(c))^{m_{\rm d}} \to \mathcal{M}^{\rm d}_{k+1,\ell}. \tag{1}$$

Here $\mathcal{M}^{\rm d}_{k+1,\ell}$ is the compactified moduli space of stable marked disk with $k+1$ boundary marked points and ℓ interior marked points:

- $\mathbf{p}$ denotes an element of $\mathcal{M}^{\rm d}_{k+1,\ell}$.
- $\mathcal{A}^{\rm s}_{\mathbf{p}}$ (resp. $\mathcal{A}^{\rm d}_{\mathbf{p}}$) is the set of irreducible components of $\mathbf{p}$ which is a sphere (resp. disk).

We denote by Σ_a the irreducible component a of $\mathbf{p}$ and use the following notations:

- $\mathbf{p}_a$ is the marked disk or the marked sphere consisting of the irreducible component Σ_a together with the marked points of $\mathbf{p}$ on Σ_a and the nodal points of $\mathbf{p}$ on Σ_a.
- $\mathcal{V}_a$ is a neighborhood of $\mathbf{p}_a$ in its associated moduli space.
- $m_{\rm d}$ (resp. $m_{\rm s}$) is the number of the boundary nodes (resp. interior nodes) of $\mathbf{p}$.
- $[0,c)$ (resp. $D^2_\circ(c)$) is the parameter space to smoothen the boundary nodes (resp. interior nodes).

Date: 2024 February.

The map $\Phi_{s,\rho}$ is determined if we fix an analytic family of coordinates of the nodal points in the sense of [FOOO3, Definition 3.1]. See [FOOO3, Lemma 3.4].

For $r_j \in [0, c)$ $(j = 1, \dots, m_d)$ and $\sigma_i \in D_0^2(c)$, $(i = 1, \dots, m_s)$, we define[1] a coordinate system given by

$$\begin{aligned} T_j^{\rm d} &= -\log r_j \in \mathbb{R}_+ \cup \{\infty\}, \\ T_i^{\rm s} &= -\log|\sigma_i| \in \mathbb{R}_+ \cup \{\infty\}, \quad \theta_i = -{\rm Im}(\log \sigma_i) \in \mathbb{R}/2\pi\mathbb{Z}. \end{aligned} \tag{2}$$

$$t_j = 1/T_j^{\rm d} \in [0, -1/\log c), \quad \rho_i = \exp(\sqrt{-1}\theta_i)/T_j^{\rm s} \in D^2(-1/\log c). \tag{3}$$

Note that $\sigma_i = e^{-z}$ is a bounded holomorphic function of the variable $z = T_i^{\rm s} + \sqrt{-1}\theta_i$ on the right half plane $\{z \in \mathbb{C} \mid {\rm Re} z > 0\}$.

Composing these coordinate changes with the map Φ, we defined

$$\Phi_{t,\rho} : \prod_{a \in \mathcal{A}^{\rm s}_{\bf p} \cup \mathcal{A}^{\rm d}_{\bf p}} \mathcal{V}_a \times [0, -1/\log c)^{m_{\rm d}} \times (D^2_\circ(-1/\log c))^{m_{\rm s}} \to \mathcal{M}^{\rm d}_{k+1,\ell}. \tag{4}$$

In [FOOO3, Lemma 9.1] it was claimed that there is a unique smooth structure on $\mathcal{M}^{\rm d}_{k+1,\ell}$ such that (4) is a diffeomorphism. There is an error in this statement. (It is true that there is a unique C^1 structure on $\mathcal{M}^{\rm d}_{k+1,\ell}$ such that (4) is a diffeomorphism.)

To motivate and explain the idea of our current corrigendum, we start with explaining the reason why the statement, as it is, of the aforementioned [FOOO3, Lemma 9.1] does not hold for $\Phi_{t,\rho}$.

We consider the case when there is only one interior node and no boundary node. Then there is only one gluing parameter which we denote by $\sigma = e^{-(T+\sqrt{-1}\theta)}$. When we change the coordinate of the gluing parameter to $\sigma' = \lambda\sigma$, where λ is a positive real number, the parameters T, θ change to $T' = T - \log\lambda$, $\theta' = \theta$. Then the coordinate $\rho = e^{\sqrt{-1}\theta}/T$ becomes

$$\rho' = e^{\sqrt{-1}\theta}/T' = \frac{e^{\sqrt{-1}\theta}}{-\log\lambda + \frac{1}{|\rho|}} = \frac{\rho}{1-(\log\lambda)|\rho|}.$$

This function $\rho' = \rho'(\rho)$ is C^1 in ρ, but not C^2, at the origin. (This has been pointed out by A. Daemi and a referee of [DF].)

On the other hand, if we use the new coordinate system of $S = \log T$ and $\phi = e^{\sqrt{-1}\theta}/S$, then we have

$$\phi' = \frac{\phi}{1 + |\phi|\log(1-(\log\lambda)e^{-1/|\phi|})}.$$

All the derivatives (of arbitrary order) of the denominator of the right hand side function vanish at $\phi = 0$. In particular $\phi \mapsto \phi'$ is smooth at the origin.

In the case where there is no interior node and one boundary node, we consider a real parameter T and $t = 1/T$. If we change e^{-T} to λe^{-T}, then t is changed to t', where

$$t' = \frac{t}{1-(\log\lambda)t}.$$

[1] We change the symbol s used in [FOOO3] to t here. This is because we use S in (5).

This is smooth with respect to the non-negative real parameter t at $t = 0$.

Remark 1. We can construct a C^1 smooth structure of $\mathcal{M}^{\mathrm{d}}_{k+1,\ell}$ still using $\Phi_{t,\rho}$. As far as we use multi-sections to perturb the moduli space of pseudo-holomorphic curves (as in [FOOO1] and others), we can use a C^1 Kuranishi structure in place of the C^∞ Kuranishi structure. However when we use the de-Rham model to obtain a virtual fundamental chain as in [FOOO4], it seems more natural to use the C^∞ structure rather than the C^1 structure. For such a purpose, the C^∞ smooth structure arising from Lemma 2 will be important.

2. Double log smooth structure of $\mathcal{M}^{\mathrm{d}}_{k+1,\ell}$

Motivated by the discussion in the previous section, we revise the statement of aforementioned lemma as follows.

We take the step of taking another logarithm,

$$S^{\mathrm{d}}_j = \log T^{\mathrm{d}}_j, \qquad S^{\mathrm{s}}_i = \log T^{\mathrm{s}}_i, \tag{5}$$

and define

$$s_j = 1/S^{\mathrm{d}}_j \in [0, 1/\log(-\log c)), \quad \phi_i = \exp(\sqrt{-1}\theta_i)/S^{\mathrm{s}}_i \in D^2(1/\log(-\log c)). \tag{6}$$

Composing these coordinate changes with the map Φ, we define

$$\Psi_{s,\phi} : \prod_{a \in \mathcal{A}^{\mathrm{s}}_{\mathbf{p}} \cup \mathcal{A}^{\mathrm{d}}_{\mathbf{p}}} \mathcal{V}_a \times [0, 1/\log(-\log c))^{m_{\mathrm{d}}} \times (D^2_\circ(1/\log(-\log c)))^{m_{\mathrm{s}}} \to \mathcal{M}^{\mathrm{d}}_{k+1,\ell}. \tag{7}$$

Now the corrected version of [FOOO3, Lemma 9.1] is the following. We call the associated smooth structure *the double log smooth structure* of $\mathcal{M}^{\mathrm{d}}_{k+1,\ell}$.

Lemma 2 (Double log smooth structure)**.** *There exists a unique structure of smooth manifold with corners on $\mathcal{M}^{\mathrm{d}}_{k+1,\ell}$ such that $\Psi_{s,\phi}$ is a diffeomorphism onto its image for each $\mathbf{p} \in \mathcal{M}^{\mathrm{d}}_{k+1,\ell}$.*[2]

Remark 3. [FOOO3, Lemma 9.1] was used in various references of the present authors to obtain a smooth Kuranishi structure of the moduli spaces of pseudo-holomorphic curves. We can replace it by Lemma 2. Then all the results and proofs (which use [FOOO3, Lemma 9.1]) go through using the smooth structure of Lemma 2 in place of [FOOO3, Lemma 9.1].

In fact, if f is a smooth function on the parameter t_j, ρ_i then it is a smooth function on the parameter s_j, ϕ_i.

Remark 4. The coordinates s_j, ϕ_i are used in [FOOO4, (25.1)]. They not only define a smooth structure but also define an admissible smooth structure in the sense defined in [FOOO4, Chapter 25] as the proof below shows.

Proof of Lemma 2. During this proof we write $\Psi^{\mathbf{p}}$ and etc. to clarify that the relevant object is associated to $\mathbf{p} \in \mathcal{M}^{\mathrm{d}}_{k+1,\ell}$. We denote by $\mathfrak{v}^{\mathbf{p}}_a$ an element of the first factor of the domain of (7) for $\mathbf{p}$. We allow the positive constants C_n and c_n to vary in the various inequalities appearing in the proof below.

[2] And for any analytic family of coordinates of the nodal points which is used to define Φ.

Suppose $\mathbf{q} \in \mathrm{Im}(\Psi^{\mathbf{p}})$. Then $m_{\rm d}^{\mathbf{q}} \leqslant m_{\rm d}^{\mathbf{p}}$, $m_{\rm s}^{\mathbf{q}} \leqslant m_{\rm s}^{\mathbf{p}}$. We may enumerate the marked points so that the j-th boundary node (resp. the i-th interior node) of $\mathbf{p}$ corresponds to the j-th boundary node (resp. the i-th interior node) of $\mathbf{q}$ for $j = 1, \dots, m_{\rm d}^{\mathbf{q}}$ (resp. $i = 1, \dots, m_{\rm s}^{\mathbf{q}}$). This enables us to construct a natural (stratawise) smooth embedding $\Phi_{\mathbf{pq}} : \mathcal{V}^{\mathbf{q}} \hookrightarrow \mathcal{V}^{\mathbf{p}}$ so that we can compare the two functions $T_{j_0}^{\mathrm{d},\mathbf{p}} \circ \Phi_{\mathbf{pq}}$ and $T_{j_0}^{\mathrm{d},\mathbf{q}}$ on $\mathcal{V}^{\mathbf{q}}$. (See [FOOO2, Proposition 8.27] for the explanation of this coordinate change map $\Phi_{\mathbf{pq}}$.) This being explicitly said, we will abuse our notation by writing $T_{j_0}^{\mathrm{d},\mathbf{p}} \circ \Phi_{\mathbf{pq}}$ just as $T_{j_0}^{\mathrm{d},\mathbf{p}}$ dropping the composition by $\Phi_{\mathbf{pq}}$ in the following calculations.

Then we can easily prove the next inequalities:

$$
\begin{aligned}
\left\| \nabla^{n-1} \frac{\partial}{\partial T_j^{\mathrm{d},\mathbf{q}}} (T_{j_0}^{\mathrm{d},\mathbf{p}} - T_{j_0}^{\mathrm{d},\mathbf{q}}) \right\| &\leqslant C_n e^{-c_n T_j^{\mathrm{d},\mathbf{q}}} \\
\left\| \nabla^{n-1} \frac{\partial}{\partial T_i^{\mathrm{s},\mathbf{q}}} (T_{j_0}^{\mathrm{d},\mathbf{p}} - T_{j_0}^{\mathrm{d},\mathbf{q}}) \right\| &\leqslant C_n e^{-c_n T_i^{\mathrm{s},\mathbf{q}}} \\
\left\| \nabla^{n-1} \frac{\partial}{\partial \theta_i^{\mathbf{q}}} (T_{j_0}^{\mathrm{d},\mathbf{p}} - T_{j_0}^{\mathrm{d},\mathbf{q}}) \right\| &\leqslant C_n e^{-c_n T_i^{\mathrm{s},\mathbf{q}}}
\end{aligned}
\tag{8}
$$

for $j_0 = 1, \dots, m_{\rm d}^{\mathbf{q}}$. Here ∇^{n-1} $(n \geqslant 1)$ is the $(n-1)$-th derivatives on the variables $\mathfrak{v}_a^{\mathbf{q}}$, $T_j^{\mathrm{d},\mathbf{q}}$, $T_i^{\mathrm{s},\mathbf{q}}$, $\theta_i^{\mathbf{q}}$ and $\|\cdot\|$ is the C^0 norm. Similar estimates hold for $T_{i_0}^{\mathrm{s},\mathbf{p}}$ and $\theta_{i_0}^{\mathbf{p}}$.

In fact, to prove the 2nd and 3rd inequalities of (8), we use the fact that the functions $\sigma_i^{\mathbf{p}}$, $\sigma_i^{\mathbf{q}}$ are holomorphic functions defining the same divisor and show that $\sigma_i^{\mathbf{p}}/\sigma_i^{\mathbf{q}}$ is a nowhere vanishing holomorphic function. Then in the same way as [FOOO2, Sublemma 8.29] we obtain the 2nd and 3rd inequalities of (8). The 1st inequality is proved in the same way by taking the double as in [FOOO3, Section 2].

Moreover in the same way we can derive the following:

$$
\begin{aligned}
\left\| \nabla^{n-1} \frac{\partial}{\partial T_j^{\mathrm{d},\mathbf{q}}} \frac{\partial}{\partial T_{i_0}^{\mathrm{s},\mathbf{q}}} (\theta_{i_0}^{\mathbf{p}} - \theta_{i_0}^{\mathbf{q}}) \right\| &\leqslant C_n e^{-c_n (T_j^{\mathrm{d},\mathbf{q}} + T_{i_0}^{\mathrm{s},\mathbf{q}})} \\
\left\| \nabla^{n-1} \frac{\partial}{\partial T_i^{\mathrm{s},\mathbf{q}}} \frac{\partial}{\partial T_{i_0}^{\mathrm{s},\mathbf{q}}} (\theta_{i_0}^{\mathbf{p}} - \theta_{i_0}^{\mathbf{q}}) \right\| &\leqslant C_n e^{-c_n (T_i^{\mathrm{s},\mathbf{q}} + T_{i_0}^{\mathrm{s},\mathbf{q}})} \\
\left\| \nabla^{n-1} \frac{\partial}{\partial \theta_i^{\mathbf{q}}} \frac{\partial}{\partial T_{i_0}^{\mathrm{s},\mathbf{q}}} (\theta_{i_0}^{\mathbf{p}} - \theta_{i_0}^{\mathbf{q}}) \right\| &\leqslant C_n e^{-c_n (T_i^{\mathrm{s},\mathbf{q}} + T_{i_0}^{\mathrm{s},\mathbf{q}})}.
\end{aligned}
\tag{9}
$$

In the same way we can show similar estimates for higher derivatives on the gluing parameters.

The first formula of (8) applied to the case $j = j_0$ implies that $T_{j_0}^{\mathrm{d},\mathbf{p}} - T_{j_0}^{\mathrm{d},\mathbf{q}}$ is bounded. Using $\log(R + c) - \log R = c/R + O(1/R^2)$ and rewriting

$$
\begin{aligned}
S_{i_0}^{\mathrm{s},\mathbf{p}} - S_{i_0}^{\mathrm{s},\mathbf{q}} &= \log T_{i_0}^{\mathrm{s},\mathbf{p}} - \log T_{i_0}^{\mathrm{s},\mathbf{q}} \\
&= \log \left(T_{i_0}^{\mathrm{s},\mathbf{q}} + (T_{i_0}^{\mathrm{s},\mathbf{p}} - T_{i_0}^{\mathrm{s},\mathbf{q}}) \right) - \log T_{i_0}^{\mathrm{s},\mathbf{q}}
\end{aligned}
$$

(for $R := T^{\mathrm{s},\mathbf{q}}_{i_0}$, $c := T^{\mathrm{s},\mathbf{p}}_{i_0} - T^{\mathrm{s},\mathbf{q}}_{i_0}$), we can apply the inequality (8) and a similar inequality for $T^{\mathrm{s},\mathbf{q}}_{i_0}$ to derive

$$\begin{aligned} \left\|\nabla^{n-1}_{\mathfrak{v}}(S^{\mathrm{s},\mathbf{p}}_{i_0} - S^{\mathrm{s},\mathbf{q}}_{i_0})\right\| &\leqslant C_n/T^{\mathrm{s},\mathbf{q}}_{i_0} \leqslant C_n e^{-c_n S^{\mathrm{s},\mathbf{q}}_{i_0}} \\ \left\|\nabla^{n-1}_{\mathfrak{v}}(S^{\mathrm{d},\mathbf{p}}_{j_0} - S^{\mathrm{d},\mathbf{q}}_{j_0})\right\| &\leqslant C_n/T^{\mathrm{s},\mathbf{q}}_{j_0} \leqslant C_n e^{-c_n S^{\mathrm{d},\mathbf{q}}_{j_0}}. \end{aligned} \tag{10}$$

Here $\nabla^{n-1}_{\mathfrak{v}}$ $(n \geqslant 1)$ is the $(n-1)$-th derivatives on the variables $\mathfrak{v}^{\mathbf{q}}_a$. (In particular, the case $n-1=0$ is included.)

We next calculate, for $j \neq j_0$

$$\begin{aligned} \left\|\nabla^{n-1}_{\mathfrak{v}}\frac{\partial(S^{\mathrm{d},\mathbf{p}}_{j_0} - S^{\mathrm{d},\mathbf{q}}_{j_0})}{\partial S^{\mathrm{d},\mathbf{q}}_j}\right\| &\leqslant \left\|\nabla^{n-1}_{\mathfrak{v}}\left(\frac{T^{\mathrm{d},\mathbf{q}}_{j_0}}{T^{\mathrm{d},\mathbf{p}}_{j_0}}\frac{\partial}{\partial S^{\mathrm{d},\mathbf{q}}_j}\frac{T^{\mathrm{d},\mathbf{p}}_{j_0}}{T^{\mathrm{d},\mathbf{q}}_{j_0}}\right)\right\| \\ &\leqslant \left\|\nabla^{n-1}_{\mathfrak{v}}\left(\frac{T^{\mathrm{d},\mathbf{q}}_{j_0}}{T^{\mathrm{d},\mathbf{p}}_{j_0}}(T^{\mathrm{d},\mathbf{q}}_{j_0})^{-1}\frac{\partial(T^{\mathrm{d},\mathbf{p}}_{j_0} - T^{\mathrm{d},\mathbf{q}}_{j_0})}{\partial T^{\mathrm{d},\mathbf{q}}_j}\right)\right\| \|T^{\mathrm{d},\mathbf{q}}_j\| \\ &\leqslant C_n e^{-c_n(S^{\mathrm{d},\mathbf{q}}_j + S^{\mathrm{d},\mathbf{q}}_{j_0})}. \end{aligned}$$

For the second line we used the identity $T^{\mathrm{d},\mathbf{q}}_j\partial/\partial T^{\mathrm{d},\mathbf{q}}_j = \partial/\partial S^{\mathrm{d},\mathbf{q}}_j$. For the third line we used the inequalities in (8). (We also use the fact that $T^{\mathrm{d},\mathbf{q}}_{j_0}/T^{\mathrm{d},\mathbf{p}}_{j_0}$ is bounded together with its $\mathfrak{v}^{\mathbf{q}}$ derivatives.)

For $j = j_0$, we have an extra term

$$\left\|\nabla^{n-1}_{\mathfrak{v}}\left(\frac{T^{\mathrm{d},\mathbf{q}}_{j_0}}{T^{\mathrm{d},\mathbf{p}}_{j_0}}(T^{\mathrm{d},\mathbf{p}}_{j_0})^{-2}(T^{\mathrm{d},\mathbf{p}}_{j_0} - T^{\mathrm{d},\mathbf{q}}_{j_0})\right)\right\| \|T^{\mathrm{d},\mathbf{q}}_{j_0}\|$$

in the second line. It can be estimated also by $C_n e^{-c_n S^{\mathrm{d},\mathbf{q}}_{j_0}}$.

We can estimate $S^{\mathrm{s},\mathbf{q}}_i$ and $\theta^{\mathbf{q}}_i$ derivatives in the same way. The estimates of higher derivatives are similar.

Furthermore we can estimate derivatives of $S^{\mathrm{s},\mathbf{p}}_{i_0} - S^{\mathrm{s},\mathbf{q}}_{i_0}$ in the same way. Thus we have

$$\begin{aligned} \left\|\nabla'^{,n-1}\frac{\partial(S^{\mathrm{d},\mathbf{p}}_{j_0} - S^{\mathrm{d},\mathbf{q}}_{j_0})}{\partial S^{\mathrm{d},\mathbf{q}}_j}\right\| &\leqslant C_n e^{-c_n(S^{\mathrm{d},\mathbf{q}}_j + S^{\mathrm{d},\mathbf{q}}_{j_0})} \\ \left\|\nabla'^{,n-1}\frac{\partial(S^{\mathrm{d},\mathbf{p}}_{j_0} - S^{\mathrm{d},\mathbf{q}}_{j_0})}{\partial S^{\mathrm{s},\mathbf{q}}_i}\right\| &\leqslant C_n e^{-c_n(S^{\mathrm{s},\mathbf{q}}_i + S^{\mathrm{d},\mathbf{q}}_{j_0})} \\ \left\|\nabla'^{,n-1}\frac{\partial(S^{\mathrm{d},\mathbf{p}}_{j_0} - S^{\mathrm{d},\mathbf{q}}_{j_0})}{\partial \theta^{\mathbf{q}}_i}\right\| &\leqslant C_n e^{-c_n(S^{\mathrm{s},\mathbf{q}}_i + S^{\mathrm{d},\mathbf{q}}_{j_0})} \end{aligned} \tag{11}$$

and

$$\begin{aligned}
\left\| \nabla'^{,n-1} \frac{\partial (S^{\mathrm{s},\mathbf{p}}_{i_0} - S^{\mathrm{s},\mathbf{q}}_{i_0})}{\partial S^{\mathrm{d},\mathbf{q}}_{j}} \right\| &\leqslant C_n e^{-c_n(S^{\mathrm{d},\mathbf{q}}_{j} + S^{\mathrm{s},\mathbf{q}}_{i_0})} \\
\left\| \nabla'^{,n-1} \frac{\partial (S^{\mathrm{s},\mathbf{p}}_{i_0} - S^{\mathrm{s},\mathbf{q}}_{i_0})}{\partial S^{\mathrm{s},\mathbf{q}}_{i}} \right\| &\leqslant C_n e^{-c_n(S^{\mathrm{s},\mathbf{q}}_{i} + S^{\mathrm{s},\mathbf{q}}_{i_0})} \qquad (12) \\
\left\| \nabla'^{,n-1} \frac{\partial (S^{\mathrm{s},\mathbf{p}}_{i_0} - S^{\mathrm{d},\mathbf{s}}_{i_0})}{\partial \theta^{\mathbf{q}}_{i}} \right\| &\leqslant C_n e^{-c_n(S^{\mathrm{s},\mathbf{q}}_{i} + S^{\mathrm{s},\mathbf{q}}_{i_0})}.
\end{aligned}$$

Here $\nabla'^{,n-1}$ is the $(n-1)$-th derivatives on the variables $\mathfrak{v}^{\mathbf{q}}_a$, $S^{\mathrm{d},\mathbf{q}}_j$, $S^{\mathrm{s},\mathbf{q}}_i$, $\theta^{\mathbf{q}}_i$. (Here again $n \geqslant 1$ is assumed and so the case $n-1=0$ is included.) We can show a similar estimate for higher derivatives of $S^{\mathrm{d},\mathbf{q}}_j$, $S^{\mathrm{s},\mathbf{q}}_i$, $\theta^{\mathbf{q}}_i$ in the same way.

We next study $\theta^{\mathbf{q}}_{i_0}$. We put

$$f_{i_0} = \lim_{T^{\mathrm{s},\mathbf{q}}_{i_0} \to \infty} (\theta^{\mathbf{p}}_{i_0} - \theta^{\mathbf{q}}_{i_0}).$$

An analogue of (8) with $T^{\mathrm{d},\mathbf{q}}_{j_0}$ replaced by $\theta^{\mathbf{q}}_{i_0}$ implies that the right hand side limit exists which is smooth on the variables $\mathfrak{v}^{\mathbf{q}}_a$, $T^{\mathrm{d},\mathbf{q}}_j$ $T^{\mathrm{s},\mathbf{q}}_i$, $\theta^{\mathbf{q}}_i$. The function f_{i_0} is independent of $T^{\mathrm{s},\mathbf{q}}_{i_0}$, $\theta^{\mathbf{q}}_{i_0}$. Then we can use (9) to derive the following collection of inequalities:

$$\begin{aligned}
\left\| \nabla'^{,n-1} \frac{\partial f_{i_0}}{\partial S^{\mathrm{d},\mathbf{q}}_{j}} \right\| &\leqslant C_n e^{-c_n S^{\mathrm{d},\mathbf{q}}_{j}} \\
\left\| \nabla'^{,n-1} \frac{\partial f_{i_0}}{\partial S^{\mathrm{s},\mathbf{q}}_{i}} \right\| &\leqslant C_n e^{-c_n S^{\mathrm{s},\mathbf{q}}_{i}} \qquad (13) \\
\left\| \nabla'^{,n-1} \frac{\partial f_{i_0}}{\partial \theta^{\mathbf{q}}_{i}} \right\| &\leqslant C_n e^{-c_n S^{\mathrm{s},\mathbf{q}}_{i}}
\end{aligned}$$

and

$$\begin{aligned}
\left\| \nabla'^{,n-1} \frac{\partial (\theta^{\mathbf{p}}_{i_0} - \theta^{\mathbf{q}}_{i_0} - f_{i_0})}{\partial S^{\mathrm{d},\mathbf{q}}_{j}} \right\| &\leqslant C_n e^{-c_n(S^{\mathrm{d},\mathbf{q}}_{j} + S^{\mathrm{s},\mathbf{q}}_{i_0})} \\
\left\| \nabla'^{,n-1} \frac{\partial (\theta^{\mathbf{p}}_{i_0} - \theta^{\mathbf{q}}_{i_0} - f_{i_0})}{\partial S^{\mathrm{s},\mathbf{q}}_{i}} \right\| &\leqslant C_n e^{-c_n(S^{\mathrm{s},\mathbf{q}}_{i} + S^{\mathrm{s},\mathbf{q}}_{i_0})} \qquad (14) \\
\left\| \nabla'^{,n-1} \frac{\partial (\theta^{\mathbf{p}}_{i_0} - \theta^{\mathbf{q}}_{i_0} - f_{i_0})}{\partial \theta^{\mathbf{q}}_{i}} \right\| &\leqslant C_n e^{-c_n(S^{\mathrm{s},\mathbf{q}}_{i} + S^{\mathrm{s},\mathbf{q}}_{i_0})}.
\end{aligned}$$

Furthermore for each $n \geqslant 1$, we can combine (10), (11), (12), (13), (14) (and similar estimates for higher derivatives) to derive

$$\begin{aligned}
\left\| \nabla'^{,n-1} (s^{\mathbf{p}}_{j_0} - s^{\mathbf{q}}_{j_0}) \right\| &\leqslant C_n e^{-c_n / s^{\mathbf{q}}_{j_0}} \\
\left\| \nabla'^{,n-1} (\phi^{\mathbf{p}}_{i_0} - e^{\sqrt{-1} f_{i_0}} \phi^{\mathbf{q}}_{i_0}) \right\| &\leqslant C_n e^{-c_n / |\phi^{\mathbf{q}}_{i_0}|}.
\end{aligned} \qquad (15)$$

$(n \geqslant 1.)$ Finally it follows from (15) that the map $(\mathfrak{v}^{\mathbf{q}}, (s_j^{\mathrm{d},\mathbf{q}}), (\phi_i^{\mathbf{q}})) \mapsto (\mathfrak{v}^{\mathbf{p}}, (s_j^{\mathrm{d},\mathbf{p}}), (\phi_i^{\mathbf{p}}))$ is a diffeomorphism (including the point where some of the coordinates are zero).[3] This implies that the coordinate change map is smooth and so follows the lemma. $\square$

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[3] It is easy to see that $\mathfrak{v}^{\mathbf{p}}$ depends smoothly on $(\mathfrak{v}^{\mathbf{q}}, (s_j^{\mathrm{d},\mathbf{q}}), (\phi_i^{\mathbf{q}}))$.